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THE OCTOBER MEETING IN CAMBRIDGE

The five hundred twenty-seventh meeting of the American Mathematical Society was held at the Massachusetts Institute of Technology on Saturday, October 27, 1956, in conjunction with a meeting of the Society for Industrial and Applied Mathematics. About 240 persons attended, including 200 members of the Society.

An address entitled *Coverings of algebraic varieties* was presented by Professor J-P. Serre of the Collège de France and the Institute for Advanced Study by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor Richard Brauer presided.

Sessions for contributed papers were held in the morning and afternoon, with Professors A. P. Calderón, J. B. Diaz, Dr. A. P. Mattuck, Professors L. N. Howard, D. J. Struik presiding.

Abstracts of the papers presented follow. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Those papers with "*t*" following their numbers were presented by title. Dr. Edwards was introduced by Mr. D. G. Kendall, and Mr. Manly by Mr. R. R. Christensen.

ALGEBRA AND THEORY OF NUMBERS

1. A. T. Brauer: *On the theorems of Ledermann and Ostrowski on positive matrices.*

Let A be a positive matrix of order n, and R_ν the sum of the elements of the νth row. We set $\max R_\nu = R$ and $\min R_\nu = r$. A well known theorem of Frobenius [Sitzungsberichte Preussische Akademie der Wissenschaften (1998) pp. 471–476 and (1909) pp. 514–518] states that A has a greatest positive characteristic root ω. Moreover, we have $R \geqq \omega \geqq r$. For $R = r$, these inequalities cannot be improved. Assume now that $R > r$. We denote the smallest element of A by m. W. Ledermann [J. London Math. Soc. vol. 25 (1950) pp. 265–268] and A. Ostrowski [J. London Math. Soc. vol. 27 (1952) pp. 253–256] obtained better bounds which are functions of R, r, and m. In this paper the best possible bounds of this kind and some similar results are obtained. (Received September 10, 1956.)

2*t*. Barron Brainerd: *On a class of lattice-ordered rings*, I.

An F-ring R is a σ-complete vector lattice which is also a commutative algebra with unit $1 > 0$ in which $x \geqq 0$, $y \geqq 0 \Rightarrow xy \geqq 0$ and $x \wedge 1 = 0 \Rightarrow x = 0$. Let R be regular, that is, for each $x \in R$ there exists $x^0 \in R$ such that $xx^0x = x$. The sub-F-ring $\overline{R} = \{x \mid x \in R, |x| \leqq \lambda \cdot 1$ *for some real* $\lambda\}$ *is a* real Banach algebra. A (ring) ideal J of an F-ring is *closed* if $a_n \in J$ $(n \geqq 1)$ and $\sup_n a_n$ exists imply that $\sup_n a_n$ belongs to J. A maximal ideal M of R is closed if and only if $R - M$ is isomorphic to the real field. There is a one-to-one correspondence between the maximal ideals of R and those of $\overline{R}$. For a maximal ideal M of R to be closed, it is necessary and sufficient that $M \cap \overline{R}$ be a maximal ideal of $\bar{R}$. There is a one-to-one correspondence between the closed maximal ideals of R and those of $\bar{R}$. If the intersection of the closed maximal ideals of R is zero, then R is (F-ring) isomorphic to the F-ring of all $(\Phi, \mathfrak{J})$-measurable functions for a pair $(\Phi, \mathfrak{J})$ which is defined in terms of R. A real function f on Φ is $(\Phi, \mathfrak{J})$-*measurable* provided $\{x \mid f(x) \leqq \lambda\}$ belongs to the σ-algebra $\mathfrak{J}$ for each real λ. These results are obtained by exploiting the concept of projector as defined by S. Kakutani [Ann. of Math. vol. 42 (1941) pp. 223–237]. (Received August 8, 1956.)

3*t*. Barron Brainerd: *On a class of lattice-ordered rings*, II.

The terminology and notions of I are used. The set I of idempotents of an F-ring A forms a σ-complete Boolean algebra (σ-B.A.) under the lattice operations of A. I is the *idempotent algebra* (i.a.) of A. Every σ-B.A. I is the i.a. of a regular F-ring $R(I)$. A result of Sikorski [Fund. Math. vol. 35 (1949) pp. 247–258] is used to construct $R(I)$ which is a uniquely determined homomorph of the F-ring of all $(\Phi, \mathfrak{J})$-measurable functions for a pair $(\Phi, \mathfrak{J})$ defined in terms of I. Olmsted [Trans. Amer. Math. Soc. vol. 51 (1942) pp. 164–193] has constructed for each σ-B.A. I an F-ring $\Omega(I)$ with I as its i.a. If R is a regular F-ring with i.a. I, then R is isomorphic to both $\Omega(I)$ and $R(I)$. Every F-ring having I as i.a. is isomorphic to a sub-F-ring of $R(I)$. Let B be a semi-ordered ring with unit $1 > 0$ which is archimedian and in which $\inf_{\gamma \in \Gamma} a_\gamma = 0$ and $b \geqq 0 \Rightarrow \inf_{\gamma \in \Gamma} a_\gamma b = 0$. From results of Nakano [*Modern spectral theory*, Chapter V], it follows that B can be imbedded in an F-ring R. Therefore there is a σ-B.A. I such that B is isomorphic to a sub-semi-ordered ring of $R(I)$. (Received August 8, 1956.)

4. Walter Feit: *Arithmetic in algebras*. Preliminary report.

Let A be a semi-simple algebra over a field K, D an integral domain whose quotient field is K. Pick a basis of A such that the D module generated by this basis forms a ring R with a unit element. The ring R can then be written as a direct sum of a finite number of indecomposable left ideals. A left ideal is a direct summand if and only if it contains an idempotent element. If K is a p-adic field and D the ring of integers in K, then there is a natural one to one correspondence between the direct summands of R and those of the algebra R-DR. If K is an algebraic number field and D is suitably chosen, then it is possible to study arithmetical properties of A by considering the decomposition of R. In particular if A is the group algebra of a finite group G, then known arithmetical properties of the characters of G can be extended in this manner. (Received September 6, 1956.)

5. I. N. Herstein: *A theorem on the characteristic roots of symmetric matrices*.

The following theorem is proved: let A be a real symmetric matrix and let $r_1(A) \geqq r_2(A) \geqq \cdots \geqq r_n(A)$ be its characteristic roots; then each $r_i(A)$ is a monotonically increasing function of each diagonal element of A, that is $\partial r_i(A)/\partial a_{ii} \geqq 0$. If B is the Jacobian matrix of the $r(A)$'s with respect to the diagonal elements of A then B is a stochastic matrix. (Received September 7, 1956.)

6*t*. M. A. Hyman: *Eigenvalues and eigenvectors of real nonsymmetric matrices*. Preliminary report.

By $(N-1)(N-2)/2$ real rotations, the real nonsymmetric matrix A of order N is transformed into matrix B with all zeros above the first super-diagonal: $B = R^{-1}AR$. The eigenvalue problem for B is then $(B - \lambda I)x = 0$ where x is a column vector with components $x_1, x_2, \cdots, x_N$. Setting $x_1=1$ and guessing λ, one calculates $x_2, x_3, \cdots, x_N$ and $F(\lambda)$ recursively; $F(\lambda)$ is the characteristic polynomial, evaluated for the trial λ. By successive approximation (see abstract below) a sequence of λ's (in general complex) is obtained which approaches an eigenvalue $\bar{\lambda}$ of $B: F(\bar{\lambda})=0$. Simultaneously, the vectors x approach the eigenvector $\bar{x}$. The corresponding eigenvalue and eigenvector of A are $\bar{\lambda}$, $R\bar{x}$. Using $\bar{x}$, B, one easily obtains matrix C, in form like B but of degree $N-1$ which has the same eigenvalues as A (less $\bar{\lambda}$ once). He repeats the process above to get an eigenvalue and eigenvector of C, then obtains matrix D of order $N-2$, etc. At any stage, the corresponding eigenvector of A is obtained by a single matrix times vector multiplication. If one has some information about the location of the eigenvalues of A, he can avoid the reduction process sketched above in whole or in part; this will save time and increase somewhat the accuracy of the results. (Received September 10, 1956.)

7t. M. A. Hyman: *Finding complex roots by interpolation*. Preliminary report.

Consider $F(\lambda)$, a general function of the complex variable λ. It is not assumed that $F(\lambda)$ is known explicitly, but that, given λ, $F(\lambda)$ can be found from a rule (for example, built up by recursion). Assuming the existence of a root of $F(\lambda)$ in a certain neighborhood, one obtains from three approximations to the root, $\lambda_1, \lambda_2, \lambda_3$, the approximation λ_4; λ_4 is a particular root of the quadratic function passing through $\lambda_1, \lambda_2, \lambda_3$. From λ_4 and the "best two" (in a certain sense) of $\lambda_1, \lambda_2, \lambda_3$, one forms λ_5. Continuing, a sequence $\{\lambda_i\}$ is constructed which usually converges to the desired root $\bar{\lambda}$. The method is designed for use with automatic computing machines. It is perhaps slower than more sophisticated methods which exploit differentiability, algebraicity, etc. of $F(\lambda)$, but appears simpler to carry through and enables the machine to handle a wider class of problems without special additional programming. The method can be speeded up using "acceleration" techniques. It causes the computing machine to use "real arithmetic" when pursuing a real root, but shift automatically to "complex arithmetic" when the root is complex (thus saving computer time). This "interpolation" method can be applied using 2, 3, 4, $\cdots$ points at each step, but to date the 3-point method described here seems the most useful. (Received September 10, 1956.)

8. H. G. Jacob: *Coherence invariant mappings of symmetric transformations.*

L. K. Hua (Amer. J. Math. vol. 50 (1949) pp. 8–31) has characterized one-to-one coherence invariant mappings of the set of $n\times n$ symmetric matrices over a field. A generalization of Hua's results is obtained by considering a nonsingular hermitian scalar product (x, y) defined on a vector space $\mathfrak{X}$ of dimension ≥ 3 over a field of characteristic $\neq 2$. A linear transformation T on $\mathfrak{X}$ of finite rank may be written as $T=\sum_1^n x_i\otimes y_i$; i.e. $xT=\sum_1^n (x, x_i)y_i$. A transformation is called symmetric if it equals its adjoint relative to the hermitian form. Under these conditions any one-to-one coherence invariant mapping σ of the set of symmetric transformations of finite rank is of the form $T^\sigma=\sum_1^n x_i\bar{A}\otimes y_iA+S$, where A is a semi-linear transformation on $\mathfrak{X}$ with an associated automorphism which commutes with the automorphism associated with the hermitian form. $\bar{A}=cA$ where c is a scalar, and S is a fixed symmetric transformation. In the event that A has an adjoint $T^\sigma=\bar{A}^*TA+S$, where $\bar{A}^*$ is the adjoint of $\bar{A}$. (Received September 10, 1956.)

9*t*. J. A. Kalman: *Lattices with involution.*

I. Nondistributive (but possibly modular) lattices with involution, or "i-lattices," are classified with respect to six laws each of which, for distributive i-lattices, is equivalent to normality (cf. Bull. Amer. Math. Soc. Abstract 61-4-536). II. It is proved that for each given normal i-lattice L there exists one and to within i-isomorphism only one normal i-lattice with zero L_Ω [closed normal i-lattice L^Ω] having the property: there exists an i-isomorphism π of L into L_Ω [L^Ω] such that every i-isomorphism τ of L into a normal i-lattice with zero M [closed normal i-lattice N] is of the form $\tau = \pi\rho$ for some i-isomorphism ρ of L_Ω [L^Ω] into M [N]. (An "i-isomorphism" is a lattice-isomorphism which preserves the involution; a normal i-lattice L is "closed" if, in L_Ω (with involution $x \to x'$ and zero θ), $x \wedge y = \theta$ implies $x = z \vee \theta$ and $y = z' \vee \theta$ for some z). III. The i-lattice with four elements and two zeros is shown to be the only subdirectly irreducible non-normal distributive i-lattice. (Research supported by the University of New Zealand Research Fund.) (Received September 11, 1956.)

10*t*. Joachim Lambek: *Subgroups of the direct product of two groups.*

Given groups A, B, C, let R and S be subgroups of $A \times B$ and $B \times C$ respectively, their *relative product* $R \cdot S$ is a subgroup of $A \times C$. Let R^- denote the *converse* of R, then $R \cdot R^- \cdot R = R$. Hence $R \cdot R^-$ and $R^- \cdot R$ are congruence relations on subgroups of A and B respectively, and R induces an isomorphism between the corresponding subfactor groups. This fact appears to have been discovered by Goursat. In particular, if R and S are congruence relations on subgroups of a given group, then $R \cdot S$ induces the isomorphism of the well-known Zassenhaus lemma. These results hold not only for groups, but for all algebras with a ternary operation $f(x, y, z)$ such that $f(x, y, y) = x$ and $f(y, y, z) = z$. Examples are loops with $f(x, y, z) = (x/y) \cdot z$ and quasigroups with $f(x, y, z) = (x \cdot (y\backslash y))/(z\backslash y)$ [see A. I. Mal'cev, Mat. Sbornik N.S. vol. 35 (1954) pp. 3–20]. To obtain Goldie's generalization of the Zassenhaus lemma [Proc. London Math. Soc. vol. 52 (1950) pp. 107–131], one must replace $R \cdot S$ by its "difunctional closure" due to J. Riguet [Comptes Rendus vol. 230 (1950) pp. 1999–2000]. (Received September 10, 1956.)

11. Marvin Marcus: *Convex functions of quadratic forms.*

Let $a = (a_1 \cdots, a_n)'$ be a vector in the real n-space, E_n, and define the convex set $M(a)$ as the intersection of the half-spaces (I) $\sum_{j=1}^{k} t_{i_j} \leq \sum_{j=1}^{k} a_j$, $1 \leq k \leq n-1$, $1 \leq i_1 < \cdots < i_k \leq n$ and the hyperplane (II) $\sum_{j=1}^{n} t_j = \sum_{j=1}^{n} a_j$. Let H be the convex hull of the points Pa as P ranges over all n-square permutation matrices. Results (i) *If f is convex on H then* $\max_{t \in M(a)} f(t) \leq \max_P f(Pa)$. (ii) *If f is convex on E_n and nondecreasing in each variable and $N(a)$ is the convex intersection of the half-spaces* (I) *then* $\max_{t \in N(a)} f(t) \leq \max_P f(Pa)$. (iii) *If A is n-square Hermitian with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and $f(t_1, \cdots, t_k)$ is convex on the hypercube C defined by $\lambda_n \leq t_j \leq \lambda_1$, $k \leq n$; and $\partial f/\partial t_j = 0$ holds for at most one value of $t_j \in (\lambda_n, \lambda_1)$ for each fixed $t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_k$ then* $\max_{(x_i, x_j) = \delta_{ij}} f[(Ax_1, x_1), \cdots, (Ax_k, x_k)] = \max_{1 \leq i_j \leq n} f(\lambda_{i_1}, \cdots, \lambda_{i_k})$ *and a maximizing set of vectors $x_1, \cdots, x_k$ spans a k-dimensional invariant subspace of A*. For $f = \sum_{j=1}^{k} c_j t_j$, $c_j \neq 0$, (iii) is pertinent to a question raised by H. Wielandt (Proc. Amer. Math. Soc. vol. 6 (1955) p. 109). For $f = \sum_{j=1}^{n} \omega(t_j)$, (i) and (ii) follow from an inequality due to G. Polya (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 49–51). (Received September 7, 1956.)

12t. D. W. Robinson: *A characterization of n-groups.*

The generalized groups of W. Dörnte [*Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Zeit. vol. 29 (1928) pp. 1–19] are systems of elements with a polyadic operation satisfying an extension of the associativity and solvability axioms for ordinary groups. This note points out that these systems can be characterized as well by replacing the solvability axiom with a generalization of the identity—inverse axiom for groups. (Received July 11, 1956.)

13. Tadao Tannaka: *On the principal ideal problem and solution of a conjecture of Deuring.*

Professor Deuring proposed a conjecture concerning the precision of principal ideal theorem, at the conference on algebraic number theory which was held in Tokyo in September 1955. His conjecture runs as follows: Let K be the absolute class field over k, then for every k-ideal $\mathfrak{a}$ we can assign suitable element $\theta(\mathfrak{a})$ in K with $\mathfrak{a} = (\theta(\mathfrak{a}))$ and $\theta(\mathfrak{a})\theta(\mathfrak{b})\sigma(\mathfrak{a})/\theta(\mathfrak{a}\mathfrak{b}) \in k$ where $\sigma(\mathfrak{a})$ is Artin-automorphism of $\mathfrak{a}$. The author reduced this to the following lemma, which itself is a general principal ideal theorem and contains several known theorems as special cases. Let K be the absolute class field over k, K the direct compositum of cyclic intermediate fields K_i ($i = 1, 2, \cdots, r$) and $\mathfrak{p}_i$ be prime ideals in k which remain prime in K. Then there are K_i-ideals $\mathfrak{a}_i$ with $\mathfrak{p}_i \sim \mathfrak{a}_i^{1-\sigma_i}$ and product of these $\mathfrak{a}_i$ ($i = 1, 2, \cdots, r$) is principal in K. This lemma and the conjecture of Deuring are also generalized to the case of "Strahl" class field. (Received October 22, 1956.)

14t. George Whaples: *Cohomology of additive polynomial and nth power mappings of fields.*

Let K be a field of characteristic p and $f(x)$ an additive polynomial. The exact sequence $0 \to \text{kernel } f \to K^+ \to f(K^+) \to 0$ induces an exact sequence of cohomology groups. This can be used (J. T. Tate) to prove a theorem of the author's (Duke Journal 21) on the index of $f(K^+)$ in K^+. It also gives the following result: A field k has the property that $f(k) = k$ for every additive polynomial $f(x)$ ("Kaplansky's Hypothesis") if and only if k has no algebraic extension of degree divisible by p. Similar methods can be used with the multiplicative group of K and the nth power mapping, with results which are useful in local class field theory. (Received September 11, 1956.)

15. Harold Widom: *Nonisomorphic approximately finite factors.*

Given cardinals $\aleph$ and $\aleph'$ satisfying $\aleph_0 \leq \aleph' \leq \aleph$, a group G is found with the properties: (1) G has cardinality $\aleph$, (2) G is locally finite, (3) given a subset S of G of cardinality $< \aleph'$ there is an element of G other than the identity commuting elementwise with S, (4) there exists a subgroup G_0 of G with cardinality $\aleph'$ such that $e \neq g \in G$ implies $[g_0 g g_0^{-1} : g_0 \in G_0]$ is infinite. Let M be the type II_1 factor obtained from G in the usual way. Then M is approximately finite (A) with density character (relative to the metric $[[\ ]]$) equal to $\aleph$, any subfactor with density character $< \aleph'$ has a commutor in M larger than the complex numbers, but there exists a subfactor with density character $\aleph'$ whose commutor is exactly the complex numbers. Thus, for a fixed $\aleph$, the M corresponding to different $\aleph'$ are nonisomorphic. This proves the existence, on inseparable Hilbert space, of factors approximately finite (A) but not (B). (Received August 7, 1956.)

ANALYSIS

16. E. J. Akutowicz: *On the determination of the phase of a Fourier integral.* I.

Let $\hat{\phi}$ denote the Fourier transform of ϕ. The problem studied in this paper is to what extent the modulus of $\hat{\phi}$ determines ϕ? The main result is THEOREM I. *Let $C(a)$ be the class of all functions ϕ fulfilling the following conditions*: (i) $\phi \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty)$, (ii) *$\phi(t)$ is equivalent to* 0 *on a half-line $t < t_0 = t_0(\phi)$*, (iii) $\hat{\phi}(x) \neq 0$, $-\infty < x < \infty$, (iv) *$a(x)$ is a fixed function such that* $|\hat{\phi}(x)| = a(x)$, $-\infty < x < \infty$. *Then if ϕ_1 and ϕ_2 belong to $C(a)$ there subsists a relation between them of the form* (1) $\exp(i\gamma + i\beta x)B_1(x)\hat{\phi}_1(x) = B_2(x)\hat{\phi}_2(x)$, *where γ, β are real numbers and $B_1(x)$, $B_2(x)$ are limits as $y \to 0+$ of certain Blaschke products in the upper half-plane. $B_1(x)$ and $B_2(x)$ are holomorphic functions of modulus identically* 1. A question of some interest is whether there exists a function $\phi \in C(a)$ such that the holomorphic extension of its Fourier transform to the upper half-plane is free of zeros. It is shown that the answer is affirmative, and an explicit construction is given, under the hypothesis that a ψ exists in $C(a)$ such that the holomorphic extension of $\hat{\psi}$ has sufficiently sparse zeros. This paper is expected to appear in a forthcoming number of Transactions of the American Mathematical Society. (Received September 6, 1956.)

17*t*. E. J. Akutowicz: *On the determination of the phase of a Fourier integral.* II.

This abstract is a sequel to the preceding one. What is investigated here is the amount that the nonuniqueness of phase expressed in (1) above is decreased if the assumption (ii) above is replaced by the condition (ii)*: *ϕ is equivalent to* 0 *outside a finite interval, depending upon ϕ*. In (1) there appear (the limits of) two Blaschke products which are defined in terms of their respective zeros $\{z_n^{(1)}\}$ and $\{z_n^{(2)}\}$. Under the conditions (i), (ii), (iii), (iv) these sequences are necessarily without finite limit points, but are otherwise more or less arbitrary, subject to the convergence of the products. Under the conditions (i), (ii)*, (iii), (iv) these two largely independent sequences of zeros reduce to $\{z_n^{(1)}\}$ and $\{\bar{z}_n^{(1)}\}$. Thus one can assert that the restriction (ii)*, being twice as severe as (ii), results in a reduction of leeway in the phase by fifty percent. (Received September 6, 1956.)

18*t*. P. R. Beesack: *A note on an integral inequality.*

The main purpose of this note is to prove the following integral inequality: *Let $F(x)$, $G(x)$, $M(x)$ be integrable functions over a measurable set A.* Let $A_1 = \{x: F(x) \leq G(x)\}$, $A_2 = \{x: F(x) > G(x)\}$, *where $A = A_1 \cup A_2$. Suppose that $\int_A G dx \leq \int_A F dx$, and that either $0 \leq M(x_1) \leq M(x_2)$ or $M(x_1) \leq 0 \leq M(x_2)$ is satisfied for every pair x_1, x_2 of points such that $x_1 \in A_1$, $x_2 \in A_2$. Then $\int_A GM dx \leq \int_A FM dx$.* This generalizes a recent theorem of Tatarkiewicz (Ann. Univ. Mariae Curie-Sklodowska, Sect. A7 (1953) pp. 83–87 (1954)) and the method of proof is the same. As an application of this inequality we prove a result comparing the first eigenvalues of two second-order linear homogeneous differential systems. (Received September 12, 1956.)

19. J. W. Brace: *The topology of almost uniform convergence.*

Consider a linear space E of scalar valued functions (for simplicity). Let $\mathfrak{A}$ be a family of subsets of the domain space for the functions. Given a positive number ϵ

and a set $A \in \mathfrak{A}$, let $V(\epsilon, A)$ be a set in E such that its complement, $\tilde{V}(\epsilon, A)$, is a maximal subset of E with the property that for every finite subset $\{f_1, f_2, \cdots, f_k\} \subset \tilde{V}(\epsilon, A)$ there exists an $x \in A$ such that $|f_i(x)| > \epsilon$ for $i = 1, 2, \cdots, k$. With the usual restrictions on the family $\mathfrak{A}$, sets of the type $V(\epsilon, A)$ form a subbase for the neighborhood system at the zero function. A net of functions converges in the topology determined by the above neighborhoods if and only if the net of functions converges almost uniformly (see author, Portugaliae Math. vol. 14 (1955) pp. 99–104) on each $A \in \mathfrak{A}$. With further restriction on the members of $\mathfrak{A}$ the neighborhood system determines a linear topology which is locally convex. This technique gives some topologies that are not directly obtainable as uniform convergence topologies. The definition of neighborhood makes it possible to carry over many of the manipulative assets of uniform convergence topologies. (Received September 12, 1956.)

20. H. J. Bremermann: *Continuation of holomorphic functions on Stein manifolds. Connectivity of the boundary of pseudo-convex domains.*

Let D be a compact subdomain of a Stein manifold. Let the boundary of D be connected. Then any function that is holomorphic on the boundary of D can be continued holomorphically into D. For the proof D is enclosed in an analytic polyhedron P which is contracted continuously onto the empty set while the "Kontinuitätssatz" is applied on the intersection $\{\text{boundary } P\} \cap D$.—For schlicht domains in the C^n this is known as the "Hartogs-Osgood theorem." It holds also if D is a bounded domain in a complex Banach space of infinite dimension. In this connection it is of interest that the boundary of a pseudo-convex domain in a complex Banach space is connected. A domain is pseudo-convex if there exists a function $V(z)$ that is plurisubharmonic in D such that for arbitrary real M the closure of the point set $\{z \mid V(z) < M, z \in D\}$ is contained in D. In particular the domains of holomorphy (existence domains of a holomorphic function) are pseudo-convex and hence their boundary connected. (Received September 13, 1956.)

21. F. E. Browder: *Eigenfunction expansions for formally self-adjoint partial differential operators.*

Let A be a formally self-adjoint partial differential operator on a domain G of Euclidean n-space, B a positive differential operator on G. Extending previous results of Garding and the writer on elliptic operators as well as results announced by Gelfand and Kostyucenko for general differential operators with self-adjoint realizations, an eigenfunction expansion theorem of the Weyl-Plancherel type is established, with an expansion in eigenfunctions of $(A - \lambda B)$ which, in general, are distributions of a certain prescribed order. (Received September 11, 1956.)

22. Chandler Davis: *A Schwarz inequality for convex operator functions.*

Let f be a real function on $[a, b]$ whose domain of definition has been extended in the usual way to include operators A on Hilbert spaces ($a \leqq A \leqq b$). Then f is a *convex operator function* provided $f(tA + (1-t)B) \leqq tf(A) + (1-t)f(B)$ for operators in the domain and $t \in [0, 1]$. Bendat and Sherman characterized such f as real functions. Theorem: f is a convex operator function if and only if $Pf(PAP)P \leqq Pf(A)P$ for A in the domain and P a projection. Proof is brief. This paper will appear in the Proceedings. (Received September 12, 1956.)

23. D. G. Dickson: *Expansions in series of solutions of difference-differential equations.*

Let $\mathcal{H}[f(s)] = \sum_{j=1}^{n}\sum_{k=1}^{m_j} A_{jk}f^{(k)}(s+\omega_j)$, with A_{jk}, ω_j, s complex. Let $m = \max m_j$, P be the smallest closed convex set containing $\omega_1, \cdots, \omega_n$, $h(s) = \sum_{j=1}^{n}\sum_{k=1}^{m_j} A_{jk}s^k e^{\omega_j s}$, $\{\zeta_k\}$ the zeros of $h(s)$, n_k+1 the order of ζ_k, Q a translation of P. If $\phi(s)$ is analytic in the interior of Q, $\phi^{(m)}(s)$ of bounded variation and continuous on every straight line segment in Q; then there exists a series, whose terms are sums of not more than $2n$ terms of the form $p_k(s)e^{\zeta_k s}$ where $p_k(s)$ is a polynomial of degree at most n_k, that converges to $\phi(s)$ in Q minus its vertices. The convergence is uniform in any closed subset of Q excluding the vertices. The coefficients may be written explicitly and the sum at the vertices determined. If Q is a line segment on which $\phi^{(m)}(s)$ is of bounded variation, similar results are obtained. If $\mathcal{H}[\phi(s)] = 0$ in a region R and $\phi(s)$ is analytic in $R+P = \{s;\ s = r+p,\ r\in R,\ p\in P\}$, then $\phi(s)$ may be expanded in such a series converging with local uniformity in $R+P$. The coefficients are unique. Similar results are obtained if $\mathcal{H}[\phi(s)] = 0$ on a curve. The method consists in expressing the partial sums of the series in terms of closed contours tending to infinity. A study of exponential sums with asymptotically polynomial coefficients is included for contour construction and convergence. (Received June 6, 1956.)

24*t*. D. G. Dickson: *Expansions of solutions of infinite order differential equations.*

Let $\mathfrak{D}[F(s)] = \sum_{k=0}^{\infty} A_k F^{(k)}(s)$ where $d(s) = \sum_{k=0}^{\infty} A_k s^k$ is of exponential type σ. For $\tau \geqq 0$ let there exist a certain sequence $\{\Gamma_p\}$ of closed contours about the origin, tending to infinity, on which $d(s) > \exp[-(\tau+\epsilon)|s|]$ for every $\epsilon > 0$ when p is large. Let R be a region and for $r \geqq 0$ let $R\oplus r = \{s \mid s = u+v,\ u\in R,\ |v| \leqq r\}$. If $\phi(s)$ is analytic in $R\oplus(\sigma+\tau)$ and $\mathfrak{D}[\phi(s)] = 0$ in R, $\phi(s)$ may be expanded in R in a series whose terms are sums of terms $c_{kh}s^h \exp(\zeta_k s)$ where ζ_k is a zero of $d(s)$ of order greater than $h \geqq 0$. The convergence is locally uniform and the coefficients may be written explicitly. If the convergence is locally uniform in $R\oplus\sigma$, the coefficients are unique. It is always possible to pick $\tau = \sigma$. If $d(s) = \sum_{j=1}^{n} A_j s^{m_j}\{1+\epsilon_j(s)\} \exp(\omega_j s)$ where $\lim_{|s|\to\infty} \epsilon_j(s) = 0$, then τ may be taken as zero. In this case the series may be bracketed so that any term arises from at most $2n$ zeros of $d(s)$. (Received September 12, 1956.)

25. D. A. Edwards: *On the continuity properties of functions satisfying a condition of Sirvint's.*

Let X be a complex Banach space and let $\mathcal{S}$ (resp. $\mathfrak{D}$) be the class of functions $x(t)$ on the real line R to X with the property that for each compact interval $[a, b]$ there is a conditionally weakly compact (resp. bounded) set $E(a, b) \subseteq X$ (depending also on $x(\cdot)$) such that $\sum_{r=1}^{n}(x(t_{2r}) - x(t_{2r-1})) \in E(a, b)$ whenever $a \leqq t_1 < t_2 < \cdots < t_{2n} \leqq b$ ($n = 1, 2, \cdots$). Then $\mathcal{S} \subseteq \mathfrak{D}$; in a weakly sequentially complete space $\mathcal{S} = \mathfrak{D}$; and in any complex Banach space X the strong limits x $(t\pm 0)$ exist at every point $t\in R$ for each function $x(\cdot) \in \mathcal{S}$, and for such a function $x(t-0) = x(t) = x(t+0)$ for all save possibly a countable set of t-values. Consequently $x(\cdot)$ is separably valued. It also follows that the classical theorem on the decomposition of a real function of bounded variation into a sum of two components, one continuous and the other a step-function, has an analogue for functions in $\mathcal{S}$. If $x(t) = x(t+0)$ for a given $x(\cdot) \in \mathcal{S}$ then there exists a vector-valued measure $\mu: \mathfrak{B} \to X$, where $\mathfrak{B}$ is the ring of bounded Borel subsets of R, such that $\mu((a, b]) = x(b) - x(a)$ when $-\infty < a < b < \infty$. Most of these results are generalizations of previous theorems of D. G. Kendall and J. E.

Moyal, who, in a forthcoming paper, have shown that the present continuity theorems can be proved in a weakly sequentially complete space for functions of the class $\mathfrak{D}(-\mathfrak{S})$ without using the notion of compactness. (Received August 27, 1956.)

26. R. E. Fullerton: *Quasi-interior points and separation properties of cones in a linear space.*

Let X be a convex linear topological space of second category and let K be a convex cone in X with vertex at the origin. If $x \in K$ we define P_x to be the set $K \cap (x-K)$ and x is said to be a quasi-interior point of K if the linear extension $[P_x]$ of P_x is dense in X. It is shown that a necessary and sufficient condition that x be interior to K is that $[P_x]=X$ and that if K has an interior all quasi-interior points are interior points. Various properties of the set K_q of quasi-interior points of K are investigated and in particular it is shown that if two cones K and K' have no quasi interior points in common there exists a hyperplane in X separating K and K' and strictly separating K_q and K'_q. (Received September 10, 1956.)

27. D. S. Greenstein: *Approximation in norm of translates by Taylor series.* Preliminary report.

Given $f(x) \in C^\infty(-\infty, \infty)$ such that $f^{(n)}(x) \in L^p(-\infty, \infty)$ $(n \geq 0)$, an important approximation problem is concerned with the approximation in norm of the translates $f(x+h)$ by finite linear combinations of $f(x), f'(x), f''(x), \cdots$. In the present report, the author considers representation of $f(x+h)$ by a "Taylor series in the mean"; i.e., by the expression (1) $\text{l.i.m.}_{n\to\infty} \sum_{k=0}^{n} (h^k/k!) f^{(k)}(x)$. The radius of convergence of (1) is easily computed from the norms of the derivatives, and an error estimate shows that (1) can only converge to the proper value. For (1) to have a nonzero radius of convergence, $f(x)$ must be extendable to an analytic function which is regular in an open strip containing the real line and symmetric about it and belongs uniformly to L^p in every closed substrip. The radius of convergence of (1) is equal to the half width of the maximal such strip. For the case $p=2$, this result is related to a theorem of Paley and Wiener on the Fourier transform of a function analytic in a strip and may be used to derive that theorem. (Received September 10, 1956.)

28. E. L. Griffin, Jr.: *A result on derivations of operator algebras.*

A derivation of an operator algebra is a linear endomorphism possessing the Leibnitz property. An operator T is said to generate the derivation if for each operator A in the algebra, the closure of the operator $(AT-TA)$ equals the derivative of A. Theorem: *Let the self-adjoint operator T generate a derivation of the finite ring of operators M. If T belongs to the ring M, then the derivation is continuous. In fact, the derivation can be generated by a bounded, self-adjoint operator.* The proof is based on the lemma: *If the T mentioned in the first paragraph of the above theorem annihilates a projection whose trace is positive definite, then T is already bounded.* (Received September 12, 1956.)

29. Sigurdur Helgason: *Lacunary Fourier series on noncommutative groups.*

The ordinary definition of a lacunary Fourier series $\sum a_k e^{in_k\theta}$, $n_{k+1}/n_k > \lambda > 1$ involves the ordering of the integers. A new definition is given which applies to all compact groups, commutative or not. The theorem of Banach stating that a lacunary L^1-series is an L^2-series is extended to noncommutative groups. According to the new definition, Fourier series of the form $\sum a_n e^{ix_n}$ ($x_1, x_2, \cdots$ are independent variables) are lacunary Fourier series on the infinite torus T^∞, and theorems of Kolmogoroff about such series are reproved and extended to groups of the form $\prod_n U(n)$ ($U(n)$ denotes the unitary group in n-dimensions). A simple application is the following theorem together with its generalization to compact abelian groups of nonzero dimension: If $\sum a_n e^{i\sigma(n)x}$ is an L^1-series for arbitrary permutation σ of the integers, then $\sum |a_n|^2 < \infty$. (Received September 14, 1956.)

30. C. S. Herz: *The closure of translations in L^p.*

A closed subset, T, of the real axis is of type U^q if $\phi \in L^q$ and $\lim_{y\to\infty} \int_{-y}^{y} \exp(-itx) \cdot \phi(x)dx = 0$ for all $t \notin T$ imply $\phi = 0$ a.e. An equivalent statement which generalizes to several variables is that there is no nontrivial $\phi \in L^q \cap L^\infty$ whose spectrum lies in T. Given $f \in L^1 \cap L^p$, $1 < p < \infty$, say that "f has property C" if its translates span L^p, "f has property U" if the set of zeros of the Fourier transform is of type U^q, $1/p + 1/q = 1$. U implies C trivially. The converse is obvious for $p \geqq 2$. We observe that if the Fourier transform of f is in Lip ϵ for some $\epsilon > 0$ then C implies U when $p < 2$. The above remarks improve and extend to several variables the results of Pollard (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 100–104). (Received August 17, 1956.)

31t. Samuel Kaplan: *On the second dual of the space of continuous functions.*

Let C denote the Banach lattice of continuous real functions on a compact Hausdorff space X. In the paper the space of all bounded real functions on X is identified with a topological direct summand of the second dual M of C. Then each bounded function f on X represents a class or elements of M—those projecting onto f. In this class there are determined two distinguished elements, denoted by f_* and f^*, such that for each Radon measure μ on X (= an element of the first dual of C), $f_*(\mu) = \int_* f d\mu$ and $f^*(\mu) = \int^* f d\mu$, where $\int_*$, $\int^*$ are the lower and upper integrals of f. Moreover if f is integrable under every Radon measure, then $f_* = f^*$, giving a unique element whose value on each Radon measure μ is $\int f d\mu$. It is shown further that the set U of all the unique elements obtained in this way is precisely the closure of C in M under order-convergence (of general directed systems). The topology defined on M by the polars of the order-bounded sets of the first dual of C is also examined, and it is shown that C is dense in M under this topology. Finally a study is made of the specific relations between some properties of M and ordinary integration theory on X. (Received July 23, 1956.)

32t. D. G. Kendall and J. E. Moyal: *On the continuity properties of vector-valued functions of bounded variation.*

Suppose that X is a weakly sequentially complete complex Banach space and that $x(\cdot)$ is a function on the real line R to X such that for each compact interval $[a, b]$ a constant $K(a, b)$ exists for which $\left\| \sum_{r=1}^{n} (x(t_{2r}) - x(t_{2r-1})) \right\| \leqq K(a, b) < \infty$ whenever $a \leqq t_1 < t_2 < \cdots < t_{2n} \leqq b$. Then the strong limits $x(t \pm 0)$ exist for all $t \in R$, and $x(t-0) = x(t) = x(t+0)$ for all save possibly a countable set of t-values. It follows that $x(\cdot)$ is separably valued, and that the classical theorem on the decomposition of a real-valued function of bounded variation into two components, one continuous and the other a step-function, has an analogue for $x(\cdot)$. The Hilbert space versions of these results have applications to the expansion theorems of stochastic process theory. (Received August 27, 1956.)

33. W. T. Kyner: *An existence proof for a Stefan-like problem*. Preliminary report.

An existence proof for a quasi-linear parabolic differential equation with a free boundary has been obtained using Schauder's fixed point theorem. The maximum principal for parabolic equations was used in deriving the necessary estimates. (Received September 13, 1956.)

34. P. D. Lax: *On Cauchy's problem*. Preliminary report.

In this note it is shown that Cauchy's problem is incorrectly posed for a linear partial differential operator with variable coefficients which has a nonreal characteristic. The proof consists of demonstrating, just as Hadamard did for operators with constant coefficients, that high frequency initial data are exponentially magnified. The tool for showing this is an asymptotic expansion of solutions with such initial data. Such an expansion in the hyperbolic case may have some use in diffraction theory. (Received June 20, 1956.)

35. C. J. Lewis: *The problem of Milloux for functions analytic in an open annulus.*

Let $A=[0<s<|z|<1]$ in the complex plane; let $\mathcal{E}$ denote a set of points in A, closed relative to A, such that $[|z|=\rho]\cap\mathcal{E}\neq\emptyset$ for all ρ satisfying: $s<\rho<1$. For $0<m<1$, let $\mathfrak{F}_m$ denote the family of functions g which are defined, analytic, and such that $|g|<1$ on A, and each one of which has an associated $\mathcal{E}$-set with the property that $z\in\mathcal{E}$ implies $|g(z)|\leq m$. The problem of Milloux is to determine, for r satisfying: $s<r<1$, max $M(g; r)$ for $g\in\mathfrak{F}_m$, and the corresponding extremal functions. It is shown that if f is such an extremal, and if $f(r)=M(f; r)$, then (a) f is unique and independent of r; (b) f has an infinite number of zeros, all simple and negative, with -1 and $-s$ as cluster points of the zeros; (c) f is real for z real; (d) on the real axis between successive zeros there exists a unique point where $f^2=m^2$; (e) $f(-s^{1/2})=m$, $f'(-s^{1/2})=0$; (f) f' has simple zeros at the points referred to in (d) and at $-s^{1/2}$; (g) $f^2=m^2$ only at the points mentioned in (f). This result is an analogue of the solution of the corresponding problem for $[|w|<1]$ due to M. Heins (Amer. J. Math. vol. 67 (1945) pp. 212–234). If f is the normalized extremal for A, ϕ is the normalized extremal for the corresponding problem for $[|w|<1]$, ψ is a certain specific (1, 2) analytic map of A onto $[|w|<1]$, then $f=\phi\circ\psi$. f may be computed by using θ-functions or S-functions of Rausenberger. (Received September 13, 1956.)

36. Walter Nef: *Axiomatic theory of invariant measure and integral.*

Necessary and sufficient conditions are given for the existence of an integral, that is a monotonous and invariant linear form on a linear set of real functions, on which a group of linear transformations is acting. Corresponding conditions for the existence of invariant measures instead of integrals are communicated. (Received October 5, 1956.)

37. S. V. Parter: *Mapping of multiply connected domains by solutions of partial differential equations*. Preliminary report.

Z. Schapiro extended the Riemann Mapping Theorem to mappings which satisfy the uniformly elliptic system of partial differential equations $u_x=a_{11}(x, y, u, v)v_x+a_{12}v_y$, $-u_y=a_{21}v_x+a_{22}v_y$. [Z. Schapiro, *Sur l'existence des representations quasi-*

conformes, C. R. (Doklady) Acad. Sci. URSS, vol. 30 (1941) pp. 690–692.] This result is here extended to multiply connected domains. Theorem: Given a multiply connected domain D and a smooth family F of multiply connected domains, there is a mapping of D onto one of the domains of F which satisfies the system of partial differential equations. The problem is first reduced to that of mappings onto special slit domains. The existence of solutions in that case is then shown by using strong a priori estimates on the Hölder continuity of solutions. These estimates are obtained from a representation theorem of L. Bers and L. Nirenberg [*On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications*, Convegno Internazionale sulle Equazioni Derivate e Parziali, Agosto, 1954, pp. 111–140]. The concept of a smooth family of domains is rigorously defined. (Received June 14, 1956.)

38*t*. I. E. Segal: *Distributions on Hilbert space and canonical sets of operators.*

Using the formulation of (probability) distribution on a linear space given in Ann. of Math. vol. 63 (1956) pp. 160–175, absolute continuity on Hilbert space is treated and the results applied to the classification and transformation of canonical sets of quantum field operators. A correspondence between quasi-invariant (abelian) distributions (of which there exist a continuum of distinct classes for an infinite-dimensional space) and canonical sets of Bose-Einstein field operators is developed. In the case of irreducible sets arising from an isotropic normal distribution, the unitary equivalence classes are in 1-1 natural correspondence with the group of closed 1-forms on the space modulo the subgroup of exact forms, which group is nontrivial in the infinite-dimensional case. Those Fermi-Dirac field operators arising from (non-commutative) distributions are also examined. In either case the condition that there exist a unitary operator transforming one canonical set into another is determined, showing that concretely this is rarely the case. (Received September 11, 1956.)

39. V. L. Shapiro: *A best possible result in the uniqueness of double trigonometric series.*

Let $M=(m, n)$ where m and n are integers, $X=(x, y)$, $MX=mx+ny$ and $|X| =(x^2+y^2)^{1/2}$. Furthermore let T_2 be the two dimensional torus $\{(x, y), -\pi<x\leq\pi, -\pi<y\leq\pi\}$. Then in this paper the following theorem is proved: Let $\sum a_M e^{iMX-|M|t}$ tend to zero with t except for possibly a finite number of points in T_2. Suppose that $a_M=o(1)$ as $|M|\rightarrow\infty$. Then the a_M are identically zero and this result is a best possible one since if it is assumed that the $a_M=O(1)$, the theorem is false. The proof of the theorem follows from a judicious use of the following lemma: Let $E(t) =\sup_{0<h\leq t} \sup_{|X-P|=h} |F_{1,X}(h)-F_{1,P}(h)|$ where $F_{1,X}(h)$ represents the mean of F in the disc of radius h with center X and $F(X)$ is the anti-Laplacian of the given double trigonometric series. Then $E(t)$ tends to zero with t. (Received August 17, 1956.)

40. G. L. Spencer, II: *An inequality and uniqueness theorem for semilinear hyperbolic systems in two independent variables.*

A comparison inequality for approximate solutions of semilinear hyperbolic systems in two independent variables is obtained analogous to the comparison inequality for ordinary first order differential equations (See Coddington and Levinson, *Theory of ordinary differential equations*, p. 8, for the inequality in the ordinary case.) The uniqueness follows directly from the inequality. (Received September 13, 1956.)

41*t*. J. L. Walsh: *On infrapolynomials with prescribed constant terms.* Preliminary report.

Every factor $z^m+\cdots$, $m\geqq 2$, of a polynomial $p_n(z)\equiv z^n+\cdots+A_n\equiv\prod_1^n(z-\alpha_k)$ of this class K on a compact point set E (containing at least n points but $0\notin E$) also belongs to K. If $p_2(z)\in K$, we have $\phi_{\alpha_1}+\phi_{\alpha_2}+\phi_0\geqq\pi$, where ϕ_z is the angle subtended at z by E; if E lies in a closed circular disc Γ not containing 0, and if one zero of $p_n(z)$ lies in no disc intersecting Γ and subtending the same angle at 0 as Γ, then all other zeros of $p_n(z)$ lie in Γ. For E in $x>0$ and symmetric in $0x$, $p_n(z)$ real, the Jensen circle on $\alpha_k\bar{\alpha}_k$ as diameter is replaced by a circle tangent to $0\alpha_k$ and $0\bar{\alpha}_k$ at α_k and $\bar{\alpha}_k$ respectively; such circles contain all nonreal zeros of $p_n(z)$. If E (real) has the convex hull H: $(0<)\ a\leqq z\leqq b$, then all zeros of a real $p_n(z)$ with $(-1)^nA_n>0$ except perhaps one lie in H, and that one lies on $(-1)^nA_n/b^{n-1}\leqq z\leqq(-1)^nA_n/a^{n-1}$; a segment of $0x$ disjoint from E contains at most one zero of $p_n(z)$. If the last ν coefficients of $p_n(z)$ are prescribed $(\nu<n)$, we deduce $\phi_{\alpha_1}+\phi_{\alpha_2}+\cdots+\phi_{\alpha_n}+\nu\phi_0\geqq\pi$. (Received September 10, 1956.)

APPLIED MATHEMATICS

42. E. H. Bareiss: *The error estimation in an approximate solution of an ordinary second-order differential equation.*

This paper develops a method for estimating the error in the approximate solution of an ordinary second-order differential equation which may be given in graphical form. Although errors often cannot be detected by a point-by-point control because their magnitude lies between the limits of tolernace, they can accumulate undesirably after, say, 50 steps. A survey of the theory and a computing scheme is given by means of which any solution in graphical (or analytical) form may be checked, and the procedure is illustrated with examples for which exact solutions are known. The influence of variations in the initial conditions and in the differential equations is also considered. First-order equations are included as a special case. (Received September 12, 1956.)

43. J. B. Diaz: *On the numerical solution of* $u_{xy}=f(x, y, u, u_x, u_y)$.

The Euler-Cauchy polygon method is available for the numerical solution of the classical ordinary differential equation problem: $y_x=f(x, y)$; $y(x_0)=y_0$. An analogous method is developed for the solution of the classical hyperbolic partial differential equation problem: $u_{xy}=f(x, y, u, u_x, u_y)$; $u(x_0, y)=\tau(y)$; $u(x, y_0)=\sigma(x)$; $\sigma(x_0)=\tau(y_0)$. By means of this approach one can prove the existence theorem for this boundary value problem given by P. Hartman and A. Wintner, Amer. J. Math. (1952) and by P. Leehey, Brown University, Ph.D. thesis, 1950, which is based on the assumption that $f(x, y, z, p, q)$ satisfies a Lipschitz condition *only* with respect to p and q (this theorem generalizes the classical Picard existence theorem based on the assumption that $f(x, y, z, p, q)$ satisfies a Lipschitz condition in all three variables z, p, q). An auxiliary finite difference inequality, which plays a rôle in the present treatment similar to that played by a "convergence inequality" in the theory of the ordinary differential equation $y_x=f(x, y)$, appears to be of independent interest. (Received September 11, 1956.)

44. D. J. Dickinson: *On a generalization of the Legendre polynomials.*

Let $\{P_n^\nu(x)\}$ be a set of polynomials, each of degree precisely n, that satisfy

$(n+\nu)P_n^\nu(x)-(2n+2\nu-1)xP_{n-1}^\nu(x)+(n+\nu-1)P_{n-2}^\nu(x)=0$ for $n\geq 1$ where $P_{-1}(x)=0$ and $P_0(x)=1$. When $\nu=0$, they are the set of Legendre polynomials. Let $\{Q_n(x)\}$ be the Legendre functions of the second kind except that $Q_{-1}(x)=1$. The $\{P_n^\nu(x)\}$ are orthogonal over any contour that includes the unit circle and with respect to the weight function $Q_\nu(x)/Q_{\nu-1}(x)$. (ν is a nonnegative integer.) (Received September 17, 1956.)

45. R. M. Durstine (p) and D. H. Shaffer: *Determination of upper and lower bounds for solutions of linear differential equations.*

Let the function u be uniquely defined by $L(u)+\phi=0$, subject to boundary conditions on u. Here L is a linear, multi-dimensional differential operator; and ϕ is a known function of the independent variables. Then, by choosing two functions w_1 and w_2, which satisfy the boundary conditions imposed on u, and certain other restrictions, a pair of linear combinations of w_1 and w_2 which bound u may be formed. The method for generating these bounding functions is restricted to those problems for which the appropriate Green's function associated with L does not change sign. Various special cases have been carried out to demonstrate this method for both ordinary and partial equations. In one such example, the true solution to the problem is generated as a result of the coincidence of the upper and lower bounds. (Received September 10, 1956.)

46t. P. S. Dwyer: *A theorem on the multi-index transportation problem*. Preliminary report.

In the multi-index transportation problem, and hence in the mathematically equivalent multi-index assignment problem and group assembly problem, it is known that the set of all $x_{i_1 i_2 \cdots i_j \cdots i_k}$ such that $\sum_{i_j \text{ fixed}} x_{i_1 i_2 \cdots i_j \cdots i_k}=f_{j i_j}$ with $x_{i_1 i_2 \cdots i_j \cdots i_k} \geq 0$ and $\sum_{i_j} f_{j i_j}=N$ gives a modified problem which may have extreme points which are not integral solutions of the multi-index transportation problem. A practical solution of this modified problem, with possible positive fractional elements, results from a recently developed method of reduced matrices, which differs operationally from the simplex method. The results of this solution process of the modified problem can be used to reduce the multi-dimensional transportation problem to one or more elemental modified problems. By reordering of rows, columns, etc., the extreme point solution of the modified problem can be presented in a diagonal block form in which the "off diagonal" components are zero. A solution of the original problem is obtained by solving the modified multi-index transportation problem within each block. If this solution is nonintegral, necessary conditions for finding extreme points of the original problem "adjacent" to those of the modified problem can be found from the details of the solution of the modified problem. (Received August 8, 1956.)

47. Gabriel Horvay: *Biharmonic eigenvalue problem.*

When a simply connected, homogeneous, isotropic, plane elastic body is subject over a small portion $2h$ of its boundary to self-equilibrating normal and shear tractions then the stresses in the body are presumably small at distances r greater than $2h$ from the load application region. This is the content of St. Venant's principle. In mathematical formulation the problem requires (a) determination of two complete sets of boundary tractions into which given normal and shear tractions may be expanded, (b) determination of eigensolutions $G_n(x, y)$, $H_n(x, y)$ of the $\Delta^4 G=0$ equation whose appropriate derivatives reduce to the two complete sets of boundary tractions, (c) investigation of the decay behavior, with r, of the second derivatives of G_n, H_n. This program is carried out for the wedge region $x \geqq 0$, $-y_0 \leqq y \leqq y_0$, $y_0 = h + x \tan \omega$ which is loaded at $x = 0$; rigorously for the semi-infinite strip $\omega = 0$ and the semi-infinite plane $\omega = \pi/2$, in variational approximation for $-\pi/2 < \omega < \pi/2$. For $\omega = 0$ the decay is of type $e^{-\alpha_n x/h}$, for $\omega \neq 0$ the decay is of type $(r/h)^{-\mu_n}$. The decay is slowest for $\omega = \pm\pi/2$. α_n and μ_n are the real parts of roots of appropriate eigenvalue equations. References: *Biharmonic Eigenvalue problem*, Quart. App. Math., 1957; *St. Venant's principle*, J. App. Mech., 1957. (Received June 8, 1956.)

48t. M. A. Hyman: *A note on best numerical integration formulas.*

The integral $I = \int_a^b F(x)dx$ is commonly approximated by $I_N = \sum_{i=1}^{N} A_i F(x_i)$, where the points x_i are evenly spaced on the interval $[a, b]$. There are a multitude of "quadrature rules" for assigning the A_i (trapezoidal, Simpson's, and Weddle's are well-known rules). Some rules are better than others in the sense that, for N and $F(x)$ fixed, they produce in general a smaller "truncation error" $E = I - I_N$. G. Birkhoff (private communication) has remarked that, in integrating periodic functions over a full period, rules such as the trapezoidal can be expected to give truncation errors of the same order of smallness as more elaborate rules. The present note asserts the same result for "transient" functions: $F(x) = 0$ for $x \leqq a$ and $x \geqq b$. Integrals of such functions occur in many applications, including the probability analysis described in the preceding abstract. Use of "simple" integration formulas somewhat facilitates machine computation. Moreover, the equality of the weights A_i, when using the trapezoidal rule, is generally desirable from the stand-point of error control. The results hold for multi-dimensional integrals. (Received September 10, 1956.)

49. M. A. Hyman: *Probability analysis of simultaneous instrument errors.*

This paper considers a system in which a physical quantity F depends on certain other quantities $\alpha, \beta, \gamma, \delta, \cdots$, whose values fluctuate within certain limits set by feed-back controllers; the probable fluctuations of F are desired. It is shown that, under rather general conditions, $F = A(\alpha)B(\beta)C(\gamma)D(\delta) \cdots$, where $A, B, C, \cdots$ are monotonic functions of their variables. Taking logarithms, $f = \log F$, $a = \log A$, $b = \log B$, etc., and choosing $a, b, c, \cdots$ as new independent variables, the problem reduces to finding the probability distribution $\phi(f)$. It is shown how $\phi(f)$ (and various cumulative distributions of f) can be accurately (and rather elegantly) calculated numerically; these computations can be readily mechanized. Usually the distributions of $a, b, c, \cdots$ within their respective ranges are not known, but it is shown how conservative results can be obtained from whatever information is available. As the number of variables increases, $\phi(f)$ approaches the Gaussian distribution. The paper concludes with certain approximations which can be used, in place of the exact analysis described above, for estimating the probable fluctuations of F. (Received September 10, 1956.)

50. L. E. Payne (p) and H. F. Weinberger: *Pointwise bounds in Neumann problems.*

Let u be harmonic in a star shaped region D with boundary B. Let the integral of u over D vanish. By an extension of previous results of the authors (Journal of Mathematics and Physics vol. 33 (1955) p. 291) a lower bound is obtained for the ratio $\int_B (\partial u/\partial n)^2 ds / \int_B u^2 ds$. (The best such lower bound is the second Stekloff eigenvalue for

D, the first being zero.) This lower bound leads to a pointwise estimate for u when $\partial u/\partial n$ is prescribed, which is simpler than that indicated in the above-quoted paper. (Received September 13, 1956.)

51. D. E. Spencer: *A postulational foundation of special theories of relativity.*

The problem is to find the most general laws of transformation between unaccelerated observers. Three postultes are taken as fundamental: I There is no privileged observer, II Relative velocities of observers are defined operationally by Doppler shifts, III A body which is unaccelerated relative to one observer A must be unaccelerated relative to all observers which are in uniform motion relative to A. From these general postulates a transformation equation is obtained which contains as special cases Galilean relativity and Einstein's special relativity, as well as many other possibilities. The role of $v=c$ as a singularity is reinterpreted. The results are in sharp contrast with the work of Pars (Phil. Mag. vol. 42 (1921) p. 249) who believed that he had proved, by a similar analysis, that Einstein's second postulate ($c=$const.) follows "from the mere hypothesis of relativity." Logically all of the relativities here are equally plausible. (Received August 31, 1956.)

Geometry

52*t*. W. R. Abel and L. M. Blumenthal: *Metric arcs with Menger curvature.* I.

The following properties are established for any metric arc with finite Menger curvature at each point: (1) for all but a finite number of positive integers n, (i) each n-lattice L_n of the arc is a homogeneous $\lambda(n)$-chain, where $\lambda(n)$ denotes the distance of two consecutive points of L_n, and (ii) each n-lattice of the arc is unique, (2) each such arc is the sum of a finite number of metrically monotone arcs. Other theorems are (1) for any metric arc A, rectifiable or not, length $A=\lim_{n\to\infty}$ length L_n, and (2) every metric ptolemaic geodesic (that is, locally straight) arc is a metric segment. (Received September 12, 1957.)

53*t*. D. W. Crowe: *On the geometry of bicomplex numbers.*

The bicomplex numbers $s+iw=x+ju+i(y+jv)$, $i^2=j^2=-1$, can be considered as points (x, u, y, v) of euclidean 4-space, E^4. The numbers whose last two coordinates are zero constitute a plane, P. Alternately it is convenient to write $s+iw$ in the more symmetric form $pZ+qW$, where $p=(1+ij)/2$, $q=(1-ij)/2$, $Z=x+v+j(u+y)$, $W=x-v+j(u-y)$. This establishes a 1:1 correspondence between pairs (Z, W) of points of P and points of E^4. For fixed $A\in P$, the E^4 locus $A'=\{pAW+qW: W\in P\}$ is a plane isocline to P at $(0, 0, 0, 0)$. The 2-parameter family of planes A' has the geometry of a 2-sphere K (Stringham, Trans. Amer. Math. Soc. vol. 2 (1901) pp. 183–214), the angle between two planes A', B' being half the central angle between their representing points A'', B'' on K. It is shown that for a natural choice of K in E^4 the mapping $m(A'')=A$ is stereographic projection. A similar result holds if the plane A' is defined by $A'=\{pA\overline{W}+qW: W\in P\}$. (Received August 20, 1956.)

54. L. E. Dubins: *On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents.*

Motivation: Let a particle pursue a continuously differentiable path from an initial point u to a terminal point v. Suppose that its speed is unity and that its velocity vectors at u and v are prescribed to be U and V respectively. Suppose its radius of curvature is never less than R. What is the nature of a path of minimal length for the particle? *Definition*: Let u, v U and V be vectors in Euclidean n space, E_n. Let $\|U\| = \|V\| = 1$, let $R > 0$. Let C be the collection of all curves X, parametrized by arc length and defined on $[0, L]$, where $L = L(X)$ varies with X, such that: $X(s) \in E_n$ for $0 \leq s \leq L$; $\|X'(s) - X'(t)\| \leq R^{-1}|s-t|$ for all s and t; $X(0) = u$, $X'(0) = U$, $X(L) = v$ and $X'(L) = V$. *Proposition* 1: For any n, u, U, v, V and R, there exists an X in C of minimal length. Such a curve of minimal length is called an *R-Geodesic*. *Theorem*: *For $n = 2$, an R-Geodesic is necessarily a continuously differentiable curve which consists of not more than three pieces, each of which is either a straight line segment or an arc of a circle of radius R.* (Received September 12, 1956.)

55. David Gale: *A theorem on flows in networks.*

A *network* $[N, c]$ consists of a finite set of *nodes* N and a non-negative (real valued) *capacity function* c on $N \times N$. A *flow* on $[N, c]$ is a function f on $N \times N$ such that (1) $f(x, y) + f(y, x) = 0$ and (2) $f(x, y) \leq c(x, y)$ for all $x, y \in N$. A function d on N is called a *feasible demand* if there exists a flow f on $[N, c]$ such that $\sum_x f(x, y) \geq d(y)$ for all $y \in N$. *Theorem*: The function d represents a feasible demand if and only if for every subset $S \subset N$, $\sum_{y \in S} d(y) \leq \sum_{x \notin S, y \in S} c(x, y)$. A special case of this theorem is the well-known theorem of P. Hall on systems of distinct representatives of subsets. Another is a theorem of R. Rado concerning nonnegative functions defined on oriented graphs. The theorem is also used to give a simple criterion for when it is possible to construct a matrix of 0's and 1's whose rows and columns shall have preassigned sums. The proof of the main result makes use of the minimum cut theorem of Ford and Fulkerson. (Received September 11, 1956.)

56*t*. Shoshichi Kobayashi: *Isotropy group and holonomy group.*

It is known that the linear isotropy subgroup H_x at x of the largest connected group of isometries of an irreducible Riemannian space M is contained in the restricted homogeneous holonomy group h_x of M with reference point x, except in the case where M is a pseudo-Kaehlerian space with vanishing Ricci curvature. It can be proved that if M is irreducible and homogeneous, then H_x is always contained in h_x. This improves a theorem of Lichnerowicz. (Received September 13, 1956.)

57*t*. August Newlander and Louis Nirenberg: *An almost complex manifold satisfying the integrability conditions is complex analytic.*

In terms of suitable local complex coordinates $z^j = x^j + ix^{n+j}$, $\bar{z}^j = x^j - ix^{n+j}$ on a $2n$-dimensional real manifold with almost complex structure the Cauchy-Riemann equations may be written $L_j w = w_{\bar{z}^j} - \sum_{k=1}^{n} a_j^k w_{z^k} = 0$, $j = 1, \cdots, n$, where the coefficients vanish at the center of the coordinate neighborhood. The integrability conditions being satisfied is equivalent to the statement that the L_j commute. The problem is the introduction of new variables ζ^j, $\bar{\zeta}^j$ that transform the system to $w_{\bar{\zeta}^j} = 0$, $j = 1, \cdots, n$. This requires that the z coordinates as functions of the ζ coordinates satisfy the nonlinear "characteristic" system $z^i_{\bar{\zeta}^j} + \sum_{k=1}^{n} a_k^i \overline{z^k_{\zeta^j}} = 0$, $i, j = 1, \cdots, n$. A corresponding system of nonlinear integral equations may be set up which can be solved by iterations, assuming the a_j^i to be only finitely differentiable. (For real analytic integrable almost complex structure the problem was solved by B. Eckmann and A. Frölicher, C. R. Acad. Sci. Paris vol. 232 (1951) pp. 2284–2286.) The integrability conditions enter in showing that the resulting solution is a solution of the characteristic system. (Received October 24, 1956.)

58. S. R. Struik: *Measure theory of area and its tools.*

To express the ratio of two areas by a parallel vector pair is the basis of affine measure theory. Given two triangles, a third one, the affine reflection of the first, can be constructed with a basis equal to the basis of the second triangle, and collinear. The vertices of the second and third triangle are made to be on opposite sides of the common basis line. Their connection is divided by the basis line in the sought ratio. To construct this affine reflection necessitates: connecting points, intersecting lines, drawing parallels. Therefore a straight edge suffices as tool, besides two fixed segments $\overline{OA}$, $\overline{OB}$, on intersecting lines, with their bisecting points A', B' resp. The Desargues theorem (an axiom in affine two-space) admits of constructing through any point a parallel line to a given one (using only the straight edge) as long as the line shows one bisected segment. The above two-segment tool admits, therefore, of constructing all possible parallel lines. The three dimensional construction is analogous and simpler, because the Desargues theorem is here provable. The analogous tool is a tripod. Each leg has two equal segments. (Received September 12, 1956.)

LOGIC AND FOUNDATIONS

59. Kurt Bing: *On the axioms for order and succession.*

Hasenjaeger (Ein Beitrag zur Ordnungstheorie, Archiv fuer Mathematische Logik und Grundlagenforschung vol. 1 (1950) pp. 30–31) has answered a question raised by Hilbert and Bernays (Grundlagen der Mathematik I, Berlin, 1934, p. 279) by showing that the axiom (1) $a<b\rightarrow a'=b\vee a'<b$ can be proved, within lower predicate calculus, from the remaining axioms of the system (B) of Hilbert and Bernays, with $0=0$ replaced by $a=a$. The axioms from which (1) can be proved define a system (B_1) which is independent and is equivalent to (B). But the axiom systems obtained from (B) and (B_1) respectively by dropping the induction axiom are not comparable in strength. The proof uses the fact, established by the reduction method described by Hilbert and Bernays, that $a=a$ cannot be proved from (B) without the induction axiom. However, $\mathfrak{z}=\mathfrak{z}$ can be so proved for every numeral $\mathfrak{z}$. (Received September 10, 1956.)

STATISTICS AND PROBABILITY

60. Bernard Friedman (p) and Ivan Niven: *The average first-recurrence time.*

A discussion of the first recurrence time of a dynamical system of k degrees of freedom, each of which is simply periodic in the time t, leads to the following problem: Suppose $\alpha_1, \cdots, \alpha_k$ are all between zero and one, and suppose t is the smallest positive integer such that, for given $\epsilon>0$, there exist integers $m_1, \cdots, m_k$ satisfying the inequalities $|t\alpha_j-m_j|<\epsilon$, $(1\leq j\leq k)$. Put $t_{av}=$ average of t over all values of $\alpha_1, \cdots, \alpha_k$ in the k-dimensional unit cube. Then there exist two constants c_1 and c_2 such that $c_1\epsilon^{-k}<t_{av}<c_2\epsilon^{-k}$. If $k=1$, we prove $t_{av}=6\pi^{-2}(\log 2)\epsilon^{-1}+O(\epsilon^{-1/2})$. (Received September 12, 1956.)

61*t*. J. C. Kiefer and Jacob Wolfowitz: *On the deviations of the empiric distribution function of vector chance variables.*

Let $X_1, \cdots, X_n$ be independent vector chance variables with the common continuous distribution function (d.f.) F, and let S_n be their empiric d.f. Let G_n be the d.f. of $n^{1/2} \sup_x |S_n(x)-F(x)|$. Without loss of generality for what follows we assume that the marginal d.f.'s of each X_i are uniform in $[0, 1]$. Let A_k be the lattice points in the unit cube on the space of the X_i's whose coordinates are multiples of $1/k$. Let $H_{n,k}$ be the d.f. of $n^{1/2} \sup_{x \in A_k} |S_n(x)-F(x)|$. The authors prove: Theorem 1. $1-G_n(r) < c_0 e^{-cr^2}$ for all $r \geqq 0$, where c_0 and c are absolute positive constants. Theorem 2. There exists a d.f. G (depending on F) such that, at every point of continuity of G, $\lim_n G_n = G$. Also $\lim_{k\to\infty} \lim_{n\to\infty} H_{n,k} = G$ at every point of continuity of G. Similar results to those of these theorems hold for the joint d.f. of the signed deviations, and also when F is not continuous. (Received August 15, 1956.)

62. Ron Manly: *Determining bounding distribution functions for nonlinear functions of random variables.*

A distribution function—(γ)—is defined as bounding another distribution function—(β)— at a point $m+\Delta$ (where m is the median for the bounded distribution function), if $\alpha(\gamma) \geqq \alpha(\beta)$, where $\alpha(\phi) = \operatorname{sgn}(\Delta) \int_{m+}^{\operatorname{sgn}(\Delta)\infty} \phi\, dx$. Given n independent random variables $x_1, \cdots, x_n$ with means μ_i, distribution functions F_i, and bounds $A_i \leqq x_i \leqq B_i$; if a function $g = g(x_1, \cdots, x_n)$ has continuous first and second derivatives in a region G ($G_1 \leqq g \leqq G_2$; $A_i \leqq x_i \leqq B_i$), a procedure is described for determining a bounding distribution function for the distribution function of g.

Procedures are also described for special cases of rectangular distributions, empirical data, correlation between the variables, and simple functions involving products, powers, and exponentials. The use of bounding distribution functions may be applied: (1) to determine reliability of complex systems such as nonlinear circuits in digital computers, (2) as an alternative to Monte Carlo methods, and (3) in other Systems Engineering problems involving nonlinear combinations of random variables (Received August 13, 1956.)

TOPOLOGY

63. I. S. Gál: *On Lindelöf spaces and L-spaces.*

We introduce the following definitions: Let X be a general topological space and let A be an arbitrary subset of X. X is called an $L(A)$-space if every covering of A by a family of open sets contains a countable subcovering. If X is an $L(A)$-space for every A in X then X is called an L-space. X is called a P_0-space if every noncountable subset S of X has a point of accumulation in X. If every noncountable subset S of X has a point of accumulation which belongs to S then X is called a P-space. The purpose of the present paper is to study the notion of $L(X)$-, $L(A)$- and L-spaces. Some results of the theory of compact and relatively compact spaces can be duplicated for these new spaces by making simple changes in the proofs and in the additional definitions. However these spaces offer also new types of results. The main result concerns pseudo-metrizable L-spaces: For uniform spaces whose uniform structure satisfies the first axiom of countability the notions of an $L(X)$-space, of an L-space, of a P_0-space and of a P-space coincide. (Received October 3, 1956.)

64. J. P. Roth: *A combinatorial topological solution of the problem of Quine.*

The *extraction algorithm* is given for solving the problem of Quine (Bull. Amer. Math. Soc. Abstract 62-3-262); actually the problem solved is more general, allowing for "don't-care" conditions. Given a cubical complex K, an elementary cocycle is a cube which has no coface. Let Z_1 be the space of all cocycles. The *-algorithm computes Z_1. An extremal is a cocycle having a vertex unshared with any other cocycle. The $\nexists$-algorithm or the $\cap$-algorithm locates the subspace E_1 of extremals. In $\hat{Z}_2 = Z_1 - E_1$ a partial order $\prec$ is introduced; Z_2 is the set of maximal elements of $\hat{Z}_2$ under $\prec$. From Z_2 is extracted the subspace E_2 of 2d order extremals to form $\hat{Z}_3$. The algorithm proceeds inductively until an irreducible subspace Z_r is reached. Here the algorithm "branches" to produce an E_r, etc. until $Z_s = \phi$, $r < s$. Theorem: $E_1 + \cdots + E_s$ is a minimum cover. (Received September 12, 1956.)

65t. Norman Stein: *The third obstruction in complex projective spaces.*

In this paper secondary cohomology operations Ψ_n, $n \geq 1$, are defined. The operation Ψ_1 is the Shimada-Adem operation. The domain of Ψ_n is $N_n^2(K) = \{h \in H^2(K, Z) \mid h^{n+1} = 0 \in H^{2n+2}(K, Z)\}$ where K is any complex and h^{n+1} denotes the multiple cup product. The pairing $Z \otimes Z \to Z$ for each component of the multiple cup product is given by $1 \otimes 1 \to 1$. The range of Ψ_n is $H^{2n+3}(K, Z_2)/\{H^2(K, Z) \cup H^{2n+1} \cdot (K, Z) + Sq^2 H^{2n+1}(K, Z)\}$ where the pairing $Z \otimes Z \to Z_2$ in the cup product term is the Whitehead product pairing $\pi_2(M_n) \otimes \pi_{2n+1}(M_n) \to \pi_{2n+2}(M_n)$ in which M_n denotes the complex projective n-space. It is shown that Ψ_n is the cohomology operation corresponding to the second nontrivial Postnikov k-invariant of M_n. A cochain formula is given for Ψ_n and using this the third obstruction for a map $K^2 \to M_n$ whose first two obstructions vanish is computed. It has been conjectured by I. M. James that the pairing $\pi_2(M_n) \otimes \pi_{2n+1}(M_n) \to \pi_{2n+2}(M_n)$ is trivial and he has proved this in case n is odd. No proof has been found for n even, Using this result where $n = 2k+1$ the range of Ψ_{2k+1} is seen to be simply $H^{4k+5}(K, Z_2)/Sq^2 H^{4k+3}(K, Z)$. (Received July 18, 1956.)

66t. Chien Wenjen: *On pseudo-compact spaces.* I.

A completely regular space is *pseudo-compact* if every real-valued continuous function defined on it is bounded. (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) p. 67). A point in a topological space is a *limit point* of a family of disjoint open sets if every neighborhood of the point intersects infinitely many members of the family. A space is *almost compact* if every family of disjoint open sets has a limit point. A family of open sets in a topological space is an *almost covering* if the union of all the sets of the family is dense in the space. Let X be a completely regular space. The following statements are equivalent: (a) X is pseudo-compact; (b) X is almost compact; (c) Every infinite covering of X has a proper almost subcovering (cf. Arens-Dugundji, Portugaliae Math. vol. 9 (1950) p. 141); (d) Every countable covering of X has a finite almost subcovering. (Received August 20, 1956.)

67t. Chien Wenjen: *On pseudo-compact spaces.* II. Preliminary report.

An open net in a space is a net, the range of which consists of nonvoid open sets (instead of points of the space). An open set is a *cluster point* of an open net if every neighborhood of any point of the open set meets the net frequently. The following hold: (1) A pseudo-compact space is bicompact iff it is paracompact (cf. Arens-Dugundji, loc. cit. p. 142); (2) A completely regular space is bicompact iff each open net has a cluster point; (3) A connected, locally pseudo-compact space is metrizable iff it has a countable base; (4) A complete uniform space is pseudo-compact iff it is bicompact. (Received August 20, 1956.)

68. E. F. Whittlesey: *Classification of finite 2-complexes.*

Form a 2-complex K by "gluing" finitely many 2-cells on a connected 1-complex so that each oriented cell-boundary is mapped piecewise into a cycle in the 1-complex. Assume each 0- or 1-cell is incident with some 2-cell (but this restriction can later be removed). By "cut-and-paste" operations like those used for surfaces (see Lefschetz, *Introduction to topology*, pp. 72–84) such a 2-complex K can be reduced to canonical form K^* consisting of one or more ***surface components*** of the type $\alpha d_1\beta_1d_1^{-1} \cdots d_r\beta_rd_r^{-1}$, where $\alpha = a_1b_1a_1^{-1}b_1^{-1} \cdots a_pb_pa_p^{-1}b_p^{-1}$ or $c_1c_1 \cdots c_qc_q$ according as the surface component is orientable or not and where the β's are blocks of alternate capital and small letters denoting vertex and edge ***singularities*** (including boundary curves as particular edge singularities). K and K^* are equivalent, combinatorially and topologically. Thus K is classified by certain invariants, partly numerical and partly of order. (Received June 25, 1956.)

69. Hidehiko Yamabe: *Hilbert fifth problem on local groups.*

Hilbert's fifth problem on local groups can be formulated as follows: "Is every locally compact connected local group locally a direct product of a local Lie group and a compact group?" First one proves the existence of Haar measure. Next, one finds a compact, normal subgroup such that the factor local group has no small subgroups, which has a locally euclidean neighborhood U of e covered by one parameter subgroups. This is proved as follows. Let $\nu(x)$ be the smallest integer such that x is outside of some neighborhood U containing no small subgroup. Suppose that the set C of points on one parameter subgroups does not cover any neighborhood of e. Then there exists a sequence x_μ converging to e, with each x_μ not in C. Take y_μ on C such that $\nu(y_\mu^{-1}x_\mu) = \sup_{y \in C}(y^{-1}x_\mu)$. By taking a subsequence if necessary, we may assume that $(y_\mu^{-1}x_\mu)^{[p_\mu r]}$ converges to $u(r)$ where $p_\mu = \nu(y_\mu^{-1}x_\mu)$. Define $s_\mu(r)$ by $\lim_m (u(r/m)y_\mu(-r/m))^m$. For large μ, $\nu(y_\mu^{-1}x_\mu) \geq \nu(s_\mu(1/q_\mu)x_\mu)$ where $q_\mu = \nu(x_\mu)$. However $(s_\mu(1/q_\mu)x_\mu)^{q_\mu}$ converges to e which contradicts the above inequality. Thus $x_\mu \in C$ for large μ. (Received September 28, 1956.)

R. D. SCHAFER,
Associate Secretary

P17573

THE NOVEMBER MEETING IN PASADENA

The five hundred twenty-eighth meeting of the American Mathematical Society was held at the California Institute of Technology in Pasadena, California, on Saturday, November 17, 1956. Attendance was about 115, including 86 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Bertram Yood delivered an address on *Semi-simple Banach algebras*. He was introduced by Professor Edwin Hewitt, and the sessions for contributed papers presided over by Professors Richard Arens and R. P. Dilworth.

Following are abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been given by title. On joint papers, the presenter's name is followed by "(p)". Mr. Bryant was introduced by Professor Roy Dubisch and Dr. Montague by Professor Alfred Tarski.

ALGEBRA AND THEORY OF NUMBERS

70*t*. B. W. Brewer: *On certain character sums and related congruences.*

The number of solutions (x, w) of the congruence $x^k+Q\equiv w^2 \bmod p$, p a prime, is given by $p+(Q|p)+\psi_k(Q)$, where $\psi_k(Q)=\sum_{x=0}^{p-1}(x^k+Q|p)$, and has been expressed in terms of certain quadratic partitions of p for $k=3, 4, 5, 6$, and 8. For $p=a^2+b^2$ $(a\equiv 1 \bmod 4)$, $\psi_4(Q)=-2a(Q^{1/2}|p)-2$ if $(Q|p)=1$, and $\psi_4(Q)=\pm 2b$ if $(Q|p)=-1$ (Jacobsthal). The ambiguity in sign in the latter case has been removed by E. Lehmer (Pacific Journal of Mathematics vol. 5 (1955) pp. 103–118) if 2 is a quartic nonresidue of p. Theorem 1: If 3 is a quadratic nonresidue of the prime $p=a^2+b^2$ $(a\equiv 1 \bmod 4$, $b\equiv a \bmod 3)$, and $(Q|p)=-1$, then $\psi_4(Q)=3b(\alpha|p)$, where $3\alpha^2+Q\equiv 0 \bmod p$. Theorem 2: If the prime $p=12k+1=a^2+b^2=s^2+3t^2$ $(a\equiv 1 \bmod 4$, $s\equiv 1 \bmod 3)$, then (1) $\psi_{12}(Q)=-2a[1+2([-3]^{1/2}|p)](\alpha|p)-4s-2$ if $Q\equiv\alpha^6 \bmod p$, (2) $\psi_{12}(Q)=-2a[1-([-3]^{1/2}|p)](\alpha|p)+2s-2$ if $Q\equiv\alpha^2\not\equiv\alpha_1^3 \bmod p$, (3) $\psi_{12}(Q)=\pm 2b[1-2([-3]^{1/2}|p)]$ if $Q\equiv\alpha^3\not\equiv\alpha^2 \bmod p$, (4) $\psi_{12}(Q)=\pm 2b[1+([-3]^{1/2}|p)]\pm 6t$ if $Q\not\equiv\alpha^2$, $Q\not\equiv\alpha_1^3 \bmod p$. Theorem 3: If the prime $p=c^2+2d^2$ (hence $p=8n+1$ or $8n+3$), then $\sum_{x=0}^{p-1}((x+2)(x^2-2)|p)=2c$, where $c\equiv(-1)^{n+1} \bmod 4$; and if $p\neq c^2+2d^2$ (hence $p=8n-1$ or $8n-3$), then $\sum_{x=0}^{p-1}((x+2)(x^2-2)|p)=0$. Theorem 3 is an exact analog of the known results for primes of the form a^2+b^2 and s^2+3t^2. (Received September 24, 1956.)

71. S. J. Bryant: *Isomorphism order for abelian groups.*

A group G is said to have isomorphism order k if G has the following property: If H is a group such that every subgroup of H which can be generated by k or fewer elements is isomorphic to a subgroup of G then H is isomorphic to a subgroup of G.

Theorem: An abelian group G has isomorphism order k if and only if G is a direct sum of two groups, one torsion the other torsion free. The torsion free summand is a vector space over the rationals of dimension less than k while the torsion summand can be written as a direct sum of fewer than k subgroups of the rationals mod one. (Received October 2, 1956.)

72. R. C. Lyndon: *Solutions of equations in free groups and semigroups.*

Problem. Given words $f(x_1, \cdots, x_n)$, $g(x_1, \cdots, x_n)$, what can be said about the set of n-tuples of words $w_1, \cdots, w_n$ such that $f(w_1, \cdots, w_n) = g(w_1, \cdots, w_n)$? *Results.* (1) $x^2y^2 = z^2$ in a free group implies x, y, z all in a common cyclic subgroup; proof by ad hoc manipulation. (Problem of Vaught) (2) It is more or less obvious that, unless $f = g$, the set $w_1, \cdots, w_n$ must lie in a free subgroup (subsemigroup) of rank less than n. (3) $x^2y^2z^2 = t^2$ in a free semigroup implies x, y, z, t all lie in a free subsemigroup of rank no greater than 2; for any quadratic equation (each x_i appearing at most twice), method extends to decide if a similar result holds. (4) Given $f(x_1, x_2, \cdots, x_n)$ in a free group, there is an effective method for obtaining a finite set of words $w_i(x_2, \cdots, x_n; t_1, \cdots, t_m)$, depending on integer parameters t_j, such that the set of values w_i for all values of the t_j is the set of words $w(x_2, \cdots, x_n)$ such that $f(w, x_2, \cdots, x_n) = 1$. (5) Proof of (4) involves the study of R-groups G, for R a ring of operators, characterized by the axioms for (noncommutative) groups together with the following: $1g = g$; $ag \cdot bg = (a+b)g$ $a(bg) = (ab)g$; $g \cdot a(hg) = a(gh) \cdot g$. For R the ring of polynomials over the integers Z, the decision what retractions ϕ, of R onto Z, map w into 1, is reduced to a system of diophantine equations. In the application to (4) these equations may be taken as linear. (Received October 25, 1956.)

73. R. S. Pierce: *A generalization of atomic Boolean algebras.* Preliminary report.

Let α be an infinite cardinal number. A partially ordered set P is called α-finitary if P is closed under meets, has a zero and enjoys the property: (*) if M is a subset of P of cardinality $\leqq \alpha$ with the finite meet property, then M has a nonzero lower bound in P. A Boolean algebra B is called α-atomic if B contains a dense subset which is α-finitary (with respect to the induced ordering). Theorems: 1. If B is α-atomic for all α, then B is atomic in the usual sense and conversely. 2. A free σ-complete B.A. is $\aleph_1$-atomic. 3. If a Boolean algebra B contains a dense subalgebra which is α-atomic, then B is α-atomic. In particular, the normal completion of an α-atomic B.A. is α-atomic. 4. If B is an α-atomic B.A., then B contains a dense subalgebra B_0 which is isomorphic to an α-atomic, α-field. 5. An α-atomic B.A. is α-distributive. 6. An $\aleph_\xi$-atomic, $\aleph_{\xi+1}$-complete B.A. is an $\aleph_{\xi+1}$-homomorph of an $\aleph_{\xi+1}$-field. 7. There exists a σ-field F (necessarily not $\aleph_0$-atomic) with the property that its normal completion is not $\aleph_0$-distributive. (Received October 4, 1956.)

ANALYSIS

74t. G. E. Cross: *On the uniqueness of multiple trigonometric series.*

Consider the multiple trigonometric series $\sum c_m e^{i(m,x)}$, where $m = (m_1, \cdots, m_n)$, $x = (x_1, \cdots, x_n)$, $(m, x) = m_1x_1 + \cdots + m_nx_n$, and the summation is over all integers $m_j \geqq 0$. The series is said to be summable (T, k) if the series $\sum C_p(x)$ is summable (C, k). Here $C_p(x)$ denotes the "triangular" sum $\sum c_m e^{i(m,x)}$, $m_1 + \cdots + m_n = p$. It is shown that if, if a multiple trigonometric series is summable (T, k) for all values of x, then the coefficients c_m are given by repeated integrals of dimension $n+1$, the inner integral being a $\mathcal{P}^{k+1}$-integral defined by James (Trans. Amer. Math. Soc. vol. 76 (1954) pp. 149–176, §8) extended in the obvious way for a complex function of a real variable. In the case of double trigonometric series $\sum c_{mn}e^{i(mx+ny)}$, $m \geq 0$, $n \geq 0$, it is shown that, if there is a countable set x_j, y_j with $x_j \neq y_j$, such that the series is summable (T, k) to zero for all $x = x_j + t$, $y = y_j + t$, $0 \leq t \leq 2\pi$, then the series vanishes identically. (Received October 4, 1956.)

75. Edwin Hewitt: *The asymmetry of certain measure algebras.*

Let G be a locally compact Abelian group, and let $\mathfrak{M}(G)$ be the algebra of all complex-valued, bounded Radon measures on G, where addition and scalar multiplication are defined set-wise, and the product $\lambda * \mu$ of $\lambda, \mu \in \mathfrak{M}(G)$ is given by $\lambda * \mu(A) = \int_G \mu(x^{-1}A)d\lambda(x)$ for all Borel sets $A \subset G$. With $\|\lambda\| =$ the total variation of λ and $\tilde{\lambda}(A) = \lambda(A^{-1})$, $\mathfrak{M}(G)$ is a commutative Banach algebra with the involution $\lambda \rightarrow \tilde{\lambda}$. Theorem. If every neighborhood of the identity in G contains an element of infinite order, then $\mathfrak{M}(G)$ is asymmetric. Specifically, there exists a measure $\nu \in \mathfrak{M}(G)$ and a multiplicative linear functional M on $\mathfrak{M}(G)$ such that $M(\gamma) = 1$ and $M(\tilde{\gamma}) = 0$. (For the case $G =$ the additive real numbers, this was proved by Yu. A. Šreĭder [Mat. Sbornik N.S. vol. 27 (69) (1950) pp. 297–318].) It follows that the usual Fourier-Stieltjes transforms are not dense in the space of all multiplicative linear functionals on $\mathfrak{M}(G)$ and that there exists a measure $\mu \in \mathfrak{M}(G)$ such that μ^{-1} does not exist, and yet the Fourier-Stieltjes transform of μ is bounded in absolute value away from zero. (Received September 19, 1956.)

76*t*. V. L. Klee, Jr. *An example in the theory of topological linear spaces.*

If E is a locally convex Hausdorff linear space, then (a) for each $x \in E \sim \{\phi\}$, E admits a continuous linear functional f such that $fx \neq 0$. In sharp contrast, there are Hausdorff linear spaces S such that (b) the only continuous linear functional on S is that which is identically zero. Henriksen has asked whether the property (a) of a Hausdorff linear space E must be inherited by its quotient space E/M, where M is a closed subspace of E. (Note that the answer is affirmative if E is locally convex, or if E is complete and metrizable and M admits in E a closed supplementary subspace.) The present paper describes a metric linear space E with property (a) and closed supplementary subspaces M and N of E such that E/M and E/N both have property (b). Some other examples are also constructed—in particular, a linear space L and total linear subspaces F and G of L^* such that L is separable under the F-topology and under the G-topology, but not under the $(F+G)$-topology. (Received October 1, 1956.)

77. A. V. Martin: *On derivatives and neighborly functions.*

W. W. Bledsoe defined a function $f: S \rightarrow T$ (where S and T are spaces having metrics d and d') to be ***neighborly*** at the point x of S if, for every $\epsilon > 0$, there is a nonempty open sphere U of S such that, for every y in U, $d(x, y) + d'(f(x), f(y)) < \epsilon$ (see Bledsoe, *Neighborly functions*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 114–115). Since U is not required to contain x, this is a generalization of a continuous function. Theorem: If f is any real-valued function of a real variable whose derivative f' is defined everywhere and is Riemann-integrable over every finite interval, then f' is neighborly. This theorem offers some prospect of being useful in connection with the unsolved problem of finding an intrinsic characterization of those functions which are derivatives. Theorem: Let k map the complete metric space S into the metric space T, and let G be the graph of k in the topological product space $S\times T$. Then k is neighborly at every point of S if and only if the set of points of $S\times T$ of the form $(x, k(x))$, where k is continuous at x, is dense in G. (Received October 2, 1956.)

78. J. C. C. Nitsche: *Remark on harmonic mappings.*

Let $x=\operatorname{Re} F(\zeta)$, $y=\operatorname{Re} G(\zeta)$, $z=x+iy$, $\zeta=\xi+i\eta$ be a one-to-one harmonic mapping of the unit disc $\Gamma[|\zeta|<1]$ onto the unit disc $C[|z|<1]$ leaving the origin fixed. (Without loss of generality one can assume $F(0)=G(0)=0$.) In C there exists a function $\phi(x, y)$ such that $\phi_{xx}=|G'|^2\cdot[\operatorname{Im} F'G']^{-1}$, $\phi_{xy}=-[\operatorname{Re} F'G']\cdot[\operatorname{Im} F'G']^{-1}$, $\phi_{yy}=|F'|^2\cdot[\operatorname{Im} F'\bar{G}']^{-1}$ and $\phi_{xx}\phi_{yy}-\phi_{xy}^2=1$ (cf. a preceding abstract and K. Joergens, Math. Ann. vol. 129, Lemma 4). According to a theorem of H. Lewy (Bull. Amer. Math. Soc. vol. 42) $\operatorname{Im}(F'\bar{G}')=x_\xi y_\eta-x_\eta y_\xi\neq 0$ in Γ. The transformation $w=u+iv$, $u=x+\phi_x$, $v=y+\phi_y$ yields a one-to-one mapping of C onto a domain of the w-plane containing at least the interior of a circle with radius 1 (cf. Techn. Rep. No. 54, 1956, Stanford Univ. and H. Lewy, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 365–374). Now observe that w is an analytic function of ζ and $dw/d\zeta=F'(\zeta)+iG'(\zeta)$. Thus because of Schwarz's lemma, at $\zeta=0$, $|dw/d\zeta|\geq 1$ and $|F'|^2+|G'|^2\geq|F'+iG'|^2/2\geq 1/2$. Since $|F'|^2+|G'|^2=x_\xi^2+x_\eta^2+y_\xi^2+y_\eta^2$, one therefore gets the inequality $(x_\xi^2+x_\eta^2+y_\xi^2+y_\eta^2)_{\xi=\eta=0}\geq 1/2$. The bound 1/2 is an improvement of the bound $2-8(\sum_{n=1}^{\infty} n^{-2})/\pi$ given by E. Heinz (see E. Heinz, Goettinger Nachr. Math.-Phys. Kl., IIa, 1952). Prof. H. Hopf kindly has communicated to the author another very simple proof of the bound 1/2. (Received October 3, 1956.)

79. Mishael Zedek: *Infra- (n, s)-polynomials on a real set.* Preliminary report.

The terminology and notation of Walsh and Zedek's *On generalized Tchebycheff polynomials*, Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) is used. If $I_n^s(z)=z^n+A_1z^{n-1}+\cdots+A_sz^{n-s}+a_{s+1}z^{n-s-1}+\cdots+a_n$ is an infra-(n, s)-polynomial belonging to a real compact set E in the complex plane consisting of at least $n-s$ points, then for any set of $s+1$ zeros of $I_n^s(z)$ not on the convex hull $C(E)$ of E, either the sum of angles subtended by E at the subset of zeros which are above the real axis, or the sum of the angles corresponding to the zeros below the real axis, or both, are at least π. If the prescribed coefficient A_1 in $I_n^1(z)$ is nonreal, then all the zeros of $I_n(z)$ lie on $C(E)$ or on the same side of the real axis as $-A_1/n$. In fact, $n-1$ of them lie on a semi-circle having the convex hull $C(E)$ as its diameter. For $I_n^2(z)$ it may be shown that if A_1 is real then $I_n^2(z)$ has no real zeros outside $C(E)$, unless A_2 is also real, in which case $I_n^2(z)$ may have a real zero there only when all its zeros are real. If A_1 is not real and has, say, a negative imaginary part, then $n-2$ of the zeros of $I_n^2(z)$ are in the union of two discs. $C(E)$ is the diameter of one of them and is also a chord of the other disc subtending an angle of $2\pi/3$ at its center. The center of the larger disc is above the real axis. (Received October 4, 1956.)

APPLIED MATHEMATICS

80. J. B. Rosen: *Nonlinear programming. The gradient projection method.*

Let R be a bounded n-dimensional convex region defined by the constraint inequalities $\lambda_i=\sum_{j=1}^{n} n_{ij}x_j-b_i\geq 0$, $i=1, 2, \cdots, k$ and $\lambda_{k+j}=x_j\geq 0$, $j=1, 2, \cdots, n$.

Let $F(x)$ be a real-valued function of class $C^1 \in R$. Given an arbitrary starting point $x^0 \in R$, the Gradient Projection method will find a point $x_{\max} \in R$ (interior or on the boundary B) for which $F(x)$ has a maximum (or local maximum). The method follows the gradient $g(x) = \nabla F(x)$ by a specified stepwise procedure and will converge to $x_{\max}$. For $x_{\max} \in B$ it is necessary to choose at each step a linearly independent subset of $l < n$ of the hyperplanes $\lambda_i = 0$, $i = 1, 2, \cdots, k+n$. The projection, u, of g on the intersection of these l hyperplanes is then followed a specified distance. This stepwise procedure is continued until $x_{\max}$ is reached where either $|u| = 0$, or for every choice of $l < n$ hyperplanes, a step in the direction of u violates at least one constraint. The latter applies if $x_{\max}$ is at a vertex of R. The method is based on an algorithm for choosing the subset of l hyperplanes and a recursion formula for an $n \times n$ projection matrix ${}_lP$ such that $u = {}_lPg$. The recursion formula is ${}_{l+1}P = {}_lP - (u_{l+1}) \cdot (u'_{l+1})$ where $u_{l+1} = {}_lPn_{l+1}/|{}_lPn_{l+1}|$, u'_{l+1} is its transpose and n_{l+1} is the normal to the hyperplane $\lambda_{l+1} = 0$. It is proved that the method will find $x_{\max}$, and an estimate is given showing that computation time is proportional to $n^2(k+n)$. A procedure is given for finding a starting point $x^0 \in B$ if no starting point $\in R$ is known. (Received July 27, 1956.)

LOGIC AND FOUNDATIONS

81. Leon Henkin: *Cylindrical algebras of dimension* 2. Preliminary report.

Tarski and Thompson have defined the notion of an *α-dimensional cylindrical algebra with diagonal elements* (CA_α), Bull. Amer. Math. Soc. vol. 58 (1952) p. 65, and raised the question of representing these abstract structures as *proper* algebras. For dimension 2, CA_2's are now found which are not isomorphic to any proper CA_2. For a CA_2 to be representable it is necessary and sufficient that it satisfy the following two identities: $C_\eta - C_\xi(C_\eta x \cdot -d_{\xi\eta}) \cdot C_\xi(x \cdot y) \cdot C_\xi(x \cdot -y) = 0$, for $\xi, \eta = 0, 1$, $\xi \neq \eta$. These identities are satisfied in every CA_3, so that a CA_2 is representable if and only if it can be neatly imbedded in a CA_3. The notion of a *neat* imbedding is defined on p. 40 of the author's monograph *La structure algébrique des théories mathématiques* (Paris, 1956). On p. 36 of this work it is erroneously stated that Tarski and Chin have shown every CA_2 to be representable. The Tarski-Chin result holds only for algebras without diagonal elements (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 80–81). (Received October 2, 1956.)

82. Richard Montague: *Independently axiomatizable theories.*

See Abstract 83 which follows and Bull. Amer. Math. Soc. Abstract 62-6-783. *Definition.* Let (Con u) be the standard formula of Peano's arithmetic expressing the consistency of the formula u. Then the theory T is *uniformly reflexive* if Peano's arithmetic, together with all the sentences (Con Δ), where Δ is the Gödel-numeral of a valid sentence of T, is relatively interpretable in T. *Theorem* 1. If T is recursively axiomatizable and uniformly reflexive, then T is independently axiomatizable. *Theorem* 2. The following theories satisfy the hypothesis of Theorem 1: (a) general set theory; (b) Peano's arithmetic; (c) Zermelo-Fraenkel set theory; (d) all recursively axiomatizable extensions without new constants of (b) and (c). *Theorem* 3. Let T be a theory with a recursive axiom system B; if T is independently (but not finitely) axiomatizable, then there is a recursive function F such that whenever ϕ is a conjunction of finitely many axioms of B, $F(\phi)$ is a valid sentence of T which is not derivable from ϕ. Theorem 1 uses the "general theorem" of Abstract 83 of which Theorem 3 is the converse. Theorem 2 (b) is due to Mostowski. (Received October 24, 1956.)

83. Alfred Tarski: *The existence of independent axiom systems for Peano's arithmetic.*

For terminology see Tarski, Mostowski and Robinson, *Undecidable theories.* The following question was communicated to the author by Kreisel: *Is theory* $\mathfrak{P}$ (*Peano's arithmetic*) *independently axiomatizable*, i.e., is there a recursive axiom system A for $\mathfrak{P}$ such that no axiom in A is derivable from the remaining axioms? *The answer to this question is affirmative.* It can be obtained with the help of the following general theorem: *Let* $\mathfrak{T}$ *be a theory with a recursive axiom system* B; *assume that there is a recursive function* F (*on and to expressions of* $\mathfrak{T}$) *such that whenever* Φ *is a conjunction of finitely many axioms of* B, $F(\Phi)$ *is a valid sentence of* $\mathfrak{T}$ *which is not derivable from* Φ. *Then* $\mathfrak{T}$ *is independently* (*but not finitely*) *axiomatizable.* If $\mathfrak{P}$ is taken for $\mathfrak{T}$, the existence of a function F satisfying the hypothesis of this theorem is assured, e.g. by a construction which Ryll-Nardzewski (Fundamenta Mathematicae, vol. 39) applied in proving that $\mathfrak{P}$ is not finitely axiomatizable. For related results see the Abstract 82 of Montague. (Received October 24, 1956.)

STATISTICS AND PROBABILITY

84. Herman Rubin and H. G. Tucker (p): *Estimating the parameters of a differential process.*

Let the value of a differential process at time t be $X(t)$. It is desired to estimate the parameters of the process from observation of a sample function over a fixed interval. One method of estimation is to estimate the expected number of jumps in any size range by the actual number, and if the sizes of the jumps are $y_1, \cdots, y_n, \cdots$, the variance of the normal component is $\int(dX(t))^2 - y_i^2$. The trend term is estimated by $X(t) - X(0) - \sum y_i^3/(1 - y_i^2)$. Another method of estimation, which with probability one yields the same estimate, and which is adaptable to discrete observations, is based on the general solution of the central limit problem. The estimate is sufficient, and the estimate of the jump density is unbiased. (Received October 5, 1956.)

TOPOLOGY

85. J. M. Gary: *Higher order cyclic elements.*

Some results of G. T. Whyburn (Amer. J. Math. vol. 56 (1935) pp. 133–146) concerning higher order cyclic elements are generalized to arbitrary compact spaces. Let M be a compact Hausdorff space and E a closed set in M containing all the $(r-1)$-dimensional cyclic elements. Then the main theorem shows the mapping $i^*: H_r(E) \to H_r(M)$, induced by inclusion, to be an isomorphism onto. Some results and examples concerning the relation between r-monotone mappings and r-dimensional cyclic elements are obtained. (Received October 3, 1956.)

86. L. E. Ward, Jr.: *Mobs, trees, and fixed points.*

Definitions of terms and symbols are given in the author's papers, Proc. Amer. Math. Soc. (1954) pp. 144–161 and pp. 992–994. A compact Hausdorff space X is a *generalized tree* if and only if X admits a continuous order-dense partial, order such that $L(x) \cap L(y)$ is a nonempty chain for each $x, y \in X$ and such that each continuum Y contained in X contains a unique minimal element. A number of characterizations of trees and generalized trees are given in terms of mobs, partially ordered spaces, and purely topological notions. It is proved that the generalized trees have the fixed point property; this is a generalization of a recent result of K. Borsuk (Bull. Acad. Polon. Sci. (1954) pp. 17–20.) (Received September 20, 1956.)

87. R. L. Wilder: *Some consequences of a method of proof of J. H. C. Whitehead.*

By abstraction and extension of a method of proof used by J. H. C. Whitehead [*Note on the condition n-colc*, to appear soon in Mich. Math. J.], two lemmas concerning dual inverse and direct systems of vector spaces are obtained which lead to (1) an affirmative solution of Problem 2.1, p. 381 of my Colloquium book [Amer. Math. Soc. Colloquium Publications vol. 32 (1949)], under quite general conditions, (2) a solution of a problem left unsettled by S. Kaplan in his dissertation [Trans. Amer. Math. Soc. vol. 62 (1947) pp. 248–271], and (3) a proof that for any point x of a locally compact space S such that x has a well-ordered (by inclusion) cofinal set of neighborhoods in S, the relation $p^n(x, S) = p_n^*(S)$ holds for all n, in contradiction of certain conclusions of Alexandroff concerning these numbers [Annals of Math. vol. 36 (1935) pp. 1–35; see Examples 1, 2, 3 and 5, p. 23]. (Received October 2, 1956.)

V. L. KLEE, JR.,
Associate Secretary

THE NOVEMBER MEETING IN EVANSTON

The five hundred twenty-ninth meeting of the American Mathematical Society was held at Northwestern University, Evanston, Illinois, on Friday, November 23, 1956. Sessions began at 10:30 a.m. and the meeting concluded with a tea at which the ladies of the Department of Mathematics entertained the Society and its guests. There were a total of 155 registrations. Among these there were 138 members of the Society.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor George Piranian of the University of Michigan to address the Society. Professor Piranian spoke on *The boundary of a simple connected domain* at 2:00 p.m. in Lecture Room 2 of the Northwestern Technological Institute Building. Professor Piranian's lecture was unquestionably the high point of the proceedings and was pleasant as well as instructive. Professor L. C. Young presided at the lecture.

There were four sections for the presentation of contributive papers. Presiding officers were Professors Gaffney, Boothby, Gillman, and Auslander.

ALGEBRA AND THEORY OF NUMBERS

88. Philip Dwinger: *Complete reducibility in complete modular lattices with an application to universal algebras.*

An element a of a complete modular lattice L is called characteristic with respect to an automorphism γ of L if $a^\gamma \leqq a$. L is characteristically irreducible with respect to some subgroup $G'[L]$ of the group $G[L]$ of automorphisms of L, if L has no element $\neq 0$ and $\neq 1$, which is characteristic with respect to all the automorphisms belonging to $G'[L]$. An element $a \in L$ is completely reducible if it is a finite sum of minimal elements of L. Theorem I. If L has finite length and L is characteristically irreducible with respect to some subgroup $G'[L]$ of $G[L]$, then every element of L is completely reducible and L is complemented. If $\mathfrak{A}$ is a universal algebra and $G[\mathfrak{A}]$ denotes the group of automorphisms of $\mathfrak{A}$, $C[\mathfrak{A}]$ the lattice of congruence relations of $\mathfrak{A}$ and $G[C[\mathfrak{A}]]$ the group of automorphisms of $C[\mathfrak{A}]$, then $G[\mathfrak{A}]$ is homomorphic to a certain subgroup $G'[C[\mathfrak{A}]]$ of $G[C[\mathfrak{A}]]$. $\mathfrak{A}$ is called characteristically simple if $C[\mathfrak{A}]$ is characteristically irreducible with respect to $G'[C[\mathfrak{A}]]$. If $\mathfrak{A}$ has a selected one-element subalgebra 1, then for every congruence relation $\theta \in C[\mathfrak{A}]$, $S(\theta)$ stands for the set of of all elements $x \in \mathfrak{A}$ for which $x \equiv 1 \pmod{\theta}$. Theorem II. If $\mathfrak{A}$ is an algebra all of whose congruence relations are permutable and with a selected one-element subalgebra 1 and if $\mathfrak{A}$ is characteristically simple and $C[\mathfrak{A}]$ has finite length, then for every congruence relation $\theta \in C[\mathfrak{A}]$, $S(\theta)$ is the finite union of simple isomorphic subalgebras $S(\theta_1), S(\theta_2), \cdots, S(\theta_n)$, every $\theta_i \in C[\mathfrak{A}]$, $i = 1, 2, \cdots, n$. If $\mathfrak{A}$ is a group, then Theorem II yields a well known theorem on direct decompositions in groups. (Zassenhaus, *Theory of groups*, 1949). (Received September 29, 1956.)

89. Casper Goffman: *A lattice homomorphism of a lattice ordered group.*

Let G be a lattice ordered group and L the set of positive elements of G. For every $x \in L$, let $D(x)$ be the set of all $y \in L$ for which $x \cap y = 0$. Let α be defined by $\alpha(x) = \alpha(y)$ if and only if $D(x) = D(y)$ and let $\alpha(x) \geqq \alpha(y)$ if $D(x) \subset D(y)$. Jaffard has shown (Journal de Math., 1953) that α satisfies the conditions (a) $\alpha(x) = \alpha(0)$ implies $x = 0$, (b) $\alpha(x \cup y) = \alpha(x) \cup \alpha(y)$, $\alpha(x \cap y) = \alpha(x) \cap \alpha(y)$, and (c) $\alpha(x + y) = \alpha(x \cup y)$ so that, in particular, α is a lattice homomorphism. It is shown here that α also satisfies (d) if $x = \sup [x_i | i \in I]$ then $\sup [\alpha(x_i) | i \in I]$ exists and $\alpha(x) = \sup [\alpha(x_i) | i \in I]$. The main result is that if G is archimedean then α is the only lattice homomorphism of L which has properties (a), (b), (c), and (d). An example is given of a nonarchimedean group for which this is false. Examples are given showing the independence of (c) and (d) with each other and with (a) and (b) for lattice homomorphisms. Pierce (Ann. of Math. (1954)) has shown that α is a lattice homomorphism for any distributive lattice with smallest element. A nondistributive lattice is given for which α is not a lattice homomorphism. (Received September 20, 1956.)

90t. H. Schwartz and H. T. Muhly: *On a class of cubic diophantine equations.*

The diophantine equation $x^3 + y^3 + z^3 - axyz = b$, has received some attention in the literature where it has unfortunately been the subject of some erroneous statements. In this note these errors are corrected and properties of this equation are investigated. For example, certain solutions from which all others can be derived in a natural way are singled out and termed fundamental. It is shown that except when $a = 1$, $b = 4$ or when $a = 2$, $b = 1$, at most a finite number of fundamental solutions can exist, while infinitely many fundamental solutions do exist in these two cases. When $a = 1$, $b = 2$ there are only a finite number of integer solutions, but otherwise when b is not a perfect square the existence of one solution implies the existence of infinitely many. (Received July 12, 1956.)

91t. W. R. Scott: *On the multiplicative group of a division ring.*

Let K be a noncommutative division ring, and let K^* be its multiplicative group. Let Z^* be the center of K^*. Then Hua (*On the multiplicative group of a sfield*, Acad. Sinica Science Record vol. 3 (1950) pp. 1–6) has shown that (i) K^*/Z^* has center 1, and (ii) K^*/Z^* is not solvable. In the present paper, it is shown that K^*/Z^* has no normal Abelian subgroups. This result contains both (i) and (ii). The following generalization is proved. Let G and H be subinvariant subgroups of K^*, $x \in G$, $y \in H$, and suppose that $1 \neq [x, y] = z \in Z^*$. Then one of G and H is not nilpotent. (Received October 1, 1956.)

92. W. R. Scott: *Solvable factorizable groups.*

Let H and K be subgroups of a finite group G, and suppose $G = HK$. During recent years, a number of theorems of the following type have been proved: if H and K satisfy certain conditions, then G is solvable. In this paper several additional theorems of this kind are given. It is shown that G is solvable under any of the following conditions: (i) H nilpotent, K Hamiltonian, (ii) H nilpotent of odd order, K contains a subgroup L of index 2 such that all subgroups of L are normal in K, (iii) H cyclic, K contains a subgroup L of index 2 or 3 such that all subgroups of L are normal in K, or (iv) H dihedral or dicyclic, K dihedral or dicyclic. (Received October 1, 1956.)

93*t*. W. R. Scott: *Half-homomorphisms of groups.*

Let G and G' be groups, and let there be given a mapping $a \to a'$ from G into G'. Call the mapping a ***half-homomorphism*** if, for all a and b in G, either $(ab)' = a'b'$ or $(ab)' = b'a'$. The following theorem is proved: every half-homomorphism is either a homomorphism or an anti-homomorphism. (Received October 1, 1956.)

94. M. F. Smiley: *Jordan homomorphisms onto prime rings.*

A brief proof is given of the following slight generalization of a theorem of I. N. Herstein (Trans. Amer. Math. Soc. vol. 81 (1956) pp. 331–341). ***A Jordan homomorphism*** (N. Jacobson and C. E. Rickart, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502) ***onto a prime ring*** (N. H. McCoy, Amer. J. Math. vol. 71 (1949), pp. 823–833) ***is a homomorphism or an anti-homomorphism.*** The proof rests on extensions of Herstein's identities and the following simple lemma. *If P is a prime ring in which the square of every commutator is zero, then P is commutative.* (Received October 2, 1956.)

95. M. L. Tomber: *Lie algebras of types A, B, C, D, and F.*

Let Φ be an arbitrary field of characteristic 0 with algebraic closure Ω. Lie algebras $\mathfrak{L}$ over Φ of types A, B, C, D, and F with the exception of D_4 are known to be derivation algebras $\mathfrak{D}(\mathfrak{J})$ of central simple Jordan algebras $\mathfrak{J}$ over Φ. A direct proof of this result is given. It is first shown that any automorphism of $\mathfrak{L}$ over Ω has the form $D \to SDS^{-1}$ for a unique automorphism S of $\mathfrak{J}$ over Ω. The methods of Jacobson (Duke Math. J. vol. 5 (1939) pp. 775–783) are then used to prove the main theorem and $\mathfrak{D}(\mathfrak{J}_1) \cong \mathfrak{D}(\mathfrak{J}_2)$ if and only if $\mathfrak{J}_1 \cong \mathfrak{J}_2$. (Received October 4, 1956.)

ANALYSIS

96. C. L. Dolph: *A saddle point characterization of the Schwinger variational principles for exterior scattering problems.*

Positive semi-definiteness of the imaginary parts of the free-space Green's function for the exterior Dirichlet and Neumann problems is established. This property is then applied to give global saddle point behavior of the reflection coefficient of the Schwinger variational principles in a manner similar to that appearing in Dolph and Ritt, Math. Zeit. vol. 65 (1956) pp. 309–326. A variant of the usual method of deriving the integral relationships for these problems is also given which illustrates more clearly the relationship between the use of single and double layer for these problems. (Received October 4, 1956.)

97*t*. R. J. Evey and P. C. Hammer: *Compositions of convex and concave functions.*

G. Szekeres recently gave necessary and sufficient conditions that an increasing function with positive derivatives and continuous second derivative be representable as a composition of two increasing functions one of which is concave the other convex. In this paper we establish necessary and sufficient conditions for any increasing function is represented as the composition of a convex function and a concave function. One result is that any continuous nondecreasing function is uniformly approximable by a concave function of a convex function over its entire range, both factors in the composition being piece-wise linear and properly increasing. (Received October 3, 1956.)

98. W. H. Fleming: *Nondegenerate surfaces of finite topological type.*

This paper establishes the following: *Theorem.* Every nondegenerate Fréchet surface in N-space ($N \geqq 2$) of the topological type of a compact 2-manifold, with or without boundary, with finite Lebesgue area has a quasi-conformal representation. For 2-cells and 2-spheres this is Morrey's theorem [Amer. J. Math. vols. 57, 58]. For higher topological types the domain of the quasi-conformal representation is a normalized parallel slit domain, of the sort used by Courant [Dirichlet's Principle, 1950] and Shiffman [Amer. J. Math. vol. 61 (1939] in connection with the Plateau problem. A surface S is nondegenerate if and only if its middle space is a 2-manifold of the given topological type. (Received September 10, 1956.)

99. R. E. Fullerton: *A geometric characterization of absolute bases in a Banach space.*

Let X be a Banach space. It is shown that if X has a countable basis there exists in X a C-cone K where K is the closure of the convex set determined by a countable set of extreme rays and where K has this further property that there exists $X \in K$ such that $P_x = K \cap (x - K)$ is compact and has its linear extension dense in the space. A necessary and sufficient condition that X possess an absolute countable basis is that there exist $x \in K$ such that P_x has linear extension dense in the space and P_x is linearly homeomorphic to the fundamental cube of Hilbert. (Received October 5, 1956.)

100. M. P. Gaffney: *The asymptotic distribution of the characteristic values and characteristic forms of the Laplacian on a Riemannian manifold.*

Results and methods obtained by Minakshisundarum for functions are generalized to differential forms. It is shown that the fundamental solution of the heat equation (a double p form) exists and equals $\sum_1^\infty \omega_i(P)\omega_i(Q)e^{-\lambda_i t}$. The behaviour of this solution is studied as t approaches zero. The knowledge of this behaviour makes it possible to obtain the asymptotic distributions from Tauberian theorems. (Received October 6, 1956.)

101. W. B. Jurkat: *On the converse of Abel's limit theorem by complex methods.*

The paper contains a direct proof of the well-known statement that (i) $f(z) = \sum a_n z^n = o(1)$ as $z \to 1$, $0 < |1 - z| / (1 - |z|) = O(1)$ implies $\sum a_n = 0$, if the Tauberian condition (ii) $na_n = O(1)$ is satisfied. The method of proof consists of evaluating the partial sums by a contour integral which can be estimated by proper choice and subdivision of the path of integration. There are several extensions: In condition (i) the variable z may be restricted to real values. The condition (ii) may be replaced by $\sum_1^n |ka_k|^2 = O(n)$. Instead of convergence it is possible to derive Cesàro summability of certain negative orders. (Received November 19, 1956.)

102. G. K. Kalisch: *On similarity and isometric equivalence of certain Volterra operators.*

Unique canonical forms are established for Volterra operators T_K operating on $L_p(0, 1) = \{f\}$ $(1 < p < \infty)$ where $T_K f = \int_0^x K(x, y) f(y) dy$ with respect to similarity and isometric equivalence. The complex kernels K are subject to various conditions: $K(x, y)=(y-x)^{n-1}L(x, y)$ for positive integral n and $L(x, x)\neq 0$; $L(x, x)$ is real; L satisfies certain regularity conditions. If E is the function identically equal to 1, then T_K is similar to a unique cT_E^n for real c; T_K is isometrically equivalent to a unique T_M where $M(x, y)=(y-x)^{n-1}N(x, y)$, $N(x, x)=c$ (c real) and $\operatorname{Im}(N_x(x, x))=\operatorname{Im}(N_y(x, x))=0$. The relevance of the above conditions imposed on K is discussed by means of examples. (Received October 4, 1956.)

103. J. H. B. Kemperman: *A property of exponential polynomials.*

Let $f_i(x)$, $a_j(x)$, $b_j(x)$, $(i=0, \cdots, p; j=1, \cdots, q)$, be defined for all real x, such that $\sum_{i=0}^{p} f_i(x+r_i y)=\sum_{j=1}^{q} a_j(x)b_j(y)$ for all x, y. Here, the r_i are $p+1$ different real constants $\neq 0$. Then, provided that $f_0(x)$ is (Lebesgue) measurable, $f_0(x)$ is an exponential polynomial, i.e. a solution of a homogeneous linear differential equation with constant coefficients. This generalizes a result due to St. Kaczmarz, Fund. Math. vol. 6 (1924) pp. 122–129. The above result fails to remain true when "measurable" is replaced by "bounded." (Received October 5, 1956.)

104. Karel deLeeuw: *Linear spaces with compact groups of operators.*

Let A be a complete locally convex topological linear space and G a compact abelian group of operators on A with the map $(x, T_\sigma)\rightarrow T_\sigma(x)$ jointly continuous. For any character χ of G, the map T_χ defined by the vector valued integral $T_\chi(x)=\int_G T_\sigma(x)\chi^{-1}(\sigma)d\sigma$, where $d\sigma$ is Haar measure, is a projection on the subspace $A_\chi=\{x: T_\sigma(x)=\chi(\sigma)x, \text{ all } \sigma \text{ in } G\}$. The *formal topology* on A is the weakest topology that agrees with the original topology on the A and which is such that the T_χ are continuous. In the spaces that occur in analysis, the formal topology is that of termwise convergence of Fourier or power series. *Theorem*: A G-invariant, convex subset of A closed in the original topology is closed in the formal topology. *Corollary*: If A is a Banach space with a G-invariant norm, the unit sphere is closed and the norm is lower semi-continuous in the formal topology. Results of a similar nature hold if G is not abelian. Sample application of theorem: If A is the space of entire functions, and $p(f)$ is the maximum modulus of f in a disk having the origin as center, p is lower semi-continuous in the topology of termwise convergence of power series. (Received October 3, 1956.)

105. Z. A. Melzak: *A scalar transport equation.*

The equation $\partial f(x, t)/\partial t=2^{-1}\int_0^x f(y, t)f(x-y, t)\phi(y, x-y)dy-f(x, t)\int_0^\infty f(y, t)\phi(x, y)dy+\int_x^\infty f(y, t)\psi(y, x)dy-(f(x, t)/x)\int_0^x y\psi(x, y)dy$ with $f(x, 0)$, $\phi(x, y)$, $\psi(x, y)$ known, is shown to possess a unique, continuous, non-negative, $L'(0, \infty)$ in x for each t and analytic in t for each x, solution $f(x, t)$ under the following hypotheses: (1) $f(x, 0)$ is continuous, bounded and non-negative on $[0, \infty)$ and in L' there (2) $\phi(x, y)$ is continuous, symmetric, bounded and non-negative in the first quadrant (3) $\psi(x, y)$ is continuous, bounded, non-negative and satisfies: $\int_0^x y\psi(x, y)dy<x$, $\int_0^x \psi(x, y)dy<c$. The equation itself occurs in a variety of physical problems, hence the non-negativity and boundedness assumptions. The proof proceeds by constructing a power-series solution valid on $0\leq t\leq t_1$, $t_1>0$, showing that this solution is non-negative and that $f(x, t_1)$ satisfies the same conditions as $f(x, 0)$, and demonstrating that the continuation process does not terminate. Uniqueness follows from another hypothesis on $\psi(x, y)$. Finally, a condition is given for strict positivity of $f(x, t)$ for $t>0$. (Received September 24, 1956.)

106. C. Y. Pauc: *Theorems of Ward for cell functions.*

A. J. Ward proved the following propositions: (W1) A finitely additive interval function is differentiable in the ordinary sense a.e. on the set at which either of its extreme ordinary derivates is finite. (W2) A finitely additive interval function is differentiable a.e. on the set at which both its extreme strong derivates are finite. D. Rutovitz (in a joint note with C. Y. Pauc in the C. R. Acad. Sci. Paris vol. 240 (1955) pp. 1956–1958) gave a version of (W2) postulating in an abstract measure space $(R, \mathfrak{M}, \mu)$ a directed set (filter) of cell partitions of R, a submetrical norm δ for the cells I, J, a grating axiom and the density axiom. Omitting the density axiom we prove: If ψ is a finitely additive cell function and E a subset of R on which both extreme δ-derivates are finite, there exists a μ-measurable function f ("Denjoy integrant") defined on a measure cover of E with the following property: any set $S \subset E$ of positive outer measure includes a bounded subset L of positive outer measure such that f is essentially bounded on a Jordan cover $\tilde{L}$ of L and $F(I) = \int_{\tilde{L} \cap I} f \cdot d\mu + \int_I \psi(J \| L)$ for any cell I, the last integral being of Burkill type and $\psi(J\|L) = 0$ if J intersects L, $= \psi(J)$ if $J \cap L$ is empty. The Denjoy integrant is defined as a fusion of Radon-Nikodym integrants. The theorem is applicable to intervals of any enumerable power of the numerical line or to any enumerable torus space. Moreover f admits a representation as δ-derivative of F if the density axiom holds. D. Rutovitz, in a paper to appear in the Annali di Matematica 1957, will give a version of (W1) for abstract free nets of Jessen type and an extension of other results of Ward's theory of interval functions. (Received September 28, 1956.)

107. I. E. Segal: *Transformation of distributions in Hilbert space.*

Let n be an isotropic normal distribution on a real Hilbert space H, with zero mean (cf. Trans. Amer. Math. Soc. vol. 81 (1946) pp. 106–134). For any closed densely-defined linear transformation T on H, the transform of n by T is absolutely continuous with respect to n if and only if T^*T is nonsingular and has the form $I+B$, where B is an operator with absolutely convergent trace. If m denotes the normal distribution on H with covariance operator $(T^*T)^{-1}$ and arbitrary mean a in H, m is likewise absolutely continuous with respect to n if T^*T has the preceding form, and the derivative is given by the equation $dm/dn = [\det (I+B)]^{(1/2)} \exp [-\{(Bx, x) -2((I+B)x, a)+((I+B)a, a)\}/2]$. This may be compared with the formula of Cameron and Martin in Trans. Amer. Math. Soc. vol. 58 (1945) pp. 184–219. (Received October 1, 1956.)

108. V. L. Shapiro. *Uniqueness of multiple trigonometric series—An n-dimensional analogue of Verblunsky's theorem.*

In this paper, two theorems on the uniqueness of multiple trigonometric series are proved. The first theorem when considered in one dimension reduces to the well-known Verblunsky result, and the second theorem when considered in one dimension is false. With $n \geqq 2$ and with $x = (x_1, \cdots, x_n)$, $ax+by = (ax_1+by_1, \cdots, ax_n+by_n)$, $(x, y) = (x_1y_1+ \cdots +x_ny_n)$, $|x| = (x, x)^{1/2}$, m an integral lattice point, and $T_n = \{x; -\pi < x_j \leqq \pi, j = 1, \cdots, n\}$, the following two theorems are proved: Theorem 1. Given the multiple trigonometric series $\sum_m a_m e^{i(m,x)}$. Set $f_*(x) = \lim \inf_{t\to 0} \sum_m a_m e^{i(m,x)-|m|t}$ and let $f^*(x)$ designate the corresponding lim sup. Suppose that (i) $\sum_{R-1<|m|\leqq R} |a_m| = o(R)$ as $R \to \infty$. (ii) $f_*(x)$ and $f^*(x)$ are finite for all x. (iii) $\bar{a}_m = a_{-m}$. (iv) $f_*(x) \geqq A(x)$ where $A(x)$ is in L^1 on T_n. Then $f_*(x)$ is in L^1 on T_n and $\sum_m a_m e^{i(m,x)}$ is its Fourier series. Theorem 2. Given the multiple trigonometric series

$\sum_m a_m e^{i(m,x)}$. Let W be a set of measure zero on T_n and q be a point on T_n not in W. Suppose that (i) $\sum_{R-1<|m|\leq R} |a_m| = o(R)$ as $R\to\infty$. (ii) $f^*(x)$ and $f_*(x)$ are finite for x in $T_n - q$. (iii) $f_*(x) = f^*(x)$ for x in $T_n-(W+q)$. (iv) $f_*(x)$ is in L^1 on T_n. Then $\sum_m a_m e^{i(m,x)}$ is the Fourier series of $f_*(x)$. (Received October 3, 1956.)

109*t*. F. M. Wright: *On Stieltjes mean sigma integrals of order p.*

The author considers a Stieltjes mean sigma integral $(m_p\sigma)\int_a^b f dg$ of order p, where p is an arbitrary integer ≥ 2. If $p=2$, this integral is the Stieltjes mean sigma integral of H. L. Smith (Trans. Amer. Math. Soc., 1925). If g is monotone nondecreasing, and if c is a number such that $a \leq c < b$ and $g(c+) > g(c)$, then $(m_p\sigma)\int_a^b f dg$ does not exist in case f is bounded near c and $f(c+)$ does not exist; a corresponding result holds for left-hand limits. If g is monotone nondecreasing, if $g(x) = g(a)$ for $x < a$ and $g(x) = g(b)$ for $x > b$, and if f is a simple step function, then $(m_p\sigma)\int_a^b f dg$ exists and equals $(LS)\int_a^b f dg + (1-1/p)\cdot \{\sum [f(x+) - f(x)]\cdot[g(x+) - g(x)] + \sum [f(x-) - f(x)]\cdot [g(x) - g(x-)]\}$. This formula may be used to give a fairly simple proof of a theorem relative to the convergence of a sequence $\{(m_p\sigma)\int_a^b f_n dg\}$, $(n=1, 2, 3, \cdots)$, of such integrals under certain hypotheses. For $p=2$, this theorem includes the principal result of P. Porcelli (Proc. Amer. Math. Soc. (1954) pp. 395–400). (Received October 3, 1956.)

GEOMETRY

110. Lawrence Markus: *Open affine manifolds.*

A complete flat (curvature and torsion zero) affinely connected manifold M^n, which is isomorphic with the affine number space R^n near infinity, is R^n. Similarly a complete Riemannian space M^n of constant negative curvature, which is isometric with its universal covering space B^n near infinity, is B^n. (Received October 4, 1956.)

111. T. S. Motzkin: *Comonotone curves and polyhedra.*

A curve without common points, except the point of contact, with its osculating hyperplanes is called comonotone; if it has no hyperosculating hyperplanes it is strictly comonotone. The convex hull of m points on a strictly comonotone curve in real n-space is a comonotone polyhedron; the m osculating hyperplanes determine a comonotone dissection of projective space (e.g. the dissection of the set of all real polynomials of given degree by a finite set of real numbers excluded as roots). Comonotone dissections and polyhedra and their duals appear to be the most important types definable for every m and n. Among their prominent features are a simple enumeration and characterization of faces of d dimensions $d=1, 2, \cdots$; highest number f of faces for polyhedra, namely $f=\sum_{j=[d/2]+1}^{n/2} (m/j)\binom{m-j-1}{j-1}\binom{j}{d-j+1}$ for even n, $f(n, m, d) = (m+1)/(d+1)f(n+1, m+1, d+1)$ for odd n; greatest skewness of f as function of d, with maximum arbitrarily near to $d=3n/4$; the fact, that every $[n/2]$ vertices are neighbors (form a simplicial face), which distinguishes these polyhedra uniquely for even n; connection with variation-diminishing transformations. (Received October 3, 1956.)

112*t*. T. S. Motzkin: *Types of dissections.*

The set of all dissections of real projective $(n-1)$-space by $m+n$ hyperplanes is decomposed by the degenerate dissections (those with concurrent hyperplanes) into $c(m, n)$ connected components. It is proved by a 1-1 correspondence between dissections and "transposed" dissections that $c(m, n) = c(n, m)$. We have $c=1$ for min (m, n)

≤ 2; $n+2$ hyperplanes determine uniquely a cyclic order and a norm curve. For $m=n=3$ and $m=3$, $n=4$ the components are well known; for $m=3$, $n=5$ they were listed by the author and A. Benhanan. Fundamental invariants of a dissection, within its component, are its genus g and class k. These are the smallest numbers for which there exist (1) a dissection in the same component with hyperplane coordinate integers absolutely $\leq k$, including an n by n unit matrix; (2) a curve osculating the given hyperplanes which can be deformed, without ever having hyperosculating hyperplanes into the norm curve $(1, t, \cdots, t^{n-1})$ traced g times. (Received October 3, 1956.)

113*t*. H. T. Muhly: *On the relative arithmetic genus of a normal surface.*

Let F be a normal surface and let P be a point of F. If L is the local ring at P and if m is the ideal of nonunits in L, then L is integrally closed in its field of quotients, but if P is singular, the various powers m^n of m need not be integrally closed in the sense of Prufer (or complete ideals in the sense of Zariski, b-ideals in the sense of Krull). If Q_n is the least complete ideal which contains m^n we show that there is an integer d such that $Q_{nd}=Q_d^n$ for all n. Thus by a theorem of P. Samuel (Thesis, Paris, 1951), the length of Q_{nd} is a polynomial $H(n)$ in n with integer coefficients. If F^* is the normal surface obtained from F by a quadratic transformation with center P followed by normalization, it is shown that the difference $p_a(F^*)-p_a(F)$ between the relative arithmetic genera of F^* and F is equal to the constant term of $H(n)$. (Received October 2, 1956.)

114*t*. A. R. Schweitzer: *Mathematics, related disciplines and Shakespearean interpretations.* Preliminary report.

The author discusses subjects which tend to emphasize Shakespeare's myriad mindedness. These subjects include mathematics, music, philosophy, metaphysics and principles which are assumed to govern an academy ("academe"). Shakespeare mentions the latter subjects in *The taming of the shrew* (Act I, Scene I, Tranio) *The merchant of Venice* (Act V, Scene I, Lorenzo) and *Love's labour lost* (Act I, Scene I, King of Navarre). The term "academy" is assumed by the author to mean an association of persons who unite to promote learning. (Received October 4, 1956.)

115. M. Z. Krzywoblocki: *Examples from statistical mechanics of a continuous medium.*

Kampé de Fériet had provided examples of a statistical mechanics for a continuous medium (Congrès Internat., Philosophie des Sciences, Paris, 1949, vol. 3 (1951) pp. 129–144, and Proc. Sec. Berkeley Symp. Math. Statist. and Probability, University of California Press, 1952, pp. 553–566]: a vibrating string of an infinite length and a vibrating string of finite length with both ends fixed. In the present paper the author furnishes proof of existence of a few more examples of a similar nature. (Received August 24, 1956.)

Topology

116. W. F. Davison: *Mosaics of curves and arcs.*

Let $\{(X_a, T_a): a \in A\}$ be a mosaic of curves (Peano spaces) on X, and let T be the ensuing mosaic topology. (For terminology see W. F. Davison, *Convergent sequences and mosaics*, Bull. Amer. Math. Soc. vol. 62 (1956) p. 180.) Then the *curve space* (X, T) is a mosaic space with the following additional properties: (i) every T-convergent sequence has a subsequence in a curve of the mosaic; (ii) every open subspace is a curve space; (iii) every component is open and arcwise connected; and (iv) the space is locally arcwise connected. A topological space is the curve space of some mosaic of curves if and only if it is mosaic space with property (i) above. If a space is locally separable then it is a curve space if and only if it is Hausdorff and locally arcwise connected. If the mosaic $\{(X_\alpha, T_\alpha): \alpha \in A\}$ consists of arcs then the *arc space* (X, T) has the properties analogous to (i) through (iv) above, obtained by replacing the word "curve" with "arc" throughout, and can be characterized as a mosaic space with the analog of property (i). Consequently, a mosaic space is strongly arcwise connected and has every point covered by an arc if and only if it is a countably compact arc space. (Received October 3, 1956.)

117. O. H. Hamilton: *Fixed points for certain noncontinuous transformations.*

A *connectedness map* is defined by John Nash to be a single valued transformation from a space A into a space B such that the induced map $A \to A \times B$ preserves the connectedness of connected subsets of A. A *peripherally continuous transformation* is defined by the author to be a single valued transformation of A into B such that if p is a point of A, and V and U are open subsets of A and B containing p and $T(p)$ respectively, then there is an open subset D of V such that T transforms the boundary of D into U. It is shown that a connectedness map of a closed n-cell I into itself is a peripherally continuous transformation. It is shown further that a peripherally continuous transformation of a closed n-cell into itself and hence a connectedness map of a closed n-cell into itself leaves a point invariant. Various examples are presented. (Received September 10, 1956.)

118. J. F. Daly (p) and L. J. Heider: *Generalized G_δ spaces*, II.

Let X denote a completely regular space, while υX denotes the Hewitt Q-space extension of X. Y is called an *imbedded* subspace of υX if Y is dense in υX and every function defined and continuous on Y can be extended as continuous over all of υX. The points of υX included in every *imbedded* subspace of υX are exactly the generalized G_δ points of υX. A generalized G_δ point p of the space X such that every bounded continuous function on $X - \{p\}$ has a continuous extension at p, but such that some unbounded continuous function on $X - \{p\}$ lacks such an extension is, in fact, a G_δ point of X. A nonisolated point p of X is such that every bounded function continuous on $X - \{p\}$ has a continuous extension at p if and only if every finite, open, normal covering of $X - \{p\}$ is the result of deleting the point p from the sets in a finite, open, normal covering of X. (Received September 28, 1956.)

119. L. F. McAuley: *An atomic decomposition of continua into aposyndetic continua.*

Suppose that M is a compact metric continuum and that $p \in M$. Let $M(p)$ denote the set of all points x in M such that there does not exist a collection G of closed separators of M such that (1) there exists an element of G which separates p from x in M and (2) for g in G, a separation $M - g = A + B$, a continuum N in A, and a point b in B, there exist (i) a continuum C containing an open set D which contains N and (ii) three disjoint elements g_1, g_2, and g_3 of G each separating C from b in M and g_2 separating g_1 from g_3 in M. It is shown that $M(p)$ is a continuum for each point p in M. The collection H of all sets $M(p)$ for the various points p in M is an upper semicontinuous collection of disjoint continua filling up M, and furthermore, with respect to its elements as points, H is a compact aposyndetic metric continuum. If M is aposyndetic, then $M(p) = p$ for each point p in M. Results are also obtained for abstract spaces. [Cf. McAuley, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 74–91.] (Received June 18, 1956.)

120. L. E. Pursell: *The ring of all real-valued continuous functions considered as a subring of the ring of all real-valued functions.*

It is shown that if X and Y are completely regular Hausdorff spaces and if there is an isomorphism of R^X onto R^Y which maps $C(X, R)$ onto $C(Y, R)$, then X and Y are homeomorphic. The proof follows readily from the fact that an ideal I in $C(X, R)$ is fixed if and only if there exists g in R^X such that $g(x) \leq 1$ for all x, $g(x) \neq 1$ for some x, and if f is in I and $f(x) \leq 1$ for all x, then $f(x) \leq g(x)$ for all x. (Received October 4, 1956.)

121*t*. Jerome Spanier: *Some effects of the σ-process on almost complex structures.*

Let M^{2n} be a compact, almost complex manifold and let V^{2r} be a compact, regularly-embedded, almost complex submanifold of M (superscripts denote topological dimension). Denote by $\bar{M}$ the *enlargement of M* obtained by replacing V by a bundle W of normal complex projective spaces of complex dimension $n-r-1$, that is, by applying the σ-process of Hopf (Rend. del. Sem. Math., Roma, vol. 10 (1951) pp. 169–182) to the pair (M, V). Then $\bar{M}^{2n}$ is an almost complex manifold, and there is a natural mapping f of $\bar{M}$ onto M which is a homeomorphism of $\bar{M}-W$ onto $M-V$, and which reduces to the fiber projection of W onto V. A study is made of the effects of enlargement on the invariants of the pair $(\bar{M}, V)$. Concerning the cohomology groups, it is shown that if $r<n-1$, for each $p \geq 0$, $H^p(\bar{M})$ is isomorphic to the direct sum of $H^p(M)$ and a certain subgroup of $H^p(W)$. Under the further restriction that $k^*: H^p(M) \to H^p(V)$ is onto for all $p \geq 0$, where k: $V \subset M$ is inclusion, $H^*(\bar{M})$ is shown to be isomorphic (as a ring) to a truncated ring of polynomials (coefficients in $f^*[H^*(M)]$) in a two-dimensional class $X \in H^2(\bar{M})$. The Chern classes of the normal bundle of V in M, as well as the characteristic class of the normal (circle) bundle of W in $\bar{M}$ play an important role in these descriptions. Finally, simple formulas relating the tangential Chern classes of M to those of $\bar{M}$ are obtained in case V is a point of M. (Received October 3, 1956.)

122. Hidehiko Yamabe: *Some type of compact transformation groups.* Preliminary report.

Let G be a compact C^2 transformation group on a euclidean space E^d, such that $g(x)^i = \sum_{j,k}^{d} [b^i_j(g)x^j + (4+|x|^2)\cdot 4^{-1}(C^i_{jk}\xi^j\xi^k + o(|\xi|^2))]$ where $|x|^2 = \sum_i^d (x^i)^2$, $\xi^i = (4+|x|^2)^{-1}(4x^i)$, $|\xi|^2 = \sum_i^d (\xi^i)^2$, $b^i_j(g)$ orthogonal. Applying suitable transformation, we may assume $\sum_j \int_G b^j_i(g^{-1}) c^j_{kl}(g)dg = 0$. v represent translations defined by vector (v). If g and h are two elements in G, $\lim_n (gv^{1/n}g^{-1}hv^{1/n}h^{-1})^n$ define a homeomorphism denoted by $gvg^{-1} \oplus hvh^{-1}$. Take the integral $J[v] = \oplus \int_G gvg^{-1}dg$ in this sense. $J[v]$ can be well defined, $v \to J[v]$ is univalent and $\{J[v]p$, for all $v\} = E^d$. From this we can construct a homeomorphism Φ from E^d onto E^d, such that $\Phi G \Phi^{-1}$ is orthogonal. From this follows that C^2 compact transformation groups on euclidean spheres with a fixed point is equivalent to an orthogonal transformation group. This is the completion for the statement the author made in winter meeting 1954 with incomplete proof. (Received October 15, 1956.)

J. W. T. YOUNGS,
Associate Secretary

THE NOVEMBER MEETING IN LEXINGTON

The five hundred thirtieth meeting of the American Mathematical Society was held at the University of Kentucky in Lexington, Kentucky on Friday and Saturday, November 30–December 1. About 95 persons registered, including 60 members of the Society.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, there were two hour addresses. Professor L. M. Milne-Thomson, of the Royal Naval College of Greenwich (visiting Professor at Brown University) spoke Friday evening on *Some hydrodynamical methods*, and Professor O. G. Harrold, of the University of Tennessee spoke Saturday morning on *Locally tame curves and surfaces in 3-manifolds*. Professors Tomlinson Fort and H. C. Griffith presided at these sessions.

There were four sessions for contributed papers, Professors R. D. Anderson, V. F. Cowling, M. L. Curtis and Frank Levin presiding.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Mr. Hunter was introduced by Professor R. J. Koch.

ALGEBRA AND THEORY OF NUMBERS

123*t*. Eckford Cohen: *Congruence representations in algebraic number fields* II. *Simultaneous linear and quadratic congruences.*

Let P be an odd prime ideal in a finite extension F of the rational field. In this paper the author determines the number of simultaneous solutions $N_s(m, n)$ of the pair of congruences $m \equiv \alpha_1 x_1^2 + \cdots + \alpha_s x_s^2$, $n \equiv \beta_1 x_1 + \cdots + \beta_s x_s$ (mod P^λ) where m and n are arbitrary integers of F and the α_i, β_i are integers of F prime to P. The case $\lambda = 1$ was treated earlier, using the terminology of Galois fields (Bull. Amer. Math. Soc. Abstract 62-1-3). The method of the paper is based on the elementary theory of Cauchy-Gauss sums in F. Explicit results for $N_s(m, n)$ are deduced, and it is shown that the least value of s such that $N_s(m, n) > 0$ for all odd prime-power ideals P^λ, for all m, n, and all α_i, β_i prime to P is the value $s = 5$. This contrasts with the minimal value $s = 4$ under the restriction $\lambda = 1$. (Received September 24, 1956.)

124. W. E. Deskins: *A note on group representations.*

A subgroup H of a finite group G is defined to have *property I* if each irreducible representation module of G (over the field F) remains irreducible when considered as a representation module for H. The following results are obtained: THEOREM 1. *Let H have property I and let g denote the order of G. Then H is a normal subgroup and contains the commutator subgroup of G provided* (1) *the characteristic of F is relatively prime to g*; (2) $g = qp^a$, $(p, q) = 1$, *p the characteristic of F, and the order of H is p^a; or* (3) *the radical of the group-ring of G is generated by the radical of the group-ring of H.* Examples which demonstrate that the theorem is not necessarily true if (1), (2) or

(3) is not satisfied exist. THEOREM 2. *If H has property I and condition* (1) *is satisfied, then not only is H normal but each conjugate class of H is also a conjugate class in G.* Examples which illustrate that the result may be false if (1) is not satisfied exist. The question, *Given a group H, does there exist a nontrivial Abelian extension G such that H possesses property I?* is answered affirmatively. (Received October 3, 1956.)

125. D. W. Wall: *A note on generalised uniserial algebras.*

Let A be an algebra over an algebraically closed field K. Let L be a primitive left ideal dual to some primitive right ideal. Thus, L has a unique minimal subideal L_1. If, in addition, it is assumed that L has only one subideal containing L_1 as a maximal subideal and that L_1 and L_2/L_1 are A-isomorphic as left A-spaces, then it can be shown that L has only one composition series and all of the constituents L_i/L_{i-1} of L are A-isomorphic to L_1. Examples are given to show that if either of the two hypotheses is dropped then neither of the two conclusions need hold. With the aid of this and additional lemmas, it is possible to extend an earlier result: If for every two-sided ideal Z, A/Z is a QF-2 algebra then A is a generalized uniserial algebra (Bull. Amer. Math. Soc. Abstract 62-2-181). The QF-2 algebras form a subclass of the QF-3 algebras which are those having unique minimal faithful representations (Thrall, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 173–183). The theorem is still true if the hypothesis that every A/Z be a QF-2 algebra is replaced by weaker hypotheses requiring only that every A/Z belong to more general classes of QF-3 algebras. (Received October 4, 1956.)

126*t*. F. B. Wright: *Ideals in a polyadic algebra.*

Halmos has shown that the Gödel completeness theorem for the first-order functional calculus is equivalent to the (polyadic) semisimplicity of a polyadic Boolean algebra. It is shown in this note that polyadic semisimplicity is a corollary of the following two results. (1) Let A be a polyadic I-algebra, with quantifier function E, and let B be the range of the quantifier $E(I)$. Then there exists a one-to-one correspondence between the ideals of B and the polyadic ideals of A. (2) Every ideal of A contains a polyadic ideal of A. (Received September 28, 1956.)

127. A. M. Yaqub: *A note on the commutativity of a certain class of rings.*

Let p be *prime*. A ring R with unit is called *p-like* if it is of characteristic p and if, for every $x, y \in R$, $(xy)^p - xy^p - x^p y + xy = 0$. p-like rings essentially generalize the Boolean rings of Stone (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 37–111), the p-rings of McCoy and Montgomery (Duke Math. J. vol. 3 (1937) pp. 455–459), and the Boolean-like rings of Foster (Trans. Amer. Math. Soc. vol. 59 (1946) pp. 166–187). It is shown that p-like rings are commutative. The proof is elementary and follows after some properties of such rings are established. An elementary number-theoretic result is also used in the proof. (Received September 24, 1956.)

Analysis

128. W. K. Ergen, H. J. Lipkin and J. A. Nohel (p): *Applications of Liapounov's second method in reactor dynamics.*

Liapounov's Second Method, which establishes sufficient conditions for stability of solutions of nonlinear differential equations, is used to obtain criteria which guarantee stability of solutions of reactor equations. Conditions for stability are obtained both for homogeneous and heterogeneous reactors, where in the latter case heat is generated in each of the n reactor media. The important special cases of two and three media as well as the general case of n media are studied. Aspects of nonlinear reactor dynamics of the type studied here were considered previously from the physical point of view by W. K. Ergen and A. M. Weinberg, *Some aspects of non-linear reactor dynamics*, Physica XX (1954), and H. J. Lipkin, *A study of the nonlinear kinetics of the Chatillon reactor*, J. Nuc. En. vol. 1 (1955). The heterogeneous reactor considered by Lipkin had heat generated in only one medium. The present paper sets these earlier investigations on a firm mathematical basis and generalizes Lipkin's work on the Chatillon Reactor. (Received October 1, 1956.)

129. Tomlinson Fort: *The five-point difference equation with periodic coefficients.*

In this paper the author considers the five-point difference equation in two independent variables over a finite domain of discrete points. The coefficients are periodic in one of the variables. The second variable is limited to a domain of extent equal to the period in the first variable. A number of theorems are proved relative to constants, ρ_i, which are roots of an equation called the characteristic equation. In particular, theorems are proved relative to the form of solutions which constitute a certain fundamental system, that is, solutions in terms of which all solutions can be linearly expressed. (Received October 4, 1956.)

130. R. M. McLeod (p), J. J. Gergen, and F. G. Dressel: *Uniqueness of mapping pairs for elliptic equations.*

Let D^* be a simple, closed Jordan curve with a representation $z=\phi(t)$, $a\leq t\leq b$ such that $\phi'(t)$ exists for $a\leq t\leq b$, $\phi'(a)=\phi'(b)$, $\phi'(t)\neq 0$ for all t in $[a, b]$, and $|\phi'(t_1)-\phi'(t_2)|\leq N|t_1-t_2|^{\nu}$ for $t_1, t_2\in[a, b]$, where $0<N$, $0<\nu\leq 1$. Let D be the interior of D^* and set $D^-=D\cup D^*$. Let α, β, γ, δ be real-valued functions of class C^0 on D^- and C' on D with bounded derivatives on D. Further suppose $\alpha>0$, $4\alpha\delta-(\beta+\gamma)^2>0$ on D^-. It is shown that, if $w=F(z)$, $F=u+iv$, is a homeomorphism of D^- onto a plane set with $F\in C'$ on D and $\alpha u_x+\beta u_y=v_y$, $\gamma u_x+\delta u_y=-v_x$ on D, then F is determined uniquely by its values at three distinct points of D^*. The case $\alpha=\delta$, $\beta=\gamma=0$, and the domain D a circular disk was treated earlier by Gergen and Dressel (Duke Math. J. vol. 19 (1952) pp. 435–444). (Received October 4, 1956.)

131. E. P. Miles, Jr. (p) and Ernest Williams: *Properties of functions associated with the Cauchy problem for the damped wave equation.* Preliminary report.

In a recent abstract (62-2-211) solutions of the form $\sum_{s=0}^{l}\nabla^{2s}F\cdot u_s$ for the damped wave equation with polyharmonic initial values F were obtained. The functions $\{u_s\}$ are shown to satisfy the recursion formulas: $u_s=1/2s(2s-2)k^2\cdot[(2s-2)(2s-3)u_{s-1}+t^2u_{s-2}]$ and $u_s=-t/2s\int_0^t u_{s-1}dt$. The polynomials $P^s(kt)$, which are the coefficients of e^{-kt} in the expansions of u_s, are shown to be solutions of a fourth order ordinary differential equation. (Received October 4, 1956.)

132*t*. B. J. Pettis: *Comment on a theorem of Jerison.*

Using a net theorem previously established by the author it is possible to rephrase and extend a result by Jerison (Proc. Amer. Math. Soc. vol. 5 (1954)) concerning the comparison of cluster points for a net in a linear topological space and for the net of closed convex covers. (Received October 8, 1956.)

GEOMETRY

133. V. G. Grove: *On closed convex surfaces.*

The purpose of this paper is to prove the theorem: let S and $\bar{S}$ be two closed orientable convex surfaces of class C''' imbedded in a Euclidean space of three dimensions, and possessing no parabolic points. Let h be a differentiable homeomorphism of S into $\bar{S}$ such that $h\, II = \overline{II}$, II and $\overline{II}$ being the second fundamental forms of S and $\bar{S}$ respectively, such that the Gaussian curvatures K and $\bar{K}$ of S and $\bar{S}$ are equal at corresponding points, and such that the orientations of S and $\bar{S}$ are preserved. Then h is a rigid motion. (Received October 3, 1956.)

TOPOLOGY

134. R. D. Anderson: *On extending homeomorphisms and a form of homogeneity.*

Let X be a compact metric continuum and let S be a collection of closed subsets of X. Then X is said to be homogeneous with respect to S if for any homeomorphism g of an element s of S onto an element s' of S there exists a homeomorphism f of X onto itself, with for $x \in s$, $f(x) = g(x)$. The principal result of this paper is the theorem that the universal curve M is homogeneous with respect to the set S of all closed subsets of M which (1) do not contain open subsets of M and (2) do not separate connected open subsets of M. It may be noted that (a) S contains all countable closed subsets of M and (b) S is maximal in the sense that M cannot be homogeneous with respect to any collection S' of closed subsets of M with S' containing S as a proper subset. That (b) is so follows from the fact that S contains a homeomorphic image of M and hence homeomorphic images of all closed subsets of M. (Received October 4, 1956.)

135. C. E. Capel (p) and W. L. Strother: *Counterexample to a theorem of Hamilton.*

In a paper of O. H. Hamilton [Duke Math. J. vol. 14 (1947) pp. 689–693] appears the statement that if T is a continuous multi-valued transformation of an n-cell into itself such that the image of each point is an $n-1$ sphere then T has a fixed point. A counterexample to this is given. (Received September 20, 1956.)

136. M. L. Curtis: *An imbedding theorem.*

R. H. Bing has defined a decomposition of E^3 into points and tame arcs such that the decomposition space M^3 is topologically different from E^3 (see page 45 of the Set Theoretic Topology Institute Summary, Madison, Wisconsin, 1955). As an application of the imbedding theorem of this paper we show that this space M^3 can be imbedded in E^4. (Received September 27, 1956.)

137. M. K. Fort, Jr.: *A note concerning a decomposition space defined by Bing.*

Recently (see pp. 44–47, *Summary of lectures and seminars*, Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955), R. H. Bing has defined a decomposition of E^3 into points and tame arcs in such a way that the decomposition space is not homeomorphic to E^3. In this note, it is shown that Bing's construction can be carried out in such a way that the tame arcs are actually broken line segments, each of which has exactly two breaks. The breaks all occur on two parallel planes in E^3; and if θ is a positive number, it is possible to require that no one of the broken line segments contain an angle that differs from a straight angle by more than θ. (Received September 28, 1956.)

138*t*. Morikuni Goto and Naoki Kimura: *Semigroups of endomorphisms of a locally compact group.*

The structures of the semigroups of endomorphisms of a locally compact group will be studied in detail. (Received October 1, 1956.)

139. H. C. Griffith: *A characterization of tame surfaces in three space.*

Let C be a 2-cell in 3-space and let $\mathfrak{I}$ be the set of all arcs which either lie in ∂C or meet ∂C only in their endpoints. Each T_i in $\mathfrak{I}$ determines two closed sets M_i and N_i such that $M_i \cap N_i = T_i$ and $M_i \cup N_i = C$, and a metric ρ on $\mathfrak{I}$ is defined by $\rho(T_1, T_2) = \min\,[\sigma(M_1, M_2) + \sigma(N_1, N_2),\ \sigma(M_1, N_2) + \sigma(M_2, N_1)]$, where σ is the Hausdorff metric. Let $\mathfrak{D}(T, \epsilon)$ denote the set of all disks such that 1° $\partial D \cap C = \square$, 2° $D \cap C \in \mathfrak{I}$, 3° $\rho(D \cap C, T) < \epsilon$, and 4° if $C \setminus D$ has two components, then D separates them in every sufficiently small neighborhood of C. If $\mathfrak{D}(T, \epsilon) \neq \square$ for all $T \in \mathfrak{I}$ and $\epsilon > 0$, then C is said to have the disk property. If to each pair of disjoint open subsets of ∂C there correspond a pair of disjoint topological rays, one emanating from each of the two sets, which are otherwise disjoint from C and have the property that to each $\epsilon > 0$ there is a 2-sphere in $S(C, \epsilon)$ enclosing C and meeting each of the rays in a single point, then C has the strong enclosure property. Some cells are such that if D_1 and D_2 are disjoint disks in $\bigcup \mathfrak{D}(T, \epsilon)$ and $T' \in \mathfrak{I}$ meets each in a single point then to each $\epsilon > 0$ there is a D in $\mathfrak{D}(T', \epsilon)$ such that $D \cap D_i \cap C$ is a point, $i = 1, 2$. It is shown in this paper that a cell with a disk property and the strong enclosure property is tame if and only if it meets this condition. (Received October 2, 1956.)

140. J. G. Horne, Jr.: *o-ideals and the ideal structure of certain rings.*

The role played by Milgram's o-ideals in studying the ring of continuous functions $C(X)$ on a topological space X has led to a study of this notion in an abstract commutative ring R with identity. (Bull. Amer. Math. Soc. Abstracts 62-1-116 and 62-2-276). If R satisfies the condition (*): every maximal (equivalently: every prime) ring ideal contains a maximal o-ideal, then the collection $\mathfrak{M}(R)$ of maximal o-ideals in R forms a compact Hausdorff space in the dual Stone Topology. In such rings, the class of maximal o-ideals coincides with the class of *prime-like* o-ideals (in Abstract 62-1-116 these were called p-ideals). In order that R satisfy condition (*) it is sufficient that R be an *r-ring*: $ef = f$ implies there are $h, e \in R$ such that (i) $e'f = f$ and (ii) $(he)e' = e'$. In general, when $\mathfrak{M}(R)$ is compact and $\bigcap \{M : M \in \mathfrak{M}(R)\}$ is zero, then for every ring ideal J, $J = \bigcap \{M + J : M \in \mathfrak{M}(R)\}$. This result implies in particular that every o-ideal is the intersection of all maximal o-ideals containing it, as well as a theorem on "belonging locally to an ideal" in commutative Banach algebra theory. (Received October 4, 1956.)

141. R. P. Hunter: *Concerning mobs.*

If S is a decomposable clan, then for any n there is a collection of n subcontinua whose finished sum is S. This generalizes a result of Koch and Wallace (Duke Math. J. vol. 21 (1954) pp. 681–686). If the clan S is nearly 2-homogeneous over x and y such that y is in $T(x)$ then S is a group. If S is indecomposable with proper kernel then one and only one composant Q, is an ideal; Q is an indecomposable connexe containing S^2; S is irreducible between a subset A and A^* if and only if A does not meet Q; S contains a dense ideal B different from Q which is an indecomposable connexe and such that $f(B)$ is widely connected where f is the canonical map mod S^2. (Swingle Proc. vol. 2 (1951) pp. 178–185). If the mob S is the finished sum of two indecomposable continua and has proper kernel then S is not irreducible between two idempotents. (Received October 4, 1956.)

142. John Jewett: *Differentiable approximations to interior functions.*

The following is proved: THEOREM. *Let D be a domain in the plane and let $f:D^- \to R^1$ be a real valued function which is continuous on D^- and interior on D into R^1. Let n be a positive integer. Then for each positive number ϵ there exists a function $g:D^- \to R^1$ such that* (1) $|f(x)-g(x)| < \epsilon$ *for all points x of D^-,* (2) *g is continuous on D^- and interior on D into R^1,* (3) *g is n times continuously partially differentiable, and* (4) *g agrees with f on the boundary of D.* The proof makes use of an approximation theorem proved by the author (Duke Math. J. vol. 23 (1956) pp. 111–124) for light interior mappings together with a deformation process. A similar theorem is proved for a class of monotone mappings of plane domains. (Received October 1, 1956.)

143. R. J. Koch (p) and A. D. Wallace: *Admissibility of semigroup structures on continua.*

A class of compact connected spaces is exhibited which do not admit the structure of a mob equal to its square except in the trivial ways $xy=x$ (all x, y) or $xy=y$ (all x, y). Included in this class are the unit circumference (when it is not a group) and the figure eight. The arguments make use of the following lemma: a compact connected mob S satisfies $S^2=S$ if and only if each dense left ideal containing K (the minimal ideal) is connected. Further, we investigate the structure of a compact connected mob S equal to its square and irreducible between K and A where A is closed and misses K. Such a mob has a unit, a group kernel, and each continuum meeting K and $S-K$ is a principal ideal. Further if cd $(S/K)=1$ then S is commutative. (Here cd represents the cohomology dimension). (Received October 4, 1956.)

144. J. D. McKnight, Jr. (p), R. W. Bagley and E. H. Connell: *Compactness conditions* I. *Pseudocompact spaces.*

Let $C(X)$ denote the set of real-valued continuous functions on a space X, and $C^*(X)$ the bounded elements of $C(X)$. A space X has the property that every sequence of $C(X)$ that converges uniformly at every point of X converges uniformly on X if and only if $C(X)=C^*(X)$. If X has this property it is called *pseudocompact* [cf. Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) p. 67]. Further, C is pseudocompact if and only if every sequence in $C(X)$ that converges locally uniformly on X converges uniformly on X. These characterizations of pseudocompactness lead heuristically to an interesting topological characterization for completely regular spaces: X is *lightly compact* if every locally finite collection of open sets of X is finite. There are, however, pseudocompact regular spaces that are not lightly compact (light compactness always implies pseudocompactness). The equivalence of pseudocompactness and light compactness for completely regular spaces furnishes an immediate proof that pseudocompactness and countable compactness are equivalent for normal T_1 spaces and makes it easy to find an example of a completely regular locally compact and pseudocompact T_1 space that is not countably compact. Moreover, compactness in completely regular spaces is characterized as pseudocompactness plus paracompactness. (Received October 3, 1956.)

145. J. D. McKnight, Jr., R. W. Bagley (p) and E. H. Connell: *Compactness conditions* II: *Lightly compact spaces.*

Light compactness is exactly the condition needed in addition to paracompactness to yield compactness. Moreover, a space is countably compact if and only if it is lightly compact and countably paracompact. The following five statements are equivalent: (i) X is lightly compact; (ii) No infinite sequence of pairwise disjoint open sets of X is locally finite; (iii) Every closed neighborhood properly contained in X is lightly compact; (iv) Every countable open covering of X contains a finite subcollection whose closures form a covering; (v) If $\{x_n\} \subset X$ is countably infinite and $\{U_n\}$ is a countable collection of open sets covering X, then there is some member of $\{\overline{U}_n\}$ containing infinitely many points of $\{x_n\}$. In a lightly compact space, the Lindelöf property is equivalent to *property*(*): Every open covering has a σ-locally finite open refinement. A regular space is compact if and only if it is lightly compact and has property(*), from which it follows that a completely regular space is compact if and only if it is pseudocompact and has property (*). In a metric space light compactness (pseudocompactness) and compactness are equivalent. In a T_1 space compactness is equivalent to light compactness plus property (*) plus the property that every point has a neighborhood whose boundary is countably compact. (Received October 3, 1956.)

146*t*. A. D. Wallace: *Remarks on the center of a lattice.*

A lattice is a Hausdorff space together with a pair of continuous lattice operations. Theorem 1. The center of a compact lattice is totally disconnected. Theorem 2. If a compact connected lattice is topologically immersed in Euclidean space then its center is contained in its boundary. Thus, far from being centrally located, the central elements are peripheral. If the lattice above is modular then any complemented element is peripheral. (Received September 14, 1956.)

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BOOK REVIEWS

Propagation des ondes dans les milieux periodiques. By Léon Brillouin and Maurice Parodi. Paris, Masson et Cie., 1956. 348 pp. 4600 francs.

Throughout the development of theoretical physics one finds the periodic structure which distinguishes itself by the fact that common mathematical methods are available for what appears to be a host of different problems in physics. In the volume under review, Brillouin and Parodi have revised the earlier text of Brillouin (L. Brillouin, *Wave propagation in periodic structures*, New York, McGraw-Hill, 1946) and they give us an interesting account of periodic structures as they arose in Newton's time to those which arise in present day physics and technology.

The simplest periodic structure arose in the study of the motion of a uniformly loaded string (neglecting the mass of the string). The difference-differential equation, describing the dynamical behavior of the individual masses, occurs in many different physical problems, such as the uniform electric filter and the periodic loaded wave-guide. For finite structures, that is a finite number of masses in the case of the loaded string, we have the advantage that we can study the motion of the particles by what amounts to "finite Fourier series" since there are only a finite number of degrees of freedom in the system and this simplifies the methods of solution considerably. The book at hand is devoted to bringing to the attention of the reader the wide variety of physical problems which fall in this general category as well as a study of the mathematical methods. The mathematical methods are applied to systems with a finite as well as an infinite number of degrees of freedom and also to systems in two and three dimensions.

ALBERT E. HEINS

Structure of Rings. By Nathan Jacobson. American Mathematical Society Colloquium Publications, vol. 37, 1956. 7+263 pp. $7.70.

In the last dozen years a great deal of progress has been made in establishing a structure theory for associative noncommutative rings without chain conditions. A large share of the basic results were obtained by the present author and he is therefore eminently fitted to present the theory in its present form to the mathematical public. He has done so very successfully in the volume under review and has given us an essentially self-contained account which should be easily accessible to a reader with only the very basic notions of modern algebra at his disposal. Professor Jacobson has also shown very ably how the general theory specializes down to the case of rings with minimum condition. In fact, the book contains essentially all the results on semisimple rings with minimum condition to be found in, say, Artin, Nesbitt and Thrall, obtained as corollaries of more general theorems with, of course, the usual gain in insight.

The first two chapters contain a detailed presentation of the basic structure theory. The radical of a ring is defined as the intersection of the kernels of all the irreducible representations and all the usual equivalent characterizations are given. A ring A is semisimple if its radical is 0 and it is shown that A is a subdirect sum of primitive rings, i.e. rings possessing faithful irreducible right modules. Primitive rings are next treated via the Chevalley-Jacobson density theorem. This is first proved as a purely algebraic theorem in a formulation which is slightly more general than that of Artin, and then, after a careful formulation of the finite topology in a set of mappings, the topological version is given.

Chapter 3 applies these results to obtain the Wedderburn-Artin structure theory for rings with minimum condition. Actually, a somewhat more general class of rings is treated: it is only assumed that the ring modulo its radical satisfies the minimum condition, while the radical is only supposed to be S.B.I. in the sense of Kaplansky, a supposition fulfilled for radicals which are nil ideals.

The next three chapters deal with deeper properties of various classes of primitive rings. Chapter 4 concerns primitive rings with minimal one-sided ideals; Chapter 5 treats the structure of tensor products of primitive algebras, and in Chapter 6 a Galois theory for the ring of all linear transformations is developed.

Chapter 4 opens with the usual material on completely reducible modules which quickly yields the theory of the socle of a ring. Dual vector spaces are then introduced both algebraically and topologically and the density theorem of Chapter 2 is used to good effect in obtaining their basic properties. Next comes the fundamental Dieudonné-Jacobson structure theorem for primitive rings with minimal ideals which shows that such rings are subrings of the ring of all continuous linear transformations on a pair of dual spaces containing the socle of the latter ring. In Chapter 5, after developing the fundamental theory of tensor products (Kronecker products) of modules, a large number of structure theorems are proven. A typical one is the following due to Azumaya and Nakayama: Let A_1, A_2 be primitive algebras with faithful irreducible modules M_1, M_2 and commuting division rings D_1, D_2. Then the lattice of $A_1 \otimes A_2$ submodules of $M_1 \otimes M_2$ is isomorphic to the lattice of right ideals of $D_1 \otimes D_2$. Chapter 6 opens by developing the two basic tools of the Galois theory: A theorem of the Jacobson-Bourbaki type for rings of endomorphisms of completely reducible modules and the independence relations of automorphisms for the ring of all linear transformations. The Galois theory proper is then given and most of the known theorems concerning a simple ring with minimum condition and a simple finite dimensional subalgebra are derived from it.

A large part of the results of the earlier chapter have reduced down to questions concerning division rings. Accordingly, in Chapter 7 the author lays a foundation for a theory of division rings which are possibly infinite over their centres. The finite Galois theory due to Cartan and the author is developed essentially without using the more general theory of Chapter 6. Next the author presents an infinite Galois theory extending Krull's theory for fields. However, all the automorphisms in this theory must be outer, so that the finite theory is not covered by this extension. The algebraic theory of finite dimensional division algebras is next presented and the various extensions of Wedderburn's theorem about the nonexistence of finite division rings up to and including Nakayama's is given. Finally, there are several very welcome examples of division rings infinite dimensional over their centres.

The next chapter treats the upper and lower nil radicals of Baer as well as the locally nilpotent radical of Levitzki. There is a very neat proof of Levitzki's theorem to the effect that the maximal condition on right ideals makes every nil right ideal nilpotent. Weakly closed nil subsystems of a ring with minimum condition are also shown to be nilpotent.

Chapter 9 is devoted to a brief survey of the theory of the structure space of a ring. This is then used to obtain some of the Arens-Kaplansky representations of a ring as the ring of all continuous functions on certain topological spaces generalizing Stone's theorem about the representation of Boolean algebras.

The final chapter deals with three applications of the structure theory developed in the first nine chapters: (A) Commutativity questions; (B) Algebras satisfying a polynomial identity; (C) Algebraic algebras. The following theorem of Herstein is a typical sample of the results of (A): If $x^{n(x)}-x$ is in the centre, the ring is commutative. (B) contains a full account of the theory as it stands at present with several applications to structure theory. (C) is mainly devoted to giving the results that have been obtained so far on Kurosch's problem: "Is every algebraic algebra A locally finite?" Kaplansky's partial solution which asserts that the answer is yes if, (i) each primitive image of A satisfies a polynomial identity and, (ii) the radical of each homomorph of A satisfies a polynomial identity, is given. The last section presents Amitsur's recent results on algebraic algebras over nondenumerable fields. For example, it is shown that a full matrix algebra over such an algebra is again algebraic.

The book also contains a large number of pertinent examples and at appropriate places challenging problems that remain to be solved are raised. The proofs are clear, well motivated, and often an improvement over those found in the literature. Moreover, there are a great many very pleasant things in this work. These range all the way from recording hitherto unpublished, more or less well known facts in a definitive form to interesting new theorems. The following few examples are typical: (1) Very careful attention is paid to deriving algebra theorems from ring theorems; (2) If A is ring with unit the ideals of A_n are all of the form I_n, I an ideal in A; (3) The derivations of a primitive ring with minimal ideals into itself are completely determined. Indeed, let T be a continuous endomorphism of the underlying vector space such that for each scalar s, $sT-Ts$ is again a scalar, then every derivation is of the form $a'=aT-Ta$; (4) As part of the infinite Galois theory for division rings there is the following theorem: Let D be a division ring, H a group of automorphisms of D, and C the subring of H invariants. Suppose further that H contains every inner automorphism of D leaving C elementwise fixed. Then, if E is a division subring of D with $[E: C]_L < \infty$, every isomorphism of E into D can be realized by an element of H.

There is, however, one criticism that this reviewer would like to express: at several places in the book it is hard for the nonexpert to realize that the results in the text are not the best ones known. Of course, it is not possible for the author to prove all the results he states in their most general form, but the usefulness of the book could have been heightened by fuller references in several spots: (1) In Chapter 3, it is shown that the faithful irreducible modules for primitive rings with minimal ideals are unique, and a footnote points out that this is no longer true if either the hypothesis of faithfulness or the existence of minimal ideals be dropped. However, no examples are constructed and no references where such examples can be found are given. Indeed, the bibliography does not contain all the papers relevant to this point; (2) The commutativity theorem of Herstein given in Chapter 10 has been very extensively generalized by Nakayama in a paper in the 1955 Hamburger Abhandlungen. Although the paper does appear in the bibliography it is not referred to in the chapter, nor is the result in it mentioned; (3) In a paper in the 1954 Transactions of the American Mathematical Society Levitzki solved Kurosch's problem for a wider class of algebras than those mentioned in the text. Although this paper appears in the references to Chapter 10 the theorem in question is not cited.

But this, of course, is not a very serious drawback and the author is to be congratulated on having written this useful and encyclopaedic volume which, this reviewer feels, will be one of the standard works on the subject for some time to come.

ALEX ROSENBERG

Zetafunktionen und L-Funktionen zu einem arithmetischen Funktionenkörper vom Fermatschen Typus. By H. Hasse. Berlin, Akademie-Verlag, 1955. Abhandlungen der deutschen Akademie der Wissenschaften zu Berlin, Klasse für Mathematik und allgemeine Naturwissenschaften, Jahrgang 1954, Heft 4. 70 pp. DM 8.00.

In this memoir, the author has translated into his own language the reviewer's paper *On Jacobi sums as Grössencharaktere* (Trans. Amer. Math. Soc. vol. 73 (1952) pp. 487–495). Some minor points are given a fuller treatment, e.g. the question of determining the exact conductor of the characters which appear in the solution of the main problem (no general answer being given for that question). By restricting himself to the curve $X^m+Y^n=1$ (instead of the more general $aX^m+bY^n=1$), Hasse has been able to give a definition of the zeta-function, based solely on the reduction modulo prime ideals, which does not make the distinction between "ordinary" and "exceptional" primes. This case, however, is undoubtedly far too special to bring out the characteristic features of this important problem, on which one may consult more profitably the recent work of Deuring (Gött. Nachr., 1956) on elliptic curves with complex multiplication. One may also observe that, so far as the "ordinary" primes are concerned, such elliptic curves, as well as the abelian varieties arising from the decomposition of the Jacobian variety of a curve of the Fermat type, are all special cases of the abelian varieties with complex multiplication, whose zeta-function has now been determined by Taniyama (Tokyo Symposium on Number-Theory, 1955); in this sense the reviewer's treatment of the Fermat curve, as reproduced in substance in the present monograph, may be said to have no more than a re[illegible]spective interest.

A. W[illegible]

Elementary topology. By D. W. Hall and G. L. Spencer, II. New York, Wiley, 1955. 12+303 pp. $7.00.

The textbook reviewed here is meant for an undergraduate course (possibly even for juniors) with this dual objective: first, to produce such understanding of the facts and techniques of elementary topology as "will help the student immeasurably in his courses in advanced calculus, real variables and complex variables . . . "; and second, to constitute that course in which the student is definitely acquainted with rigor. It does present the basic material carefully and patiently, with many exercises, a good part of which requires much reexamination of the proofs given; and there are twenty-six figures. Using the book in the way suggested would involve (in most schools) serious revision of the program of courses. The book ought also to be considered simply as an introduction to topology.

The following topics are included in the first four and one-half chapters, which are suggested, on the basis of experience, for an undergraduate year course: (1) Introductory set theory; (2) The real number system; (3) Concepts such as metric, closure, compactness, etc.; (4) Metric spaces more intently, including metrics specially related to local connectedness, completion, the space (C) and its completeness, Urysohn's metrization theorem, etc. Among the exercises we find Directed sets, the Hilbert cube and Baire's density theorem (not involved in the regular text) suggested with references as topics for a series of short papers at this point; (5) The study and characterization of arcs, curves, the 1-, and 2-sphere, and (other) spaces which are continuous images of the unit interval, to the extent of fifty pages, which is nearly half of the fifth chapter. This chapter is the real core of the book. In a graduate course it could probably be covered together with the remaining two: (7) Partitionable spaces (R. H. Bing's fruitful concept); (8) The axiom of choice, containing also infinite topological products (that is, those with more than a finite number of factors) and Tychonoff's theorem.

In my estimation this book would be very useful in a graduate semester course in set theoretic topology where it is desired to include the carefully prepared material of the fifth chapter. The authors deserve praise for exhibiting a teeming arena for the application of easily intelligible topological techniques, to say nothing of the inherent interest of the topics already mentioned (and Jordan's curve theorem, to mention another).

They contend that the treatment has been kept especially elementary through the careful avoidance of algebraic arguments. Indeed, no groups of any sort are introduced. It thus opposes the widely held opinion that algebra can clarify and simplify topology. However that may be, modern topology does involve a lot of algebra; and if indeed there is anything tricky about the application of algebra to topology, then the sooner its utility is gently illustrated to the student the better. There is, of course, considerable topological activity concerned with phenomena in lower-dimensional Euclidean spaces in which it appears possible to do quite well without some of the tools being sharpened in algebraic topology; and for such research this book will prepare the student quite well.

If the needs of the future algebraic topologist were thus as a matter of pedagogic (not sectarian) policy left to one side, the needs of the future topologic algebraist were overlooked altogether. Uniform structures are presented only with a countability restriction; function spaces are treated almost not at all. Infinite topological products are presented merely as an application of the axiom of choice, and Tychonoff's theorem appears as a variant of this axiom. (The treatment of the axiom of choice is marred by a preoccupation with what seems to be an unnecessary axiom of finite choice.) Metrizability of product spaces is not discussed (and therefore the Hilbert cube has to be carefully circumvented in the earlier section of metrization). One would have expected the theorem on the connectedness of product spaces. These defects can easily be repaired by a competent instructor, and are thus minor. We invite anyone preparing a course in topology to consider this work on the merit of the substance and treatment of the fifth chapter which is even physically half the book.

RICHARD ARENS

Plane waves and spherical means applied to partial differential equations. By Fritz John. New York, Interscience Publishers, Inc., 1955. 8+172 pp. $4.50.

This book is dedicated, as the title indicates, to an exposition of the author's results on various problems concerning partial differential equations, results obtained by using relations between a function in an n-dimensional Euclidean space and its integrals over planes and spheres. While treatises and encyclopedia articles supposedly strive for completeness either in the extent of subject-matter treated or in its detailed presentation, a tract has no such pretension and may present a slice of the subject-matter of its field organized [illegible] author's taste. The present example, the second of a new se[illegible] Interscience Tracts, has been fitted to a very elegant taste [illegible] Yet while concentrating on techniques and ideas which he has [illegible]sonally developed and finds congenial, Fritz John has touched upon many of the most important topics of the present-day field of partial differential equations.

It would be somewhat difficult to conceive of this book being used in any general way as a textbook on partial differential equations. The very stylistic economy which makes reading it so pleasant for anyone acquainted with the field has as its compensating disadvantages an allusiveness and a lack of discursive explanations and pedagogic emphases which would make the book difficult for students. An unfortunate consequence of the book's ultra-clear but Spartan style could be that it may be passed up by analysts outside the circle of workers in partial differential equations who might profit from becoming familiar with its elegant analytical technique. As the author shows by example, there are many analytical problems in which simplifications can be achieved and convergence difficulties avoided by substituting his Radon transform for the more familiar Fourier-Laplace transformation.

By a plane wave in E^n is meant a function $g(y\cdot x)$ of x, where $g(s)$ is a function of the scalar variable s, y is a fixed vector in E^n, and $y\cdot x$ the Euclidean scalar product in E^n. Chapter I discusses the expansion of a C^1 function f in E^n as a superposition of plane waves of the form $|(y-s)\cdot x|^k$ or $|(y-s)\cdot x|^k \log |(y-s)\cdot x|$. This expansion is equivalent to formulae of Radon expressing f in terms of its integrals over planes. These formulae are applied in Chapter II to obtain a solution of the Cauchy problem on a half-space for homogeneous hyperbolic equations with constant coefficients. After the geometry of the normal surface has been studied in a projective representation, an explicit integral formula is obtained for the solution of the problem in which the kernel of the integral is given in terms of an integral over the normal surface. In this and further results on domains of dependence, the author re-derives results previously obtained by Herglotz, Petrovsky, Gårding, Leray, and Bureau using variously Fourier methods, Riesz integrals, and the finite and logarithmic parts of divergent integrals.

Chapter III is dedicated to the author's construction, first published in 1950, of a fundamental solution in the small for general linear elliptic equations with analytic coefficients. Here the Radon formulae are applied in combination with the Cauchy-Kowalewski theorem and a method of Holmgren first used to prove the uniqueness of solutions of the Cauchy problem for equations with analytic coefficients. The results are extended to the case of systems and explicit formulae given for a fundamental solution in the large for the constant coefficient case.

Means over the surface of spheres of various radii enter the discussion in Chapter IV which is concerned with obtaining a representation of a rather complicated kind for a function f in terms of its iterated means over spheres with radii bounded away from zero. By an elementary argument, iterated spherical means are used to obtain solutions of the Euler-Poisson-Darboux equation. Chapter V is dedicated to Asgeirsson's mean-value theorem with applications to the Darboux and wave equations. A generalization to ellipsoidal means due to A. S. Howard is discussed. Chapter VI contains a study of the problem of determining a function from its integrals over spheres of a fixed radius, prefaced by a formal discussion of a more general class of convolution equations.

The differentiability theorems for solutions of various classes of partial differential equations by the use of spherical means which are presented in the last two chapters are the culmination of the techniques expounded in this book. In Chapter VII differentiability theorems are established for solutions of linear and nonlinear elliptic systems by means of the identities of Chapter IV. In Chapter VIII, similar regularity properties are established for the integrals over a family of time-like curves of solutions of nonelliptic partial differential equations. The proofs are carried through in a strikingly explicit and concrete analytical fashion. In the elliptic case, no fundamental solutions are utilized. The results include the differentiability of weak solutions of linear elliptic equations, a fact which plays a crucial role in the L^2-theory of elliptic boundary value problems.

It seems certain that this work by Professor John will occupy an important and permanent position in the literature on partial differential equations.

F. E. Browder

Topological transformation groups. By D. Montgomery and L. Zippin. New York, Interscience Publishers, Inc., 1955. 11+282 pp. $5.50.

In writing this book the authors have had two purposes: (1) to present, in connected form, the recent solution by the authors and A. Gleason, based on the work of many mathematicians, of the famous 5th problem of Hilbert, stating that a topological group which is locally homeomorphic to Euclidean n-space E^n is a Lie group; and (2) to report on the work done during the past twenty years on transformation groups, the emphasis being on the way a group acts on a space as a group of transformations.

The first topic represents the final step in a long developme [illegible] which many outstanding mathematicians have contributed: [illegible] duction of the idea of Lie groups by Lie, via sets of transformations which depend (differentiably) on a number of parameters; intensive study of Lie groups, in particular the classification of all semisimple Lie groups (Killing, Cartan); representation theory of Lie groups (Cartan, Weyl); introduction of the concept of topological group (O. Schreier, 1925); analysis of compact groups (von Neumann, Pontryagin, 1933–1934); analysis of locally compact abelian groups (Pontryagin, 1934); invariant measure in locally compact groups (Haar, 1933); and then about twenty years of study of general locally compact groups (the compactness restriction seemed necessary to get significant results) by many mathematicians; finally, the solution of Hilbert's 5th problem, which actually involves a much more general statement—it might be formulated, somewhat loosely, as saying that in a sense (namely up to certain limit processes) there are no other locally compact groups but our old friends, the Lie groups.—We have sailed across a wide ocean; many times the course was not clear at all; we finally arrived—at our own shores; from a different direction to be sure, but it is the old country. One might almost be a bit disappointed at the outcome; but one should remember that in the years of work on these questions many and very fruitful ideas have been discovered which have enriched all of mathematics tremendously, in addition to the many direct applications of the results.

The first two chapters bring a general introduction to the "classical" theory of topological groups, about what was known in 1935, roughly comparable to the corresponding chapters of the books of Pontryagin and Weil (with some new material added), including things like Haar measure, von Neumann's theorem that a locally compact group which admits a continuous univalent finite dimensional representation is itself a Lie group, the Peter-Weyl theorem, the nature of compact groups, Pontryagin's main theorems on abelian groups. Lie groups are defined but their theory is more or less taken for granted.

Chapters III and IV contain the solution of Hilbert's 5th problem. The chain of reasoning is quite intricate and uses a wide variety of arguments, geometrical and analytical; all the things developed in the introductory section are brought to play. Some of the main steps are: the extension theorem that a group with normal Lie subgroup and Lie factor group is itself a Lie group; the existence of 1-parameter groups in metric connected locally compact groups; the use of these 1-parameter groups for groups without small subgroups (i.e. for groups with a neighborhood of e containing no nontrivial subgroup) to construct a sort of finite dimensional tangent space, in which the group admits a representation [image and kernel are Lie groups, and the extension theorem becomes applicable]; groups without small subgroups are thereby shown to be Lie groups; in the general case, each neighborhood of e contains a subgroup which contains all sufficiently small subgroups of G. The final result can be stated as follows: A locally compact group contains an open subgroup that has arbitrarily small compact normal subgroups with Lie groups as factor groups. Many consequences are derived concerning the topological (local) structure of such groups, etc.

Chapters V and VI are on transformation groups, a subject which, as the authors state, is far from its final phase; in fact the purpose of this part is to make widely scattered material available in convenient form as background for further work. Some knowledge of things like inverse limits, covering spaces, properties of manifolds, fiber bundles etc. is assumed in this part. It is in the nature of the material that there is not so much a systematic development here, but rather a collection of various topics.

Chapter V includes the following (and more): The theorem (Bochner-Montgomery), related to Hilbert's 5th problem, that a locally compact group of C^n-transformations of a C^n-manifold is a Lie group, with the operation being of class C^n in group and space variables simultaneously; Bochner's theorem that a compact group of differentiable transformations with a stationary point can be rendered linear near the stationary point by proper choice of coordinates; Gleason's theorem that there exist local cross sections to the family of orbits under a compact Lie group without fixed points, an important fact for the study of the behavior of the orbits.

Chapter VI is more specialized and concerns mostly the nature of the orbits under a compact group operating on a manifold. Example: If (effective) group and manifold are connected and the orbits are locally connected, then the group is a Lie group (and the orbits are manifolds). The authors' theorem on compact groups acting on Euclidean E^n with at least one $(n-1)$-dimensional orbit is presented: there is a stationary point, and all other orbits are $(n-1)$-spheres. Whether the group is actually linear in suitable coordinates is not quite decided. This is an instance of a question which is the motivation for much of this work: To what extent does the general compact transformation group differ from the "regular" case of a compact group of linear transformations? In contrast to the groups themselves, where, by the solution of Hilbert's 5th problem, irregular behavior is just a matter of change of coordinates, new phenomena do appear for transformation groups (Bing's example). It is still unknown whether an infinite compact totally disconnected group act effectively on a manifold. The last topic is the authors' theorem that a compact effective group on 3-space E^3 is linear (orthogonal) in suitable coordinates, with an interpretation of this result for the axiomatic foundation of Euclidean geometry of E^3.

The first four chapters are essentially self-contained, up to general mathematical education and some references to special topics. But it is true of course, no surprise with a subject as complex as the one under consideration, that actually a good deal of sophistication and preparation (or perseverance) will be required for appreciation of the material.

Usually things are spelled out in detail, in an almost conversational style. The authors have not aimed at maximum elegance or brevity in their presentation; e.g., some theorems carry unnecessarily strong hypotheses. Occasionally, particularly in the last part, the reader is asked to supply a good deal of the argument.

In both of its main parts the book leads close to present day work; it constitutes a rich source of facts, techniques of all kinds, and references, for anybody who is actively interested in the subject; it will be of great value to beginning and mature mathematicians alike, and is therefore a very welcome addition to the mathematical literature.

HANS SAMELSON

Théorie globale des connexions et des groupes d'holonomie. By A. Lichnerowicz. Edizioni Cremonese, Roma, 1955. 15+282 pp. 4000 lire.

This is a very timely book on the modern theory of connections. The classical theory was mainly initiated by Levi-Civita and Schouten and received, partly because of its applications to the general theory of relativity an extensive development. Elie Cartan observed, from his effective applications of the method of moving frames to various geometrical problems, that the group concept is the basic underlying idea. He knew many examples and had on the basis of this knowledge all the important notions of a general theory, but did not have the tools and terminology to express them. In fact, his "tangent space" is a fiber in the modern terminology, and his space of moving frames is what is now called a principal fiber bundle, etc. This is not to minimize the contributions of modern geometers (Ehresmann, Weil, H. Cartan, Chern, Ambrose, Singer, etc.), whose efforts have made what was once a difficult subject into a beautiful theory. It is now the considered opinion that in differential geometry a connection is a concept pertaining to a principal fiber bundle.

The book is divided into five chapters. Chapters I and II give a lucid account of the basic notions of manifolds and connections. The treatment seems to take the middle road between modern abstraction and classical nonintrinsic formulation. Whenever necessary the author never hesitates to use local coordinates and to make a choice of coordinate frames. In view of applications this attitude can be fully justified. Chapter III studies holonomy groups, to whose foundations the author himself has made important contributions. The holonomy group is a very natural notion in the theory of connections. However, recent investigations by M. Berger and I. M. Singer (Berger, Bull. Soc. Math. France vol. 83 (1955) pp. 279–330; Singer, to be published) have shown that its possibilities are rather limited. Except for homogeneous spaces it is perhaps not a strong invariant. The chapter ends with a theorem of de Rham which states that a complete simply-connected reducible Riemannian manifold is a topological product.

Chapter IV gives an introduction to the theory of harmonic forms of Hodge. The emphasis is on the formal and algebraic aspects of the theory and does not include a proof of the fundamental theorem. The author is mainly interested in the applications of the fundamental theorem to differential geometry. As a generalization of Kähler geometry the author studies the case of a compact orientable Riemannian manifold on which there exists an exterior differential form with covariant derivative equal to zero. If k is the degree of this differential form F, there is defined a sequence of operators K_h, $0 \leq h \leq k$, such that K_0 is the exterior product and K_k the interior product by F. The aim of the computations is to prove that they commute with the Laplacian operator. From this, results are derived on properties of the Betti numbers of a compact Riemannian manifold on which such a form F exists.

As Hodge's pioneering work has shown, a finer analysis of the global properties of a compact manifold is possible, if the manifold has a Kähler metric. A Kähler metric is usually defined to be a positive definite Hermitian metric $ds^2 = \sum_{k,l} h_{kl} dz^k d\bar{z}^l$ with the property that the associated exterior two-form $i \sum_{k,l} h_{kl} dz^k \wedge d\bar{z}^l$ is closed. What accounts for Hodge's theorems on compact Kähler mainfolds can, in the reviewer's opinion, be briefly described by the commutativity of the following diagram

$$\begin{tikzcd}
\text{Hermitian metric} \arrow[r] \arrow[d] & \text{Riemannian metric} \arrow[d] \\
\text{Associated connection} \arrow[r] & \text{Levi-Civita connection}
\end{tikzcd}$$

In fact, one defines from an Hermitian metric a connection with the unitary group as the structure group; the condition that it reduces to the Levi-Civita connection of the Riemannian metric arising from the given Hermitian metric is equivalent to the Kähler condition. That such commutativity should be basic follows from the fact that the harmonic forms are defined relative to the Riemannian metric. Chapter V starts with an introduction to almost complex and complex manifolds and has as main purpose the derivation of Hodge's theorems. They are formulated in terms of pseudo-Kählerian manifolds; the proofs are identically the same as in the Kählerian case. Recently, L. Nirenberg and A. Newlander succeeded in proving that local complex coordinates can be introduced in a differentiable integrable almost complex structure. As a side result it seems that the adjective "pseudo" can now be dropped in the terminology of these classes of manifolds.

The book should be considered as an account of the author's own researches with sufficient introductory material and does not aim at completeness. For instance, two important topics, characteristic classes and locally flat spaces, are hardly touched. With this in mind the book is highly recommendable as an introduction to modern differential geometry.

S. S. CHERN

RESEARCH PROBLEMS

1. Richard Bellman: *Differential equations.*

Consider the Sturm-Liouville problem:

$$u'' + \lambda(f(x) + \epsilon g(x))u = 0, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

where $f(x)$ and $g(x)$ are continuous functions over $[0, 1]$ with positive minima.

Let us regard λ_1, the smallest characteristic value, as a function of ϵ. Is it true that λ_1 is an analytic function of ϵ for $R(\epsilon) \geq 0$? In general, where is the singularity nearest the origin ($\epsilon = 0$)? (Received September 24, 1956.)

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MATHEMATICS AND THE FUTURE OF SCIENCE

MARSHALL H. STONE

It is indeed a great honor to be asked to deliver the Gibbs Lecture this year, and one which I have accepted in a humble spirit. The work of Josiah Willard Gibbs comprised a series of achievements in applied mathematics of such beauty and perfection that the most fitting tribute which could be offered to him here would be a scientific paper written in a spirit of emulation. Such a tribute lies beyond my powers. My interests as a mathematician have been directed toward pure mathematics; and I have never entertained the ambition of contributing explicitly to the advancement of applied mathematics or cherished the illusion that I might have such a contribution to make. I have not, however, been one of those mathematicians whose joy in their mathematical achievements is intensified by the belief that these are to remain forever useless and unused outside the happy realm of pure mathematics. On the contrary, the satisfaction which I have derived from working in mathematics has been increased by the knowledge that what I have done could be seen, with few exceptions, to have some bearing upon mathematical physics or upon some other branch of applied mathematics. I have taken much pleasure in acquainting myself with the ways in which the results of pure mathematics could be turned to good account in probing Nature's secrets and rendering them intelligible—and, eventually, useful. My tribute to Gibbs will therefore take the form of an expression of faith in the growing importance of mathematical thought for the future of science. I wish on this occasion to speak of my reasons for believing that the wonderful development of mathematics which we have witnessed in our time holds the seeds of brilliant scientific progress in time to come.

In a sense, all that can be said upon this theme is summed up in a syllogism: science is reasoning; reasoning is mathematics; and, therefore, science is mathematics. Because this simple argument seems to me to express so well the essence of the inescapable connections between science and mathematics, I wish to devote a few words to clarifying and justifying the premisses upon which it rests. The minor premiss raises a point which has been at issue between logicians and

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The thirtieth Josiah Willard Gibbs Lecture, delivered at Rochester, New York on December 27, 1956, under the auspices of the American Mathematical Society; received by the editors December 27, 1956.

mathematicians since the publication of *Principia Mathematica*. I think both parties are right—the logicians when they claim that mathematics can be formalized in certain systems of formal logic, the mathematicians when they rejoin that the systems of formal logic are themselves special mathematical systems which should be treated by the methods of mathematics. While I have tried in my own mathematical work, some of which is to be read by title at this meeting, to provide evidence for the mathematicians' side in this debate, I believe that the issue should be resolved by equating formal logic with mathematics, as has been done here. It is of some importance for our later considerations that this equation recognizes mathematics as embracing much more than the study of quantity and number. Indeed, it requires us to conceive of mathematics as the study of general systems comprising designated objects and designated relations among them; and to regard the quantitative or numerical aspects of particular mathematical systems as accidental, rather than as essential or characteristic, phenomena for mathematics as a whole. It would, for instance, be wrong from this point of view to reject a scientific theory as nonmathematical merely because the theory is nonquantitative.

Our major premiss clearly hinges upon a question of definition: what are we to mean by the term "science"? The answer which we propose to give here is precisely the one which validates this major premiss—namely, that science comprises those and only those disciplines in which reasoning plays a predominant and essential part. While this is hardly the sense in which the term "science" is used in popular American speech or writing, weightier objections can be raised against our definition and are entitled to a thoughtful review. Should not the definition of "science" be so framed as to refer explicitly to the roles of observation, experiment, and prediction? Is it not necessary to take into account the distinction between inductive and deductive reasoning? Does not the subject matter of a discipline have some bearing upon its classification in the present context? Is not our definition too inclusive in some respects, too exclusive in others? Such queries as these plainly deserve some kind of comment, even though the circumstances compel me to be succinct in making it. Any intellectual discipline whatever deals with recorded observations, seeking to find in them some sort of order or pattern, and to view them in the light of reason. It is surely not until observation becomes something more than the casual or the systematic collection of facts that it can be thought of as a characteristic feature of science. Otherwise we would have to regard history and literary or aesthetic criticism as sciences, for example, and would soon realize that we had appealed to a criterion without value. On the other hand, if we were to require that the term "science" be applied to no disciplines except those in which systematic, directed, and controlled observations—in a word, experiments—are an essential element, we could then not include geology or meteorology among the sciences. We might try to avoid these unsatisfactory extremes by choosing a definition in terms of directed observations. It seems to me however that this attempt must fail because of the difficulty in deciding what shall be meant by the phrase "directed observation." How can we differentiate the kinds of observation made by the historian, the systematic botanist, and the meteorologist by the criterion of "directedness"? I would say that we can do so only by determining to what extent observations in these different disciplines are guided by and related to the reasoned theoretical structures which they have built up over the years. But if this is so we have returned to a characterization of science based upon the prominence of the rational element. On another occasion [1] I suggested that concern with prediction distinguishes the scientific from the nonscientific disciplines. Without altering the position which I took then, I must now observe that this criterion too leads us back once more to the definition with which we started here. Indeed, the possibility of making predictions, either deterministic or statistical, rests squarely on the ability to reason closely from such general principles as may have emerged from previous observation and study. The more we seek to refine our predictions, the more we must refine both these principles and the arguments by which we discover their implications. From our point of view, to say that history is not a science because it does not predict is merely one way of calling attention to the fact that in the study of history reasoning plays neither a predominant nor an essential role.

Although this discussion of the meaning which is to be attached to the term "science" is far from complete, enough has perhaps been said to qualify as a usable analytical tool the definition we have proposed here and to elucidate our conclusion that science is mathematics. Nonetheless, a strictly logical analysis of the connections between science, reasoning, and mathematics leaves much to be desired because its scope is too narrow to include any account of the growth and development of those disciplines which we accept as sciences. Looking backward, we see that the sciences we call astronomy, physics, and chemistry have passed through various stages, beginning as practical arts in very remote times, becoming in due course subjects of scholarly study and speculation, and emerging at last as intellectual disciplines centered about elaborate mathematical theories. Indeed, in each of these three cases we have little difficulty in identifying the moment at which the transformation into the scientific phase began. These moments are marked, respectively, by the contributions of Eudoxus, of Galilei and Newton, and of Lavoisier and Priestley. While these examples may serve to strengthen our conviction that it is the rational component of an intellectual discipline which identifies it as a science, they also force us to recognize that in every branch of knowledge this component has to develop, often very slowly, from a few small seeds. As we look around us today we see such seeds vigorously sprouting in many fields, and on the point of germinating in as many others. These fields of knowledge are not only those which deal with the physical and biological worlds, but also those which are concerned with the mind of man and with his behavior as a social being. We are without any doubt witnesses to the birth of new sciences, and we can be sure that in many of them we shall see rapid development along theoretical, mathematical lines. Indeed, we have the strongest logical and historical grounds for believing that the tendency towards the mathematicization of knowledge, which began with the Greeks, will be broadened and accelerated in the coming century. That this tendency will require and stimulate advances in mathematics itself is inevitable. To some extent it will even be directed and conditioned by the progress which can be made by mathematicians, as we can infer from a consideration of the history of modern physics. The future of science is in this sense tightly bound together with the future of mathematics. The extraordinarily luxuriant growth of pure mathematics in our time encourages us to look with optimism and enthusiasm to a future in which science will draw bountiful nourishment from the fruits of our husbandry in the mathematical field.

In spite of logic and the lessons of history it would be very foolish to expect that mathematics will be able to make its fullest contribution to the advancement of science unless mathematicians themselves take an active and well-informed interest in its applications to the multiplying fields of science, old and new. Even though the proliferation of mathematical modes of thought into the most diverse domains of learning may rightly be regarded as inevitable, it can easily be retarded or deflected by obstacles which will have to be identified and removed if the development of the sciences is to be accelerated. While it may be true that the world will beat a path to the door of the man who builds a better mouse-trap, as Emerson once said, we mathematicians must recognize that, however good the mouse-traps we invent, the world may be very slow to realize its need of them and equally slow to discover the most convenient and satisfactory paths to our door. The task of attracting the world's attention to our stock of mouse-traps and marking out some of the approaches to it is primarily the task of mathematicians. It is a difficult, challenging task which cannot be satisfactorily handled until we ourselves have gained a clearer and more profound understanding of the changes wrought by mathematical, scientific, and technological progress in the nature of applied mathematics and its relation to pure mathematics on the one hand and to science and technology on the other. Of all the different aspects of our central theme which might be discussed in greater detail, there is none which seems to me so timely or of such obvious importance as the evolution now taking place in applied mathematics. I propose therefore to speak during the remainder of my allotted time about the subject of applied mathematics.

This subject is one which has evoked a great deal of discussion and writing in recent years. It is therefore impossible to avoid going over a good deal of familiar ground in what I am about to say here. Where I can do so, I shall, of course, touch only lightly on matters which have been thoroughly handled in other places. For more extensive details relating to such matters reference may be made to one of the most elaborate of the recent treatments of our subject, the report of the National Research Council's Committee on Training and Research in Applied Mathematics [2], on which I had the privilege of serving. I shall try to emphasize instead those considerations and points of view which seem to me to possess some degree of freshness, if not novelty. It is, indeed, my feeling that for a pure mathematician to offer a frank discussion of the current problems of applied mathematics is in itself something of a novelty.

Since the time when I first attempted to clarify my own point of view about the pursuit of applied mathematics, in an unpublished address delivered at the celebration of the University of Chicago's fiftieth anniversary, I have become increasingly aware of the tensions created by the existence of two different and philosophically opposed attitudes toward the applications of mathematics. In most discussions of the subject these tensions can be sensed even when no direct allusion is made to them. I have learned too that the attempt to discuss applied mathematics without frank statements of position from the participants is to invite confusion and ill-feeling. I therefore wish to preface my own remarks upon this subject with a clear indication of my own attitude. It took me rather a long time to understand the nature and the true significance of the two different attitudes which

I have mentioned, though there was never any doubt in my mind as to my choice between them. It would have taken me even longer to arrive at a satisfactory statement of these matters, if I had not had the good fortune to come upon an essay of Whitney Griswold, President of Yale University, in which he probes the nature of the fundamental controversy in contemporary American educational circles [3]. He describes the conflict as one between liberal education and utilitarian education, and traces its origins to ancient times. The opposition is between that point of view which regards as good whatever develops the intellectual and spiritual powers of the individual and the point of view which regards as good whatever works or accomplishes useful results. Griswold quotes a wonderful passage from Francis Bacon, who says, "First, therefore, amongst so many great foundations of colleges in Europe I find it strange that they are dedicated to professions, and none left free to arts and sciences at large. For if men judge that learning should be referred to action, they judge well, but in this they fall into the error described in the ancient fable, in which the other parts of the body did suppose the stomach had been idle, because it neither performed the office of motion as the limbs do, nor of sense, as the head doth, but yet notwithstanding it is the stomach that digesteth and distributeth to all the rest. So if any man think philosophy and universality to be idle studies, he doth not consider that all professions are from thence served and supplied. And this I take to be a great cause that hath hindered the progression of learning, because these fundamental knowledges have been studied but in passage. For if you will have a tree bear more fruit than it used to do, it is not anything you can do to the boughs, but it is the stirring of the earth and the putting new mould about the roots that must work it." Is this not admirably put indeed? I find it so, and am quite content to take my stand with Francis Bacon. I hold, as he does, that utility alone is not a proper measure of value, and would even go so far as to say that it is, when strictly and short-sightedly applied, a dangerously false measure of value. For mathematics, which is at once the pure and untrammelled creation of the mind and the indispensable tool of science and modern technology, the adoption of a strictly utilitarian standard could lead only to disaster: it would first bring about the drying up of the sources of new mathematical knowledge and would thereby eventually cause the suspension of significant new activity in applied mathematics as well. In mathematics we need rather to aim at a proper balance between pure theory and practical applications, as Bacon's words counsel us to do. The peculiar importance which maintaining such a balance may have for our times was emphasized not very long ago by Alan Waterman, Director of the National Science Foundation, in an address delivered before this Society and the Mathematical Association [4]. As he pointed out, "... mathematics, in a sense, bridges the gap, real or imaginary, which exists between the sciences and the humanities. The exigencies of modern technology have attracted many of the sciences away from their original orbits in the realm of natural philosophy. Mathematics, too, has had its practical part to play in the modern world, but in the process it has never lost its scholarly aura. It occupies an honored place perhaps equally among the humanities as among the physical sciences." It is my fondest and most cherished hope that mathematics, despite the strains to which it is subjected by the expansion of its own boundaries and the demands of the modern world, may forever preserve its essential unity and may thus continue to aspire to this position of twofold honor, always setting for itself those high standards of intellectual achievement which alone can qualify it as worthy to occupy such a place.

Let me now turn to a consideration of those developments in mathematics which compel us to revise and to reappraise our estimates of the relations between pure and applied mathematics. For this purpose it is necessary to examine some of the historical factors which have combined to produce the present situation. Approximately at the turn of the century mathematics entered upon a new phase of its development, destined to bring about a profound transformation of its substance and structure. The changes which have been wrought over the short period of less than sixty years are little short of amazing, in variety as well as in magnitude. Their extent can be quite accurately gauged by comparing the curricula of 1900 or 1920 with those of today; or by noting the long list of new mathematical concepts, methods, and disciplines, hardly known fifty years ago, which form an essential part of the intellectual equipment of any modern mathematician who pretends to a general knowledge of his subject. This flowering of pure mathematics in the twentieth century has involved the specialization inseparable from an intensive exploration of new possibilities, the continual testing of new ideas for their viability, and a ceaseless struggle to comprehend in universal terms the rapidly accumulating products of fruitful research. It was an essential condition of this flowering that mathematicians should first have recognized and acted upon the fact that mathematics is not closely bound to the material world or to physical reality—if, indeed, it is bound at all. As a necessary consequence of this recognition, the pure mathematicians of our century have sought the future of mathematics not in technical virtuosity alone but also in abstraction and universality. Throughout its long historical evolution, Western mathematics had dealt, consciously or unconsciously, with mathematical aspects of the real world—the properties of numbers, the geometry of the space in which we move, the dynamics of the moving things about us. Mathematicians thought of the theories which they constructed about these aspects of reality as calculated to yield a philosophical understanding of nature as well as some practical mastery of it. All this appears clearly in the achievements of such great mathematicians as Archimedes, the Bernoullis, Euler, Laplace, Gauss, and Cauchy, whose contributions are honored today by both the applied mathematician and the pure. It appears equally in the uneasiness occasioned by the discovery of noneuclidean geometry early in the nineteenth century, when it began to be plain to everyone that mathematics would henceforth have to deal with two geometries, only one of which could govern the space of our experience—and, worse, that skillful measurements such as those carried out by Gauss might be insufficient to decide which of the two applies to the physical world. The full implications of this profound new insight into the freedom of mathematical theory from physical necessity made themselves felt but slowly, partly because the nineteenth century still had an immense task to perform merely to build upon the foundations which had already been laid, partly because the techniques developed in an earlier time were in urgent need of overhauling and refinement before they could be deemed satisfactory for the work which was to be done. Thus it fell in large measure to the twentieth century to make the most of an insight which was gained in the early part of the preceding one. The instrument fashioned for the accomplishment of this task was the axiomatic or postulational method, now so well known that I do not need to describe it here. There are those who profess to see in the use of the postulational method the essence of modern pure mathematics. To my mind it is a mistake to suppose or to suggest that a mere instrument, however powerful or characteristic, can contain in itself the essence of such an intellectual movement as that which has swept the mathematicians of this generation forward to achievements certain to stamp an indelible mark upon the mathematics of the future. Rather it is what this instrument has made possible—the dissection of mathematical concepts into their elemental components, the recombination of these components into new constructs of intrinsic interest, the critical evaluation of alternative approaches to the important mathematical theories, the unification of hitherto unconnected branches of mathematics—that best expresses the spirit of modern pure mathematics. Though concentration of one's attention on the details might at first seem to indicate the contrary, pure mathematics does in fact continue to revolve about the great central problems of number theory, of geometry, and of analysis, which deal with matters fully as concrete as the abstractions of the atomic or nuclear physicist. Pure mathematics, despite an altogether natural preoccupation with the very fruitful independent development which has been in progress in our times, has never ceased, in the words of Francis Bacon, to "refer learning to action." In truth, it has continued to draw inspiration from the oracle of nature, and has remained constantly aware of the role it plays in increasing the applied mathematician's resources for understanding the world in which we live. Interesting testimony to this effect is offered in the National Research Council report mentioned above [2] when it notes that "insofar as lines of communication with modern physics are being maintained at all by mathematical activities in this country, some pure mathematics faculties have managed to tend them better than any of the applied mathematics centers . . . ." But, to repeat what I have insisted upon above, pure mathematicians cannot accept "reference to action" as the sole criterion by which their work is to be judged.

While pure mathematics was growing in this rapid and remarkable fashion for more than five decades, applied mathematics too has experienced highly significant changes—on the one hand a radical reorientation, and on the other an extensive enlargement of its scope. The most spectacular developments, generally agreed to have been of a genuinely revolutionary character, were those involving the foundations of physics. The abandonment of the classical views of dynamics which stemmed from Galilei and Newton in favor of relativity theory and quantum theory opened up to mathematical analysis great masses of data which had proved refractory to treatment by the classical methods of mathematical physics. It was virtually inevitable that theoretical physics should have shifted the center of its chief interest into the exciting new fields which begged for exploitation, leaving to a sort of marginal cultivation many traditional domains of the science—in particular, hydrodynamics, the theory of elasticity, and other aspects of the mechanics of continua. These fields were henceforth of greater concern to engineers and a few mathematicians than they were to physicists. At the same time chemistry, which in the nature of things had hitherto been but lightly touched by mathematics—and this mainly through the intervention of thermodynamics and statistical mechanics—now felt the heavy impact of the new atomic theories. Despite the grave imperfections which have since been revealed through the accumulation of new data about the elementary particles and the nucleus, these theoretical developments undoubtedly constitute the crowning achievement of applied mathematics. This achievement would have been impossible without the use of mathematical tools, such as the tensor calculus and the theory of continuous or topological groups, which the physicist of 1900 would presumably have regarded as mere mathematical curiosities. For the most part these tools were already available in the stocks prepared by the mathematicians of an earlier day. One of the reasons why pure mathematics has tended to maintain a certain amount of contact with modern theoretical physics, as noted above, is that the new uses to which these tools were put posed novel, intrinsically interesting, and relatively difficult questions about traditionally important branches of mathematics. Much effective mathematical research has been devoted in recent years to just such questions, particularly to those concerning the representations of topological groups and the theory of operators in Hilbert space. On the other hand, theoretical physics, already highly mathematical in form, has been drawn into abstractions and speculations of a more or less metaphysical character for the exploration of which the axiomatic or postulational method peculiar to modern pure mathematics has proved itself to be a most powerful instrument. Many illustrations could be cited, but the two most striking are perhaps the papers of Dirac [5] and Yukawa [6] in which the positron and the meson were postulated and subjected to mathematical investigation before either elementary particle was known to have physical existence. At the present time the mathematical and conceptual difficulties of field theory have apparently produced a new crisis in theoretical physics which seems to call for new physical insights and new mathematical tools. From a mathematician's point of view it is not unlikely that part of the physicist's present troubles arises from his inadequate mastery of modern mathematical methods and concepts. It is true that most of what has been accomplished in modern theoretical physics has been based on nineteenth century mathematics, and it is conceivable that this is not going to be sufficient for further progress.

If the mathematical successes in the field of physics have been the most spectacular and impressive, the twentieth century has witnessed many other advances in applied mathematics. As in the case of quantum theory, many of these advances depend in one way or another upon the use of mathematical statistics. We have learned that many of the most fundamental processes of nature seem to defy any exact deterministic characterization and must be thought of in statistical terms. This is true not only of the behavior of the elementary particles of physics, but also of the mechanisms governing heredity. The modern theory of genetics, which is perhaps the most highly developed and mathematical part of theoretical biology, is thus founded on statistical principles. Because of the tremendous scope of its applications, ranging all the way from theoretical physics to the social sciences, mathematical statistics has undergone a rapid and extensive development so that it now enjoys the status of an independent discipline. Mathematically we now know that it is a branch of measure theory, which is linked with the real world through a few simple principles embodying the essence of inductive reasoning. There is, of course, some disagreement as to how these principles should be formulated. It has always seemed to me that they all have to be based on a single rule of thumb, "A sufficiently improbable event may be ignored." In making decisions according to this rule, the role of mathematics is to provide the measure-theoretic calculations of interrelated probabilities and the role of practical insight is to determine for each concrete situation which probabilities are sufficiently small. Why the real world should be amenable to such a rule is, I think, a philosophical question no more—and no less—mysterious than the problem of why it should be amenable to logic. The fact that these questions are but two aspects of the ultimate problem of the connections between mind and matter leads me to think that distinguishing between inductive and deductive reasoning can be of little help in defining what we mean by "science." In any case, so far as techniques are concerned, the development of mathematical statistics in its modern form has shown that virtually all the detailed procedures of inductive reasoning are deductive in character.

It would be illuminating, if I had the knowledge and the time for it, to depict in detail the way in which applied mathematics has ramified in the course of the present century. Apart from what has taken place in the physical sciences other than physics and chemistry, we should have to examine at some length the attempts to fashion mathematical theories of certain biological phenomena. Closely related to the work in genetics, to which we have already alluded, are the mathematical studies of the growth of populations and of competing species, with their obvious practical significance. We should mention also the mathematical treatment of various physiological problems, including the theory of nerve-networks and the brain. Then we should go on to survey the emergence of new mathematical theories of various psychological and social phenomena, such as the deduction from communication theory of the observed statistical regularities in language, the interpretation of certain competitive situations in terms of game theory, or the application of linear programming to managerial problems. The list of such theories would already be a long one. It seems to me that a rather special interest attaches to the novel and promising applications of mathematics currently being made in fields which were once deemed inaccessible to mathematical treatment, either on grounds of principle or because they had already shown themselves recalcitrant to it. More often than not it is the possibility of using novel mathematical tools—as, for example, the refinements of mathematical statistics or the theory of games—which has permitted some progress to be made in these new directions. I believe that we are entitled to expect a great deal of further progress in the immediate future, but that we must be prepared to see it coupled with the development of new mathematical methods as well as with direct applications of those already available. Nevertheless, optimism in regard to potential advances of this nature should be tempered with a considerable dose of caution. Even though there seems to be very good reason for optimism, it must be admitted that the role of mathematics outside the exact sciences is far from being solidly established by its achievements, save possibly in the field of genetics. There is in fact a large body of opinion, particularly in the social studies, which considers that the really important phases of the behavior of living beings will never yield to mathematical treatment—and in some cases not even to scientific treatment in the broadest sense of the term. Some of the most fascinating and significant intellectual adventures of the decades just ahead of us are likely to be experienced in attempts to determine the extent to which mathematical penetration of such unexplored domains is possible.

The technological demands of our age have begun to exert an increasingly strong pressure upon the development of applied mathematics, as it becomes necessary to answer more and more precisely a greater and greater variety of difficult questions arising in connection with the design, construction, or operation of novel, complex, or versatile machines. World War II presented some of these demands with a peculiarly high degree of emphasis and urgency, which have in the meantime been underlined, if anything, by the ensuing cold war. It seems plain that in the long run these demands will be just as insistently forced upon our attention by the competition between man and nature as by that between man and his fellow-man. The technological demands of the time have, in particular, played a major role in the revival of a branch of applied mathematics—the dynamics of continuous media—which for some time had suffered neglect at the hands of both mathematician and physicist, as we have already observed earlier in our discussion. The fact that new mathematical and physical insights seem to be imperatively needed in this field suggests that one reason for past neglect may have been the conviction that for the moment science was not yet ready to cope with the recognized obstacles to further progress. Certainly such a feeling could only have enhanced the attractions exerted by other domains—attractions which were in themselves strong enough to draw most mathematicians and physicists away from this classical domain of common interest. The technological applications of mathematics characteristically require numerical answers in quantity and thus place a premium upon computational routines and the mathematical skills involved in setting them up. The mechanization of these routines, itself dependent upon a high degree of technological proficiency, has been successfully undertaken in response to this obvious need and has important implications for the future. Not only is the utilitarian scope of applied mathematics very substantially enlarged because necessary computations can now be performed at such high speeds and at such relatively low cost in terms of effort, but in addition the mathematical art of computation has been lifted up to a higher and much more interesting level. It is not easy to foretell what varied influences the introduction of modern high-speed computing machines will have upon mathematics or upon the mutual relations of mathematics and industry. There is a distinct possibility that industry, guided by a thoroughly utilitarian outlook, will tend on the whole to confine its interest in applied mathematics to the organization of working teams composed of applied mathematicians and computing experts charged with the task of turning out, with the highest possible degree of proficiency, specific answers to a limited range of specific questions. A heavy demand along these lines appears to be building up today in the United States. If this tendency should not be counterbalanced by the parallel development of a deeper interest in fundamental mathematics such as some of the leading industries here and abroad have consistently shown in the past, then there would be reason for viewing the future with certain misgivings.

The response of American mathematics to the broad trends and developments which we have been seeking to describe seems to me to have been significantly affected by certain circumstances peculiar to the American scene. These circumstances are to be explained in part, I think, by interpreting them as the product of America's experience as a nation of pioneers cut off by distance from the historic centers of their culture. Perhaps another factor, as suggested by President Griswold in the article cited above, has been the recent immigration of large numbers of men and women drawn from those elements of European society having the most tenuous contacts with the main intellectual centers of the continent. In America the creation of centers of research and advanced training in higher mathematics lagged far behind the developments of the eighteenth and nineteenth centuries in Europe. Despite the pioneer influence of such essentially isolated figures as Sylvester and Benjamin Peirce, no sustained interest in higher mathematics marked the growth of the American university until the closing decade of the last century. Thus the entire development of the great mathematical centers which now dot the country has taken place in little more than sixty years. A very significant feature of this development, apart from the fact that it coincided almost exactly with the great flowering of pure mathematics which we have described above, was the almost complete freedom accruing to the mathematicians of America from the cultural lag just mentioned. They were not bound by strong academic traditions nor hampered by powerful vested interests in their task of setting the course which higher mathematics should follow in America. At the same time the highly pragmatic outlook of the American industrialist and business man—a product also of the pioneer experience—prevented American mathematics from being called on to play anything but the most modest utilitarian role during this critical period. In consequence, the mathematicians of this country were free to direct their efforts towards the great central themes of modern mathematics and thus came to participate both as teachers and investigators in the exciting progress of pure mathematics in the twentieth century. Indeed, American mathematics has been conspicuously marked by tendencies towards abstraction and generality which must seem very surprising to anyone convinced of the utilitarian orientation of American culture. If American mathematics has enjoyed a singular independence from utilitarian demands and has thus been enabled to achieve important intellectual progress, it has also been placed in a false position both intellectually and professionally. The utilitarian philosophy is by no means confined to the United States, though it is perhaps as strong here as anywhere in the world. It has had an influence not only upon the industrial applications of science, including mathematics, but also upon the relations between the various branches of science, and perhaps nowhere a stronger influence than in the United States. Insofar as mathematics is concerned the effects of the utilitarian attitude are particularly significant because all the sciences, as we have tried to show here, exhibit a common trend towards an increased dependence upon mathematical reasoning and thus need an increasingly deep and intelligent appreciation of the nature and resources of modern mathematics. It need hardly be pointed out that in this era of specialization much of the best work in applied mathematics is done by men who regard themselves not as mathematicians at all but as physicists, chemists, or biologists. And this, it might be added, is bound to be the case under any circumstances because good work in applied mathematics must be concerned at least as much with nature as with mathematics. The majority of scientists, at least in this country, fall easily into a strictly pragmatic and utilitarian point of view towards mathematics, which they finally come to consider merely as a more or less useful tool concerning which it is unnecessary to know much more than the immediately useful features. Quite literally the scientist who slips into this attitude accepts what works as true, so far as mathematics is concerned. In this way communication has broken down to a very grave extent between pure mathematics and many branches of applied mathematics. If mathematics is to be restored to its true position and thus enabled to play its part intellectually and professionally in the development of science and its industrial applications, then these lines of communication must be reëstablished within the learned world and extended also into all parts of the world of technology.

While the means for redressing the balance between the pure and the applied aspects of mathematics and for restoring these lines of communication lie outside the purview of this address so far as any detailed discussion is concerned, I shall direct my closing remarks to stressing once more that we mathematicians have the primary reresponsibility for identifying and using them. This is all the more true because the most important means of influencing the relations between mathematics and man's intellectual and industrial activities is mathematical education. Financial means can properly be of concern to us; but if in the future industry were to contribute to the promotion of mathematics as generously as private individuals and the government are now doing, our worries on that score would speedily be put to rest. What we must realize above all is that the mathematical education of the past has to a disturbing extent failed to lay the groundwork for the kind of intercommunication among mathematics, the various sciences, and engineering, which we now see to be necessary. By and large mathematical instruction has been little touched, except at the graduate level, by the mathematical advances of this century. Until it is, such improvements as are made in it will

WITOLD HUREWICZ, IN MEMORIAM

SOLOMON LEFSCHETZ

Last September sixth was a black day for mathematics. For on that day there disappeared, as a consequence of an accidental fall from a pyramid in Uxmal, Yucatan, Witold Hurewicz, one of the most capable and lovable mathematicians to be found anywhere. He had just attended the International Symposium on Algebraic Topology which took place during August at the National University of Mexico and had been the starting lecturer and one of the most active participants. He had come to Mexico several weeks before the meeting and had at once fallen in love with the country and its people. As a consequence he established from the very first a warm relationship between himself and the Mexican mathematicians. His death caused among all of us there a profound feeling of loss, as if a close relative had gone, and for days one could speak of nothing else.

Witold Hurewicz was born on June 29, 1904, in Lodz, Russian Poland, received his early education there, and his doctorate in Vienna in 1926. He was a Rockefeller Fellow in 1927–1928 in Amsterdam, privaat docent there till 1936 when he came to this country. The Institute for Advanced Study, the University of North Carolina, Radiation Laboratory and Massachusetts Institute of Technology (since 1945) followed in succession.

Mathematically Hurewicz will best be remembered for his important contributions to dimension, and above all as the founder of homotopy group theory. Suffice it to say that the investigation of these groups dominates present day topology.

Still very young, Hurewicz attacked dimension theory, on which he wrote together with Henry Wallman the book *Dimension theory* [39].[1] We come to this book later. The Menger-Urysohn theory, still of recent creation was then in full bloom, and Menger was preparing his book on the subject. One of the principal contributions of Hurewicz was the extension of the proofs of the main theorems to separable metric spaces [2 to 10] which required a different technique from the basically euclidean one of Menger and Urysohn. Some other noteworthy results obtained by him on dimension are:

(a) *A separable metric n-space* (= *n dimensional space*) *may be topologically imbedded in a compact metric n-space* [7].

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[1] Square brackers refer to the bibliography at the end.

(b) *Every compact metric n-space Y is the map of a compact metric zero-space X in such a manner that no point of Y has more than $n+1$ antecedents, where n cannot be lowered, and conversely where this holds* dim $Y=n$. *In particular one may choose for X a linear set containing no interval* [6].

(c) Perhaps his best dimension result is his proof and extension of the imbedding theorem of compact spaces of dimension $\leqq n$ in Euclidean E_{2n+1} which reads: *A compact metric n-space X may be mapped into E_{n+m} ($m=1, 2, \cdots$), so that the points which are images of k points of X make up a set of dimension $\leqq n-(k-1)m$* [26].

This proposition may also be generalized as follows: *Any mapping $f\colon X\rightarrow E_{n+m}$ may be arbitrarily approximated by one behaving as stated.* Special case: *X may be mapped topologically into E_{2n+1}*. Earlier proofs of this last theorem existed. The wholly original proof of the main theorem by Hurewicz rests upon the utilization of the space E_{n+m}^X of mappings of $X\rightarrow E_{n+m}$, as defined by Fréchet and the proof that the mappings of the desired type are dense in E_{n+m}^X.

A more special but interesting dimensional result is:

(d) *Hilbert space is not a countable union of finite dimensional spaces* [10].

Recall R. L. Moore's noteworthy proposition: a decomposition of the two-sphere S^2 in upper semi-continuous continua which do not disconnect S^2 is topologically an S^2. Hurewicz showed [17] that *for S^3 no such result holds and one may thus obtain topologically any compact metric space*. This shows that R. L. Moore's results describe a very special property of S^2.

Another investigation of Hurewicz marked his entrance into algebraic topology. The undersigned had introduced so called LC^n spaces: compact metric spaces locally connected in terms of images of p-spheres for every $p\leqq n$. One may introduce HLC^n spaces with images of p-spheres replaced by integral p-cycles and contractibility to a point by ~ 0 in the sense of Vietoris. Hurewicz proved this very unexpected property: *N.a.s.c. for X as above to be LC^n is HLC^n plus local contractibility of closed paths* [33]. An analogous condition will appear in connection with homotopy groups.

We come now to the four celebrated 1935 Notes on the homotopy groups, of the Amsterdam Proceedings [29; 30; 34; 35]. The attack is by means of the function spaces X^Y. Let Y be a separable metric space which is connected and locally contractible in the sense of Borsuk. Let S^p denote the p-sphere. Let x_0 be a fixed point of S^{n-1}, $n\geqq 1$, and y_0 a fixed point of Y. Let N be the subset of $Y^{S^{n-1}}$ consisting of the mappings F such that $Fx_0=y_0$. The group of the paths of N

is the same for all components of M. It is by definition the nth homotopy group $\pi_n(Y)$ of Y. For $n=1$ it is the group of the paths of Y, and hence generally noncommutative but for $n>1$ the groups are always commutative. Hurewicz proved the following two noteworthy propositions:

I. *When the first* $n-1$ ($n\geqq 2$) *homotopy groups of the space* Y (*same as before*) *are zero then the nth* $\pi_n(Y)$ *is isomorphic with* $H_n(Y)$, *the nth integral homology group of* Y.

II. *N.a.s.c. for a finite connected polyhedron* Π *to be contractible to a point is* $\pi_1(\Pi)=1$ *and* $H_n(\Pi)=0$ *for every* $n>1$.

For many years only a few homotopy groups were computed successfully. In the last five years however great progress has been made and homotopy groups have at last become computable mainly through the efforts of J.-P. Serre, Eilenberg and MacLane, Henri Cartan, and John Moore.

Many other noteworthy results are found in the four Amsterdam Proceedings Notes but we cannot go into them here. We may mention however the fundamental concept of homotopy type introduced by Hurewicz in the last note: Two spaces X, Y are said to be of the same homotopy type whenever there exist mappings f: $X\rightarrow Y$ and g: $Y\rightarrow X$ such that gf and fg are deformations in X and Y. This concept gives rise to an equivalence and hence to equivalence classes. This is the best known approximation to homeomorphism, and comparison according to homotopy type is now standard in topology. Identity of homotopy type implies the isomorphism of the homology and homotopy groups.

At a later date (1941) and in a very short abstract of this Bulletin [40] Hurewicz introduced the concept of exact sequence whose mushroom like expansion in recent topology is well known. The idea rests upon a collection of groups G_n and homeomorphism ϕ_n such that

$$\cdots \rightarrow G_{n+2} \xrightarrow{\phi_{n+1}} G_{n+1} \xrightarrow{\phi_n} G_n \rightarrow \cdots$$

and so that the kernel of ϕ_n is $\phi_{n+1}G_{n+2}$. This was applied by Hurewicz to homology groups and he drew important consequences from the scheme.

Still another noteworthy concept dealt with by Hurewicz is that of fibre space. In a Note [38] written in collaboration with Steenrod there was introduced the concept of the covering homotopy, its existence was established in fibre spaces, the power of the method was made clear. He returned to it very recently [45] to build fibre spaces on a very different basis. In another recent Note [46] written in collaboration with Fadell there was established the first fundamental advance beyond the theorem of Leray (1948) about the structure of spectral sequences of fibre spaces.

Hurewicz made a number of excursions into analysis, principally real variables. A contribution of a different nature was his extension of G. D. Birkhoff's ergodic theorem to spaces without invariant measure [42].

During World War II Hurewicz gave evidence of surprising versatility in distinguished work which he did for the Radiation Laboratory. This led among other things to his writing a chapter in the Servo Mechanisms series issued by the Massachusetts Institute of Technology.

The scientific activity of Hurewicz extended far beyond his written papers important as these may be. One way that it manifested itself is through his direct contact with all younger men about him. He was ready at all times to listen carefully to one's tale and to make all manner of suggestions, and freely discussed his and anybody else's latest ideas. One of his major sources of influence was exerted through his books. *Dimension theory* [39] already mentioned is certainly the definitive work on the subject. One does not readily understand how so much first rate information could find place in so few pages. We must also mention his excellent lectures on differential equations [41] which has appeared in mimeographed form and has attracted highly favorable attention.

On the human side Witold Hurewicz was an equally exceptional personality. A man of the widest culture, a first rate and careful linguist, one could truly apply to him ***nihil homini a me alienum puto.*** Tales were also told of his forgetfulness—which made him all the more charming. Altogether we shall not soon see his equal.

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THE ANNUAL MEETING IN ROCHESTER

The sixty-third Annual Meeting of the American Mathematical Society was held at the University of Rochester on Thursday through Saturday, December 27–29, 1956, in conjunction with meetings of the Mathematical Association of America on Saturday, December 29 and the Association for Symbolic Logic on Thursday, December 27. The registration was 615 including 508 members of the Society.

The thirtieth Josiah Willard Gibbs Lecture, entitled *Mathematics and the future of science*, was delivered by Professor M. H. Stone of the University of Chicago on Thursday evening. Professor Richard Brauer, President Elect of the Society, presided.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professor D. C. Spencer of Princeton University delivered an address, entitled *On complex structures*, on Thursday afternoon at a session presided over by Professor A. C. Schaeffer.

On Friday morning the Frank Nelson Cole Prize in Number Theory was awarded to Professor J. T. Tate of Harvard University for his paper *The higher dimensional cohomology groups of class field theory* which appeared in volume 56 of the Annals of Mathematics. Professor Tate addressed the Society briefly on the subject of his prize-winning memoir.

Tea was served on Thursday afternoon in the Faculty Club, and there was a beer party in the Men's Dining Hall after the Gibbs Lecture on Thursday evening. A conducted tour of the George Eastman House Museum of Photographic Art was held on Friday afternoon.

A banquet was held in the dining room of the Women's Residence Hall on Friday evening. Professor J. F. Randolph of the University of Rochester acted as toastmaster. The speakers were Mr. Sol Linowitz, a trustee of the University of Rochester, Professor Richard Brauer, President Elect of the American Mathematical Society, Dean W. L. Duren, President of the Mathematical Association of America, Professor S. C. Kleene, President of the Association for Symbolic Logic, and Professor M. H. Stone. A resolution of thanks to the University of Rochester was presented by Professor R. L. Jeffery.

The Annual Business Meeting of the Society was held on Friday, December 28, 1956. The Secretary reported that at this time the ordinary membership of the Society is now 5271, including 572 nominees of institutional members and 31 life members. There are also

136 institutional members. The total attendance of members at all meetings in 1956 was 2376; the number of papers read was 884; there were 14 hour addresses; 1 Gibbs Lecture and 9 papers at an Applied Mathematics Symposium. The number of members attending at least one meeting was 1520.

Professor J. F. Randolph reported for the tellers that a total of over 1300 ballots had been cast in the election and the following were elected:

Vice President, Professor Salomon Bochner.

Treasurer, Dean A. E. Meder, Jr.

Secretary, Professor J. W. Green.

Associate Secretary, Professor R. D. Schafer.

Member of the Editorial Committee of the Bulletin, Professor B. J. Pettis.

Member of the Editorial Committee of the Proceedings, Professor Irving Kaplansky.

Member of the Editorial Committee of the Transactions and Memoirs, Professor S. S. Chern.

Member of the Editorial Committee of the Colloquium Publications, Professor E. J. McShane.

Member of the Editorial Committee of Mathematical Reviews, Professor R. P. Boas, Jr.

Member of the Editorial Committee of Mathematical Surveys, Dr. S. M. Ulam.

Member of the Committee on Printing and Publishing, Professor E. G. Begle.

Representative on the Board of Editors of the American Journal of Mathematics, Professor Harish-Chandra.

Members of the Board of Trustees, Professor Deane Montgomery, Dean Mina Rees, and Professor G. A. Hedlund.

Members-at-large of the Council, Dr. R. E. Bellman, and Professors Lipman Bers, A. M. Gleason, P. R. Halmos and D. H. Lehmer.

The Council met on Thursday afternoon, December 27, 1956.

The Secretary announced the election of the following fifty-two persons to ordinary membership in the Society:

Miss Esther Adler, Columbia University;
Professor O. R. E. Barker, Hampton Institute;
Mr. George Biriuk, Naval Avionics Facility;
Mr. R. E. Bolger, Fairfield University;
Mr. R. E. Briney, Massachusetts Institute of Technology;
Dr. Johanna H. M. Brunings, Ramo-Wooldridge Corporation;
Sister M. L. A. Colbert, Marylhurst College;

Professor R. B. Crouch, New Mexico College of Agriculture and Mechanic Arts;
Dr. D. A. Edwards, Yale University;
Mr. Herbert C. Field, Purdue University;
Dr. Seymour Haber, Weizmann Institute of Science;
Miss Susan G. Hahn, Institute of Mathematical Sciences, New York University;
Mr. Milton Halem, New York University;
Dr. Haim Hanani, Israel Institute of Technology;
Mr. D. M. Hess, Fordham University;
Professor C. V. Holmes, San Diego State College;
Reverend F. A. Homann, S. J., University of Pennsylvania;
Mr. R. H. Homer, University of California, Berkeley;
Mr. T. J. Horrigan, Cook Research Laboratories, Skokie, Illinois;
Mr. J. E. Houle, Georgetown University;
Mr. L. A. Johnson, Taylor Pipe and Forge Works;
Mr. J. J. Kaganove, Armour Research Foundation;
Professor C. E. Kerr, Lafayette College;
Dr. Lester Kraus, Convair, San Diego, California;
Mr. P. V. LaGrange, Jr., The RAND Corporation;
Mr. Eric Liberman, Elite Furniture Company, Bulawayo, S. Rhodesia;
Mr. W. E. Loper, U. S. Naval Ordnance Test Station;
Reverend Anthony Mardellis, Long Beach State College;
Dr. R. L. Marti del Castillo, Banco Nacional de Cuba, Habana, Cuba;
Mr. Jack Minker, RCA Victor, Camden, New Jersey;
Professor T. F. Mulcrone, S. J., Loyola University;
Mr. P. D. Oyer, National Security Agency;
Mr. P. C. Patton, Harvard University;
Miss Nelly E. Reitlinger, University of Michigan;
Dr. Nicholas Rescher, RAND Corporation, Santa Monica, California;
Professor B. E. Rhoades, Lafayette College;
Dr. W. J. Riordan, Bell Telephone Laboratories, Whippany, New Jersey;
Dr. R. C. Roberts, Naval Ordnance Laboratory, Silver Spring, Maryland;
Miss Grace A. Sacks, International Business Machines, New York, New York;
Mr. P. J. Sally, Jr., Parke Mathematical Laboratories, Carlisle, Massachusetts;
Professor Diran Sarafyan, Lamar State College of Technology;
Dr. Berthold Schweizer, Illinois Institute of Technology;
Dr. R. G. Segers, Bell Telephone Laboratories;
Professor D. R. Sudborough, Central Michigan College;
Mr. G. M. Taylor, North American Aviation, Los Angeles, California;
Mr. F. A. Varrichio, St. Peter's College;
Dr. H. M. Wachowski, The Ramo-Wooldridge Corporation;
Professor E. E. Walden, New Mexico College of Agriculture and Mechanic Arts;
Dr. A. D. Wasel, Lockheed Aircraft Corporation;
T/Sgt. R. E. Wheeler, U. S. Air Force, Seattle, Washington;
Dr. Morris Yachter, The M. W. Kellogg Company, New York, New York.

It was reported that the following one hundred and ninety persons had been elected to membership on nomination of institutional members as indicated:

Alabama Polytechnic Institute: Professor D. E. Johnson.
University of Alabama: Professor R. N. Festa.

University of British Columbia: Mr. Bomshik Chang, Mr. Jih-ou Chow, and Mr. I. H. Mufti.

Brooklyn College: Mr. C. F. Ricciardelli.

Brown University: Dr. A. G. Arpeitia, Professor Maurice Holt, and Professor L. M. Milne-Thomson.

California Institute of Technology: Mr. George Gerson, Mr. Morton Lowengrub, Mr. J. R. Rice, and Mr. P. M. Weichsel.

University of California, Berkeley: Mr. R. O. Abernathy, Mr. H. S. Bear, Mr. Lensey Chao, Mr. R. P. Holten, Mr. H. H. Johnson, Miss Eva M. Kallin, Mr. Milton Lees, Mr. A. W. McKinney, III, Mr. J. D. Monk, Mr. J. E. Ohm, Mr. M. R. Porter Mr. J. A. Simmons, Mr. S. F. Tuan, and Mr. C. E. Watts.

University of California, Los Angeles: Mr. A. M. Bruckner, Mr. Seymour Goldberg, Mr. K. M. Larsen, Mr. P. A. Nickel, and Mr. J. F. Seewerker.

Case Institute of Technology: Mr. W. O. Portmann.

The University of Chicago: Mr. J. D. Halpern, Mr. M. W. Hirsch, Mr. R. C. O'Neil, Mr. J. J. Rotman, Mr. J. G. Thompson, Mr. Jacob Towber, and Mr. J. A. Wolf.

Columbia University: Mr. Arthur Babakhanian, Mr. E. W. Chamberlain, Mr. N. P. Herzberg, Mr. J. D. Lubin, Mr. Robert Morixe, Mr. W. V. Petryshyn, Mr. S. D. Ross, and Mr. Jerrold Rubin.

Cornell University: Dr. Harry Kesten, Mr. D. B. Lissner, Dr. Wolfgang Rindler, Mr. J. D. Rutledge, and Professor Ambikeshwar Sharma.

Duke University: Dr. A. M. Chak, Mr. Auguste Forge, Mr. G. E. Goode, Mr. A. S. Joyner, Mr. W. L. Messmer, and Mr. E. H. Moore, Jr.

University of Florida: Mr. J. B. Wilson.

Harvard University: Mr. Michael Artin, Mr. J. A. Cohn, Mr. P. L. Falb, Mr. Paul Fong, Mr. M. H. Greenberger, Mr. M. E. Harris, Mr. E. S. Loebenstein, Mr. D. M. Mandelbaum, Mr. Albert Marden, Mr. A. P. Ogg, and Mr. V. C. Williams.

University of Illinois: Mr. D. W. Dean, Mr. C. C. Farrington, Jr., Mr. J. L. Goldberg, Mr. M. I. Knopp, Mr. E. A. Newburg, Miss Rosemarie S. Stemmler, Mr. L. T. Wos, and Mr. Peter Yff.

Indiana University: Mr. R. P. Kanwal.

Institute for Advanced Study: Dr. A. E. Dold, Dr. H. B. Griffiths, Dr. K. W. Gruenberg, Dr. W. P. A. Klingenberg, Dr. H. W. Leopoldt, Dr. F. M. Ragab, Dr. Frank Rhodes, and Dr. Sadayuki Yamamuro.

Iowa State College of Agriculture and Mechanic Arts: Mr. R. D. Low.

State University of Iowa: Mr. G. S. Rogers.

University of Kansas: Mr. C. R. Deeter, and Mr. S. A. Khabbaz.

Kenyon College: Mr. T. M. Jenkins.

Lehigh University: Miss Hanna I. Nassar, and Mr. R. C. Scott.

University of Maryland: Mr. H. C. Berry, Mr. D. W. Fox, Professor W. H. Greub, Dr. H. R. A. Holmann, Mr. P. H. Maserick, Mr. R. E. McGill, and Mr. G. H. Weiss.

Massachusetts Institute of Technology: Mr. R. R. O'Brien, Dr. K. F. Roth, Mr. J. R. Schue, and Dr. J. T. Stuart.

University of Michigan: Mr. J. P. Benkard, Mr. G. T. Cargo, Mr. C.-Y. Chao, Mr. P. T. Church, Mr. Oswald Petrucco, Mr. Arnold Steiken, Mr. E. S. Simons, and Mr. W. G. Weideman.

University of Minnesota: Mr. S. D. Burgstahler, Mr. R. D. Driver, Mr. J. A. Lindberg, Jr., Mr. D. E. Varberg, and Mr. J. J. Yeh.

University of Nebraska: Mr. L. D. Fountain.

University of New Hampshire: Mr. F. J. Lorenzen, Jr.

Northwestern University: Miss Ethel J. Allard, Mr. D. M. Danvers, and Mr. K. E. Hillstrom.

Oklahoma Agricultural and Mechanical College: Mr. T. K. Boehme, Mr. T. W. Cairns, and Miss Leone Yarborough.

University of Oregon: Mr. J. R. Borsting.

Pennsylvania State University: Mr. R. T. Heimer.

University of Pennsylvania: Mrs. Bonnie Averbach, Mr. S. J. Einhorn, Mr. B. I. Gross, Mr. D. J. Ostroff, and Mr. M. W. Pownall.

Purdue University: Mr. D. L. Phillips, and Mr. J. H. Stapleton.

Queens College: Dr. C. R. Chester.

Rutgers University: Mr. W. E. Gould, Mr. W. R. Jones, Mr. D. R. King, Mr. R. A. McHaffey, and Mr. John Weissman.

College of St. Thomas: Professor R. J. Dowling.

University of Southern California: Mr. J. W. Capps, Professor H. O. Cordes, Mr. E. J. Eckert, Mr. Isidore Eisenberger, Mr. Humberto Gutierrez, and Mr. K. F. Wilson.

University of Tennessee: Mr. P. H. Doyle.

University of Texas: Mr. J. L. Cornette, Mr. J. W. Neuberger, Mr. J. H. Nicholson, and Mr. Dave Pandres, Jr.

University of Toronto: Mr. D. F. Clapp, Miss Diane M. Johnson, Mr. J. H. Lindsay, Jr., Mr. E. S. Lowry, Mr. C. A. Pegis, and Mr. F. A. Sherk.

University of Virginia: Mr. S. H. Coleman, Mr. W. D. Googe, Mr. E. H. Greene, Mr. S. H. Lawrence, and Mr. C. N. Lee.

State College of Washington: Professor Hans Schneider.

University of Washington: Mr. C. W. Clark, Mr. E. T. Kobayashi, Mr. R. D. Mayer, Mr. R. B. Paine, and Mr. K. R. Stromberg.

Wayne State University: Sister Mary Catharina Bereiter, and Mr. Louis Sucheston.

University of Wisconsin: Mr. H. F. Bechtell, Jr., Mr. W. G. Collar, Mr. R. C. Courter, Mr. G. W. Hedstrom, Mr. J. M. Kister, Mr. H. J. Rebassoo, Mr. D. A. Robinson, and Mr. R. H. Rosen.

Yale University: Mr. R. L. Adler, Mr. R. T. Barnes, Miss Eleanor Killam, Mr. P. E. Miles, Mr. J. M. Osborn, Jr., Mr. M. J. Poliferno, Mr. S. L. Salas, Miss T. R. Spiselman, and Mr. Donald Wehn.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker-Verinigung: Dr. E.-A. Behrens, Mathematicsches Seminar, Dr. G. Mayer-Kalkschmidt, University of Malaya, and Professor C. E. Mueller, Technical University; Société Mathématique de France: Professor Manuel Balanzat, Instituto de Fisica, and Dr. Paul Dedecker, Université de Liège; Svenska Mathematikersamfundet: Dr. E. H. Bareiss, David W. Taylor Model Basin, and Dr. F. A. Ryde, State Secondary School of Eksjoe; Wiskindig Genootschap te Amsterdam: Mr. P. H. Krijgsman, and Professor J. de Groot, Netherlands Postal and Telecommunications Services, The Hague, Netherlands.

The following appointments by the President were reported: as a committee to look into the relations between the Transactions and Proceedings: J. L. Doob, Chairman, R. P. Boas, Jr., Richard Brauer, and G. T. Whyburn; as a committee to consider the establishment of annual examinations at the master's level: Leonard Gillman, Chairman, R. P. Dilworth, and D. E. Richmond; as a program committee for a Symposium on Orbit Theory to be held in April 1957: J. B. Rosser, Chairman, Garrett Birkhoff, W. Eckert, Philip Hartman, and H. Newell; as members of Committees to Select Hour Speakers (terms to expire December 31, 1958): Summer and Annual Meetings: Einar Hille (Committee now consists of J. W. Green, Chairman, G. P. Hochschild, and Einar Hille); Eastern Sectional Meetings: Warren Ambrose (Committee now consists of R. D. Schafer, Chairman, Warren Ambrose, and E. R. Kolchin); Western Sectional Meetings: A. C. Schaeffer (Committee now consists of J. W. T. Youngs, Chairman, E. H. Spanier, and A. C. Schaeffer); Far Western Sectional Meetings: Charles Loewner (Committee now consists of V. L. Klee, Chairman, Arthur Erdelyi, and Charles Loewner); Southeastern Sectional Meetings: O. G. Harrold (Committee now consists of J. H. Roberts, Chairman, E. E. Floyd, and O. G. Harrold); as a member of the Committee on Visiting Lectureships for a term of three years beginning January 1, 1957: M. R. Hestenes (Committee now consists of J. M. Thomas, Chairman, Samuel Eilenberg, and M. R. Hestenes); as the Society's representative on the U. S. National Committee for Theoretical and Applied Mechanics for a term of four years beginning January 1, 1957: William Prager; as tellers for the 1956 election: David Gale, W. G. Lister, and J. F. Randolph.

The following appointments to represent the Society were reported: at the inauguration of Reverend Louis Melbourne Hirshson as President of Hobart and William Smith Colleges on October 12, 1956: Professor T. R. Hollcroft; at the inauguration of G. C. Simpson as Chancellor of Mary Washington College on October 19, 1956: Dr. Hobart C. Carter; at the exercises dedicating the Classroom-Laboratory Building and Marking Twenty Years of Progress at Northeastern University on October 24, 1956: Professor J. L. Walsh; at the convocation in honor of Dr. Frank L. Griffin at Reed College on November 4, 1956: Professor T. S. Peterson; and at the inauguration of Warmoth Thomas Gibbs as fourth President of the Agricultural and Technical College of North Carolina on November 9, 1956: Professor W. M. Whyburn.

The Secretary reported that Professor L. M. Milne-Thomson had accepted an invitation to deliver an address at the November 30–

December 1, 1956 meeting held at the University of Kentucky; that Professor D. C. Spencer had accepted an invitation to deliver an address at the 1956 Annual Meeting of the Society in Rochester, New York; that J. T. Tate had accepted an invitation to deliver an address at the meeting to be held in February 1957 in New Haven, Connecticut; that Professors Bernard Friedman and R. V. Kadison had accepted invitations to deliver addresses at the April 1957 meeting to be held in New York City.

The Executive Director reported that he had conducted a major membership campaign during the fall and that letters of invitation to membership had been sent out to about 15,000 people.

The Executive Director reported on the reasons for the late arrival of the October NOTICES and on the steps that had been taken to make sure that there would be no repetition of this delay.

After reviewing the make-up, printing, and distribution schedule for the NOTICES, the Council requested that the present deadline for abstracts be kept, but that preliminary announcements, as complete as possible and including reservation cards, should appear in the issue of the NOTICES just before the issue containing the program of the meeting.

The Council voted to set a meeting at the State College of Washington, Pullman, Washington, on June 15, 1957.

The Council voted that the Annual Meeting for 1958 should be held some time during the last two weeks of January 1959.

The Council voted to set Council meetings April 19, 1957, in Chicago, and at the Summer and Annual Meetings.

The following resolution was adopted by the Council:

Dr. J. V. Wehausen served as Executive Editor of Mathematical Reviews from the summer of 1950 until the summer of 1956. This was a period of rapid growth of the mathematical literature: in these six years the amount of material reviewed was considerably more than in the preceding ten and one-half years. Dr. Wehausen coped with this flood with skill, devotion, and unfailing tact and good humor. He was resourceful in finding reviewers and in extracting reviews from them, and in extracting journals from the often (surprisingly enough) uncooperative people who issue journals. A considerable part of his success was due to the devotion which he inspired in his office staff. As the amount of material increased, he worked longer and longer hours, until at the end he was doing much more than anyone could properly have demanded. It is a tribute to his energy and resourcefulness that Mathematical Reviews today enjoys a reputation as the most complete and most informative of the mathematical abstracting services. It is fitting that Dr. Wehausen's services to Mathematical Reviews should be formally acknowledged. The Council, therefore, on behalf of mathematicians everywhere, extends its hearty thanks to Dr. Wehausen for his work on Mathematical Reviews.

The Council voted to elect Professor Hans Samelson to the Editorial Committee for the Proceedings to replace Professor S. S. Chern.

The Bulletin Editorial Committee reported that 652 pages had been used in 1956. The Council voted to recommend to the Board of Trustees that the Bulletin be authorized to print 625 pages in 1957.

At the request of the Editors of the Bulletin, the Council discussed the advisability of discontinuing the Research Problems section. It was the sense of the Council that this section should be continued.

The Transactions and Memoirs Editorial Committee reported that the interval between the receipt and publication of a manuscript is now approximately one year. The Council voted to request the Trustees to authorize three volumes of 550 pages each of the Transactions in 1957.

The Editorial Committee for the Proceedings reported that 1168 pages had been published in 1956 and that the Trustees had authorized the same number of pages for 1957, instead of the usual 1006 pages. The extra pages authorized by the Trustees have reduced the backlog so that the interval between receipt and publication of a manuscript is now approximately one year.

A committee consisting of J. L. Doob (Chairman), R. P. Boas, Richard Brauer, and G. T. Whyburn recommended that the Council request the Editorial Boards of the Transactions and Proceedings to interchange manuscripts as necessary in order to stabilize the Transactions publication rate at three volumes per year, with the present rough distinction between periodicals. that the Transactions prints the longer papers, serving as the guiding principle in the interchanges. The Council voted to approve this recommendation.

The Editorial Committee for Mathematical Reviews reported that the 1956 volume would be almost exactly the same size as the 1955 volume. The Committee reported finding that some members of the Society wondered whether the Council felt that Mathematical Reviews should still be considered an experiment, which could be terminated if it continues to cost too much. At the request of the Committee, the Council voted to go on record as favoring the indefinite continuation of Mathematical Reviews.

The Secretary reported that Dr. S. H. Gould had agreed to continue as Executive Editor of Mathematical Reviews.

The Joint Committee on Employment Opportunities reported that the number of listings in the Employment Register continues to rise.

The Committee on Translations reported that during 1956 about 900 papers had been selected for translation and were being prepared for publication. At the request of the Committee the Council voted to recommend that an increased appropriation be sought for 1957.

There were twenty sessions for contributed papers presided over by Professor R. D. Anderson, Dr. E. H. Batho, Professors Dorothy L. Bernstein, R. H. Bruck, D. J. Dickinson, S. H. Gould, Marshall Hall, Jr., G. K. Kalisch, W. S. Loud, Mr. J. C. Mairhuber, Drs. E. W. Marchand, Steven Orey, Professors L. J. Paige, R. A. Raimi, Mary E. Rudin, Dr. Lowell Schoenfeld, Professors C. H. W. Sedgewick, M. L. Tomber, W. G. Warnock, G. W. Whitehead.

The abstracts of the papers follow. Those having the letter "t" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Dr. Friedman was introduced by Professor Nachman Aronszajn, Professor Kallianpur by Professor Ingram Olkin, Dr. Klingenberg by Professor M. H. Heins, Professor Meyer-Koenig by Professor G. M. Merrmian, Mr. Osborn by Professor A. A. Albert, Mr. Pakshirajan by Mr. A. T. Bharucha-Reid, Mr. Pucci by Professor Alexander Weinstein, Dr. Rector by Professor D. M. Young, Jr., Professor Reid by Professor J. S. Frame, and Professor Sarafyan by Professor Selby Robinson.

ALGEBRA AND THEORY OF NUMBERS

147. Shreeram Abhyankar: *Unaffected equations.*

Let V be a normal r-dimensional algebraic variety with quotient field K/k where k is algebraically closed and of characteristic p and let P be a simple point on V. It is proved in [Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. vol. 77 (1955) pp. 575–592] that if Q is a point corresponding to P on a normalization of V in a finite algebraic extension L of K and if the branch locus D on V for the extension L/K has an s-fold normal crossing at P then the local galois group $G(Q/P)$ is a p_s-group (definition in the above paper). Now the following *construction problem* is raised: Given an $(r-1)$-dimensional subvariety D of V having an s-fold normal crossing at P and given a p_s-group G does there exist Q (in some extension L of K) such that D is the branch locus at P and $G(Q/P)$ is isomorphic to G? (For $r=1$, $G(Q/P)$ is a p-group and the converse is easily settled in the affirmative, so we assume $r\neq 1$). The essential part of this question is the case $s=1$ and G being a quasi p-group i.e. G is generated by its p-sylow subgroups. Observe that for $p=2$ the symmetric group S_n on n letters is a quasi p-group; for this case the construction problem is settled. (Received November 13, 1956.)

148. A. A. Albert (p) and Nathan Jacobson. *On the isomorphisms of the exceptional Jordan algebra.*

Let $A=A(\Gamma, C)$ be the algebra of all three-rowed J-Hermitian matrices with elements in the Cayley algebra C of dimension eight where Γ is the diagonal matrix defining the involution J. Then we prove that two such algebras A and $A_1=A(\Gamma_1, C_1)$ are isomorphic only if C and C_1 are isomorphic. (Received November 6, 1956.)

149t. A. A. Albert and L. J. Paige. *Some properties of the algebras of Malcev.*

Malcev has defined a Moufang-Lie algebra M to be an algebra satisfying the identities (i) $x^2=0$; (ii) $xy\cdot xz=(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y$. It is proved that the second identity may be replaced by the identity $w(xy\cdot z)=wy\cdot zx+(yz\cdot w)x+(wx\cdot y)z+(x\cdot wz)y$. If the linear transformation S_x is defined by the equation $ax=aS_x$, it is shown that trace $(S_xS_y)=F(x, y)$ is an invariant form for the algebra M. Moreover, the algebra M is a shrinkable algebra in the sense of Albert. (Received November 13, 1956.)

150. Rafael Artzy: *Loops with identities.*

In a loop $(L, +)$ let J be the mapping of each element on its right inverse so that $x+xJ=0$ and $(xJ^k)J=xJ^{k+1}$, all $x\in L$, all integers k. Loops with the identity $(x+y)J=xJ+yJ$ are called *automorphic-inverse*. If, for some $z\in L$, $zJ^n=z$, but $zJ^m\neq z$ whenever $0<m<n$, then the zJ^i, $i=1, 2, \cdots, n$, are said to form an *inverse-cycle* of length n. *A necessary condition for an automorphic-inverse loop to consist only of the zero and r inverse-cycles of equal length n is* $2^{-1}r^2n(n+1)-r=0 \pmod{n}$. This is a stronger requirement than that stated previously by the author (Proc. Amer. Math. Soc. vol. 6 (1955) p. 449). Furthermore a general method is proposed for constructing uncountably many monogenic automorphic-inverse loops of infinite order. Each of these loops satisfies one of certain additional identities, examples of which are $x+(xJ+(xJ^2+\cdots+(xJ^n+yJ^{n+1})\cdots))=y$, all nonzero x, y; $n>0$; or $(z+w)+zJ=w$ (the *crossed-inverse* identity). A loop thus constructed is always the additive loop of a neofield, a generalization of Bruck's statement for the crossed-inverse case (Proc. Amer. Math. Soc. vol. 6 (1955) p. 57). (Received November 13, 1956.)

151. S. G. Bourne: *On a Wedderburn-Artin structure theory of potent semirings.* Preliminary report.

In a paper [S. Bourne, *On multiplicative idempotents of a potent semiring*, Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) pp. 632–638] it was shown that if S is a potent semiring, in which each two-sided ideal contains a minimal right ideal and a minimal left ideal of S, then any right ideal $R\neq(0)$ contains a multiplicative idempotent. Using the existence of this idempotent it is proved that if S is a simple semiring, which contains a minimal right ideal and a minimal left ideal, then $\bar{\rho}=($[illegible]$)$ where $\rho\in S$, $r\in R$ a minimal right ideal of S and $\rho R\neq(0)$ has an inverse. Hence, if in addition S possesses an identity, S is isomorphic to a semiring of matrices over a division semiring. Thus, if S is a potent semiring with identity and $S=\sum_{R_i\in\Delta}R_i$, where Δ is the set of all minimal right ideals $\neq(0)$ in S and the decomposition is direct, then S is a direct sum of semirings isomorphic to semirings of matrices over division semirings. (Received November 5, 1956.)

152. A. T. Brauer: *A new proof of theorems of Perron and Frobenius on positive matrices.*

A well known theorem of Perron and Frobenius states that a positive matrix has a positive characteristic root ω which is greater than the absolute value of the other roots. In this paper a new proof for this result is obtained. Moreover, not only the existence of ω is proved, but the proof gives also a simple method to compute ω and a characteristic vector belonging to ω as exactly as needed without determining the characteristic equation. The theorem of Perron and Frobenius can be formulated as follows. Every positive matrix A has a greatest positive root ω. This root is simple and all the other roots lie in the circle $|z| < \omega$. In this form the theorem can be improved. It will be shown that all the roots lie in the interior or on the boundary of the oval of Cassini $|z-a_{ii}|\,|z-a_{jj}| \leq (\omega-a_{ii})(\omega-a_{jj})$ where a_{ii} and a_{jj} are the two smallest elements of the main diagonal. (Received November 13, 1956.)

153. Bailey Brown and N. M. McCoy (p): *Prime ideals in nonassociative rings.*

If A and B are ideals or, more generally, any sets of elements of the nonassociative ring R, by AB is meant the set of all elements of R of the form ab, $a \in A$, $b \in B$. Let $\mathfrak{A}$ denote the set of all finite formal nonassociative products of the indeterminates $x_1, x_2, \cdots$, and let $u = u(x_1, x_2, \cdots, x_n) \in \mathfrak{A}$. An ideal P in R is u-prime if $u(A_1, A_2, \cdots, A_n) \subseteq P$ implies that some $A_i \subseteq P$, where the A_i are ideals in R. Let $u^* = u(x_1, x_1, \cdots, x_1)$. Then if u contains at least two distinct indeterminates, a u-prime ideal is prime and a u^*-prime ideal is semi-prime, but examples show that the converses need not be true. The theory of the u-radical parallels much of the theory of the prime radical in the associative case. In particular, the u-radical always coincides with the u^*-radical. This has been established by Amitsur (Amer. J. Math. vol. 76 (1954) pp. 126–136) for the case in which $u = x_1x_2$, and is well known in the associative case. Various properties of these radicals are obtained, including the relation to a certain type of nilpotence and to some other radicals. For example, a primitive ideal is u-prime for each $u \in \mathfrak{A}$, and hence the Jacobson radical is u^*-prime for each $u \in \mathfrak{A}$. (Received November 13, 1956.)

154. A. T. Butson: *Simply ordered ideal preserving groups.* Preliminary report.

In a recent paper (*Ideals in partially ordered sets*, Amer. Math. Monthly vol. 61 (1954) pp. 223–234), Frink proposed a definition of ideal in a poset, and suggested the possibility of generalizing ordered algebraic systems by requiring the algebraic operations to preserve ideals rather than the order relation itself. The generalization obtained of a simply ordered group is as follows. A simply ordered ideal preserving group (soip-group) G is a system which is (i) simply-ordered, (ii) a group, in which (iii) J an ideal implies $a+J+b$ is either an ideal or a dual ideal for all a, b in G. The only nontrivial finite soip-group is the group B containing two elements b, e where $b+e = e+b = b$, $b+b = e+e = e$, and $b < e$. Every infinite soip-group G is either (1) a simply ordered group, or (2) isomorphic algebraically to the direct sum $B \oplus F$ and orderwise to the ordinal sum $\tilde{F} \oplus F$, where F is some simply ordered group and $\tilde{F}$ its dual. (Received November 13, 1956.)

155*t*. Bomshik Chang and S. A. Jennings: *On certain free groups generated by two matrices.*

Let $F_{\alpha,\beta} = \{A, B\}$ where $A = \left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix}1 & 0\\ \beta & 1\end{smallmatrix}\right)$, α and β being complex numbers. We prove that if either (i) $|\alpha\beta| \geqq 4$ or (ii) $I(\alpha\beta) \geqq 2$ (where $I(z)$ is the imaginary part of z), then $F_{\alpha,\beta}$ is a free group generated by A and B. If $A_1 = \left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix}1 & \alpha i\\ 0 & 1\end{smallmatrix}\right)$, $B_1 = \left(\begin{smallmatrix}1 & 0\\ \alpha & 1\end{smallmatrix}\right)$, $B_2 = \left(\begin{smallmatrix}1 & 0\\ \alpha i & 1\end{smallmatrix}\right)$ with $|\alpha| \geqq 2$, then $F = \{A_1, A_2, B_1, B_2\}$ is the free product of the abelian groups $F_A = \{A_1, A_2\}$ and $F_B = \{B_1, B_2\}$. These results generalize those of Sanov

[Doklady Akad. Nauk SSSR (NS) vol. 57 (1948) 657–659] and Brenner [Bull. Amer. Math. Soc. vol. 62 (1956) 149]. (Received November 5, 1956.)

156. H. S. M. Coxeter: *Groups generated by unitary reflections of period two.*

Among the unitary groups generated by reflections [G. C. Shephard and J. A. Todd, Canadian J. Math. vol. 6 (1954) pp. 274–304], those whose generators are involutory are found to belong to a single family. Such a group is denoted by a symbol $[p\ q\ r^l]^m$, where $p+q+r$ is the number of generators while l and m are the periods of certain products of generators. For instance, $[1\ 1\ 2^4]^4$, in unitary 3-space, is an infinite group which transforms the origin into a lattice consisting of the points whose coordinates are mutually congruent (mod c), while their sum is congruent to zero (mod $\bar{c}$), in the domain of algebraic integers generated by the roots, c and $\bar{c}$, of the equation $x^2+x+2=0$. The corresponding lattice in real Euclidean 6-space represents a new extreme senary quadratic form, recently discovered by E. S. Barnes. (Received November 13, 1956.)

157. R. B. Crouch (p) and W. R. Scott: *Normal subgroups of monomial groups.*

Let U be a set, H a group, B an infinite cardinal, and B^+ the successor of B. Let $o(U)=B$, where $o(U)$ means the number of elements of U. The set of monomial substitutions on the elements of U with coefficients in H forms a group $\sum(H; B, B^+, B^+)$. The set $\sum(H; B, C, D)$ of substitutions of the form $y=vs$, where v is a multiplication with less than C nonidentity factors, s is a permutation that permutes less than D elements of U, is a subgroup. The normal subgroups of $\sum(H; B, \aleph_0, \aleph_0)$ are known (R. B. Crouch, Trans. Amer. Math. Soc. vol. 80 (1955) pp. 187–215). This paper extends the results by determining all the normal subgroups of $\sum(H; B, \aleph_0, C)$ for $\aleph_0 < C \leqq B$. (Received November 5, 1956.)

158. R. L. Davis: *Torsion structure in Engel modules of arbitrary exponent.*

Following the discovery of torsion in the Engel module of exponent 5 (Bull. Amer. Math. Soc. Abstract 62-6-605), this paper studies structure questions for Engel modules of arbitrary exponent. Let an Engel module E on 3 or more generators have exponent m (prime or composite). Then the submodule of E consisting of forms of degree $m-2$ in x, 1 in y and 1 in z has the module structure of $Z_p+Z+\cdots+Z$ ($m-3$ copies of Z) if $m=p^k$ is a prime power (here Z is the integers and Z_p the integers mod p); for any other m it is simply the direct sum of $m-3$ copies of Z. The methods are applied to study Engel modules of exponent 4 in all dimension submodules. (Received November 13, 1956.)

159t. John De Cicco: *Some theorem concerning commutative rings with unit which admit involutorial automorphisms.*

Characterizations of quadratic extensions Q of a commutative ring X with unit 1 in which $2=1+1$, is regular in X, are obtained within isomorphisms. Any such ring Q admits an involutorial automorphism. Applications are given to the case where X is a field F. The elements of such a ring Q may be represented as points of a z-plane. This plane is a two dimensional vector space relative to the ring X. Inner product and norm are studied. Under certain general conditions, isotropic coordinates can be introduced in this z-plane. Relations are developed between isotropics and residue classes relative to certain principal ideals. Polynomial forms as well as polynomial functions form rings which admit involutorial automorphisms. Theorems are developed concerning the roots of an algebraic equation over such a ring Q. (Received September 10, 1956.)

160. Philip Dwinger: *A note on direct sums and direct products in completely modular complete lattices.*

Let L be a completely modular complete lattice (Kurosh, Theory of Groups, Vol. II). A product $\prod_{\alpha\in A} x_\alpha$, $x_\alpha \in L$ is called a direct product if $x_\alpha + x_\alpha^* = 1$ for every $\alpha \in A$, where $x_\alpha^* = \prod_{\beta\in A, \beta\neq\alpha} x_\beta$. A direct product is called regular if for every $A_1 \subset A$, $\prod_{\alpha\in A_1} x_\alpha + \prod_{\alpha\in A-A_1} x_\alpha = 1$. In the present paper it is shown that if $\{x_\alpha\}$, $\alpha \in A$, is a set of elements of L the sum of which is a direct sum and which is equal to 1 and $x'_\alpha = \sum_{\beta\in A, \beta\neq\alpha} x_\beta$ for every $\alpha \in A$, then the product of the set $\{x'_\alpha\}$ is a regular direct product which is equal to 0 and $x'^*_\alpha = x_\alpha$ for every $\alpha \in A$. Conversely if $\{x_\alpha\}$, $\alpha \in A$, is a set of elements the product of which is a direct product (not necessarily regular) and which is equal to 0, then the sum of the set $\{x_\alpha^*\}$ is a direct sum and $x_\alpha^{*\prime} \leq x_\alpha$ for every $\alpha \in A$, while $x_\alpha^{*\prime} = x_\alpha$ for every $\alpha \in A$ if and only if $\sum_{\alpha\in A} x_\alpha^* = 1$. The first of these results was proved by the author in a previous paper (Indag. Math. vol. 18, p. 435) for a smaller class of lattices (see also P. Dwinger and J. de Groot, *On the axioms of Baer and Kurosh in modular lattices*, to appear in Ind. Math. vol. 18). The theorems are applied to the lattice of congruence relations of a universal algebra any two congruence relations of which are permutable and with a selected one-element subalgebra. (Received November 7, 1956.)

161. B. M. Dwork: *A model for multiplicative characters of local number fields.*

Let k be a $\mathfrak{p}$-adic number field, i.e. a finite extension of a rational p-adic field Q. The characters of the multiplicative group, $1+\mathfrak{p}$ are described in terms of additive characters and polynomials, $F_r(x) = \sum_{j=1}^{p^r-1} x^j/j$. Let θ be a (continuous) additive character of Q and let $\Theta = \theta \circ S_{k/Q}$. An infinite sequence $A = (a_i)_{i=0}^{\infty}$ of elements of k is said to be an n-sequence if for all $r \geq 0$, $\Theta(\sum_{i=0}^{r} a_i x^{p^{r-i}}/p^{r-i}) = 1$ for $p^r \operatorname{ord}_{\mathfrak{p}} x \geq n$. For $p^r \operatorname{ord}_{\mathfrak{p}} x \geq n$, let $X_A(1-x) = \Theta(\sum_{i=0}^{r-1} a_i F_{r-i}(x))$. The basic result is that $A \rightarrow X_A$ is a well defined homomorphism of the group (under componentwise addition) of n-sequences onto the character group of $(1+\mathfrak{p})/(1+\mathfrak{p}^n)$, $(n \geq 1)$, and the kernel is the set of all A such that $(0, a_0, a_0+a_1, a_0+a_1+a_2, \cdots)$ is also an n-sequence. The theory is applied to complete the explicit reciprocity law for cyclic, ramified extensions of k of degree p (cf. MacKenzie and Whaples, Amer. J. Math. vol. 78 (1956) pp. 473–486). This is done by determining an A such that X_A is a nontrivial character of the Galois group. The treatment hinges upon the self-duality of k as an additive group. (Received October 15, 1956.)

162. C. C. Faith: *Normal extensions in which every element with nonzero trace is a normal basis element.*

Let K/k be a normal extension, where $K = K_1 \times K_2$ over k, for subfields K_1, K_2, where $K_1 \neq k$. Let G, $\mathfrak{G}$, and $\mathfrak{H}$ denote the Galois groups of K/k, K/K_2, and K_2/k, resp. $w \in K$ (resp. K_2) is a ***normal basis element*** (n.b.e.) in K/k (resp. K_2/k) if the set $\{w^S\}$, $S \in G$ (resp. $\mathfrak{H}$), forms a basis of K/k (resp. K_2/k). The ***trace of w in K/K_2*** is $T_{K/K_2}(w) = \sum_{s\in\mathfrak{G}} w^s$. The equivalence of (1) and (2) is demonstrated: (1) $T_{K/K_2}(w)$

is a n.b.e. in K_2/k implies w is a n.b.e. in K/k; (2) k has prime characteristic p and K_1/k has degree p^e. Letting $K_2=k$ so that $K=K_1$, one sees that (2) is equivalent to: $(1')$ $T_{K/k}(w)\neq 0$ implies w is a n.b.e. in K/k. Perlis [Theorem 1, Duke Math. J. vol. 9 (1942) pp. 507–517] proved (2) implies $(1')$ assuming K/k is cyclic. Thus, this result is extended to the case that k has prime characteristic p, and K/k is an arbitrary normal extension of degree p^e. Moreover, our results show that these are the most general normal extensions satisfying $(1')$. Let $G(k)$ be the group algebra defined by G and k. The elements of G are right operators on $G(k)$ and K. An (operator) module isomorphism between $G(k)$ and K enables us to apply structure theorems for $G(k)$ to obtain our results. (Received November 13, 1956.)

163*t*. C. C. Faith: *On decompositions of certain subgroups of abelian groups*. Preliminary report.

(A) If S is any subgroup of a finite, p-primary, abelian, additive group G satisfying $p^\alpha G\geq S\geq p^{\alpha+1}G$, then there exists a decomposition of G into the direct sum of cyclic subgroups G_i such that S is the direct sum of the subgroups $S\cap G_i$. The proof is obtained by the proof of (A) when $\alpha=0$ by induction on the order of G, followed by an induction on α. If K/k is an abelian extension, application of (A) via the Galois Theory yields new results concerning factorizations over k of certain intermediate fields of K/k. In the remainder, M always denotes a module over a discrete valuation ring R with unique prime p. A direct calculation shows: (B) If S is any submodule of M obeying (1) $p^\alpha M\geq S\geq p^{\alpha+1}M$, then S is regular (in the sense of Vilenkin, i.e., $p^{n+k}S\cap p^kM=p^n(S\cap p^kM)$, for any n, k). A characterization of regularity in Exercise 78(b) in Kaplansky's "Infinite abelian groups," produces a kind of generalization of (A): (C) If M has bounded order, and if S is any submodule of M satisfying (1), then there exists a basis $\{x_i\}$ of M such that $\{r_ix_i\}$ is a basis for S, for suitable r_i in R (also see Ex. 79, loc. cit., in this connection). (Received November 13, 1956.)

164. Walter Feit: *Integral representations of finite groups*.

Let A be the group algebra of a finite group G of order g over the field K of gth roots of unity. Let S be a set of prime ideals in K, D the ring of elements in K which are local integers at each prime in S, and let R be the D module whose basis consists of the elements of G. Corresponding to the decomposition of R into a direct sum of indecomposable left ideals (Bull. Amer. Math. Soc. Abstract 63-1-4) there is a decomposition of A into a direct sum of left ideals, each summand of A corresponds to representation U of G with trace Φ. It is shown that each Φ is a character of G which vanishes for all S-singular elements, i.e. elements which are p-singular for some p which has a divisor in S. The number of Φ's equals the number of S-regular classes of G. If S consists of only one prime ideal p, then the Φ's are the principal indecomposable characters of G mod p. The dual basis of the Φ's corresponds to the irreducible characters mod p. Various relationships involving generalized Cartan invariants and decomposition numbers are obtained. (Received October 24, 1956.)

165. J. K. Goldhaber: *The nonexistence of certain finite projective planes possessing a polarity*.

Using results of R. Baer (*Polarities in finite projective planes*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 77–93) concerning the absolute points of a polarity, a canonical form for the incidence matrix of a projective plane possessing a polarity and having an odd number of points on a line is derived. On the basis of this canonical form and with the aid of the method of R. H. Bruck and H. J. Ryser (*The nonexistence of certain finite projective planes*, Canadian J. Math. vol. 1 (1949) pp. 88–93) the following theorem is proved: If $n \equiv 0 \pmod 4$ and if a prime of the form $4k+3$ divides the squarefree part of n then a projective plane which possesses a polarity and which has $n+1$ points on a line does not exist. (Received October 24, 1956.)

166. Juris Hartmanis: *A note on the lattice of geometries.*

It has been shown (Proc. Amer. Math. Soc. vol. 7 (1956) pp. 571–577) that every finite lattice can be embedded in the lattice of all geometries on a finite set. Since the corresponding problem for partition lattices is still an unsolved problem, the common properties of these lattices are investigated. (cf. ORE, Duke Math. J. vol. 9 (1942) pp. 573–627). It is shown that they are complemented, have only trivial homomorphisms and that the group of automorphisms is a symmetric group. Finally, it is noted that the quotient lattices of the partition lattices are isomorphic to direct products of partition lattices. The corresponding result does not hold for the lattice of geometries. (Received November 9, 1956.)

167. Edwin Hewitt (p) and H. S. Zuckerman: *The irreducible representations of a semigroup related to the symmetric group.*

Let $\mathfrak{T}_n$ be the semigroup of all single-valued mappings of a set consisting of n elements into or onto itself: for $f, g \in \mathfrak{T}_n$, $fg(x) = f(g(x))$. An explicit method is given for computing all irreducible representations of $\mathfrak{T}_n$ by semigroups of complex matrices. There are several steps. Let $\mathfrak{B}_p = \{f : f \in \mathfrak{T}_n$, range of f contains just p elements$\}$ $(p = 1, 2, \cdots, n)$. For every irreducible representation M of $\mathfrak{T}_n$, there is a p such that $M(f) = 0$ for $f \in \bigcup_{j=1}^{p-1} \mathfrak{B}_j$ and $\{M(f)\}_{f \in \mathfrak{B}_p}$ is irreducible. Every M is completely determined by its values on $\mathfrak{B}_p$. Every irreducible representation of $\mathfrak{B}_p \cup \{z\}$ (this is $\mathfrak{B}_p$ with a zero added to make it a semigroup) can be extended to a representation of $\mathfrak{T}_n$. The irreducible representations of $\mathfrak{B}_p \cup \{z\}$ are computed explicitly in terms of the irreducible representations of the symmetric group $\mathfrak{S}_p$. (Received October 22, 1956.)

168*t*. J. H. Hodges: *Weighted partitions for hermitian matrices over a finite field.*

Let $q = p^n$, q odd, and suppose that $\theta \in GF(q^2)$, $\theta^2 = \nu \in GF(q)$, but $\theta \notin GF(q)$. Then $\alpha = a + b\theta \in GF(q^2)$ if $a, b \in GF(q)$. $\bar{\alpha} = a - b\theta$ is called the conjugate of α. If $A = (\alpha_{ij})$ is a square matrix, $\alpha_{ij} \in GF(q^2)$, let $A^* = \bar{A}' = (\bar{\alpha}_{ij})'$ where the prime denotes transpose. Then A is said to be hermitian if and only if $A^* = A$. For $a \in GF(q)$, let $e(a) = e^{2\pi i t(a)/p}$ where $t(a) = a + a^p + \cdots + a^{p^{n-1}}$. If $A = (\alpha_{ij})$, $\alpha_{ij} \in GF(q^2)$ is hermitian, then $\sigma(a) = \sum_i \alpha_{ii}$, the trace of A, is a number of $GF(q)$. Consider the sum $S = S(B, X, A) = \sum e\{\sigma(X^*U + U^*X)\}$, where B, X, A and U are matrices over $GF(q^2)$, A is hermitian and nonsingular of order m, B is hermitian of order t, X and U are $m \times t$ and the sum is over all U satisfying $U^*AU = B$. If $X = 0$, S is the number of solutions U of $U^*AU = B$ which is given in *Representations by hermitian forms in a finite field* (L. Carlitz and J. H. Hodges, Duke Math. J. vol. 22 (1955) pp. 393–405). In the present paper it is shown that S can be expressed in terms of certain Kloosterman sums defined for hermitian matrices over $GF(q^2)$. A number of the properties of these Kloosterman sums are also given. The analogous sums for symmetric, skew, and general square matrices over $GF(q)$ have been considered previously by the author. (Received November 13, 1956.)

169. D. R. Hughes: *Generalized incidence matrices over group algebras.*

Let π be a (v, k, λ) configuration, $\mathfrak{G}$ a collineation group of π, and let $\mathfrak{A}$ be the group algebra of $\mathfrak{G}$ over an appropriate field F (e.g., F can be any field of characteristic zero). Then there exist diagonal matrices C_1, C_2 with entries from F, and (square) matrices D, B_1, B_2 with entries from $\mathfrak{A}$, such that $DC_1D^* = B_1$, $D^*C_2D = B_2$, where D^* is the conjugate transpose of D. In the cases of interest (e.g., $\mathfrak{G}$ has prime order), the B_i and C_i are completely known. These matrix equations imply the author's earlier results on generalized incidence matrices with rational entries, but also yield new results, such as: if π is a projective plane of order $n \equiv 2 \pmod 4$, $n \neq 2$, then π possesses no collineations of even order. (Received October 19, 1956.)

170. R. E. Johnson: *Rings with unique addition.*

A ring R is said to have unique addition if every multiplicative mapping of R onto a ring S is also additive. Rickart has shown (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 758–764) that a semi-simple ring R satisfying certain minimum conditions has unique addition. The present paper extends Rickart's results to rings with zero singular ideal (Proc. Amer. Math. Soc. vol. 2 (1951) p. 894). (Received November 5, 1956.)

171. C. W. Kohls: *Ideals in rings of continuous functions.* III.

Let $C(X)$ be the ring of all continuous real functions on a completely regular Hausdorff space X; βX, the Stone-Čech compactification of X; υX, the largest subspace of βX over which *all* f $C(X)$ can be extended; $N^p = \{f \in C(X): f = 0$ on some X-nbhd. of $p\}$; and $M'^p = \{f \in C(X)$: the zero-set of f meets every deleted nbhd. of $p\}$, if p is a nonisolated point of X, $= M^p$ otherwise. For any $p \in \beta X$, p is a βF-point with respect to X if whenever $f \in C(X)$ is extendable to p and vanishes there, then $f \geqq 0$ or $f \leqq 0$ on some X-nbhd. of p. (1) If P is any prime ideal of $C(X)$, then $C(X)/P$ is an ordered integral domain containing the reals. Let $p \in \beta X$ be the unique point such that $P \supseteq N^p$. Then $C(X)/P$ has infinitely large elements iff $p \notin \upsilon X$. (2) Let p be a βF-point. Then for each pair of subsets D, D' of $C(X)/N^p$ of order type ω, ω^* resp., with $D < D'$, there is an element $\delta \in C(X)/N^p$ such that $D < \delta < D'$. The ring $C(X)/N^p$ has no countable cofinal subset iff $p \notin \upsilon X$. (3) An example is given of a space X and a βF-point $p \in \beta X - \upsilon X$ such that the positive subset of $C(X)/N^p$ has countable coinitial subset. (This corrects an assertion of Bull. Amer. Math. Soc. Abstract 62-6-622.) (4) Let $p \in X$ be a βF-point. Then every proper open subinterval of $C(X)/M'^p$ is an η_1-set iff $M'^p \neq M^p$. (Received December 10, 1956.)

172*t*. C. W. Kohls: *Ideals in rings of continuous functions.* IV.

For any $p \in \beta X$, p is a V-point if $C(X)/N^p$ is a valuation ring (see previous abstract for terminology); equivalently, if every finitely generated ideal of $C(X)$ containing N^p is principal. If p is a V-point it is a βF-point. Several sufficient conditions that a βF-point be a V-point have been obtained; whether every βF-point is a V-point is undetermined. Let p be a V-point which is not a P-point. For each $a \in M^p/N^p$, $a > 0$, we call $Q^a = \{b \in C(X)/N^p: |b| \leqq a^r$ for some real $r > 0\}$ an upper ideal, and $Q_a = \{b \in C(X)/N^p: |b| < a^r$ for all real $r > 0\}$ a lower ideal. (1) Q^a and Q_a are prime ideals. (2) A prime ideal I of $C(X)/N^p$ has an immediate successor J under set inclusion iff for some $a \in M^p/N^p$, we have $I = Q_a$ and $J = Q^a$. (3) If $Q^a \subseteq Q_b$, there exists

$c \in C(X)/N^p$ such that $Q^a \subseteq Q_c \subseteq Q^c \subseteq Q_b$. (4) Let I be an ideal of $C(X)$ such that I/N^p is a lower ideal. Then neither I nor I/N^p is countably generated. (5) We call an ideal I a $\mathfrak{Z}$-ideal if whenever f and g have the same zero-sets and $g \in I$, then $f \in I$. If I is a $\mathfrak{Z}$-ideal containing N^p, then I/N^p is not a lower ideal, nor an upper ideal if $I \neq N^p$. (6) There are uncountably many prime ideals in $C(X)/N^p$. (7) Given $a \in M^p/N^p$, $a>0$, for each $b \in Q^a - Q_a$, we have $b = ha^s$ or $a^s = hb$ for some real $s>0$ and $h \in C(X)/N^p - Q^a$. (Received December 10, 1956.)

173*t*. Georg Kreisel: *Hypotheses on algebraically closed extensions.*

Let $P(a_1, \cdots, a_m, x_1, \cdots, x_n, t_1, \cdots, t_p)$ $[P(\mathfrak{a}, \mathfrak{x}, \mathfrak{t})]$ be a formula built up of the field operations and the variables indicated, with the property: in the (commutative) field considered, (a) if $(\mathfrak{x})[P(\mathfrak{a}, \mathfrak{x}, \phi^{(1)}) \vee \cdots \vee P(\mathfrak{a}, \mathfrak{x}, \phi^{(k)})]$ for algebraic functions $\phi_i^{(j)}$, then (b) $(\mathfrak{x})P(\mathfrak{a}, \mathfrak{x}, \tau)$ for suitable polynomials $\tau_i(a_1, \cdots, a_m, x_1, \cdots, x_n)$. Suppose H is a hypothesis formulated in the first-order predicate calculus from which $(\mathfrak{x})(Et)P(\mathfrak{a}, \mathfrak{x}, \mathfrak{t})$ follows by (a necessarily finite subset of) the axioms for algebraically closed fields (ACF); then (b), where the construction of τ is effective provided the step from (a) to (b) is effective.—By ACF, $H \leftrightarrow H_0$, H_0 quantifier-free (Tarski), where the axioms $(Es)(s^n + c_1 s^{n-1} + \cdots + c_n = 0)$ are replaced by $\rho_n^n + c_1 \rho_n^{n-1} + \cdots + c_n = 0$, $\rho_n(c_1, \cdots, c_n)$ being new function symbols (for algebraic functions). By Hilbert's first ϵ-theorem, if $(\mathfrak{x})(Et)P(\mathfrak{a}, \mathfrak{x}, \mathfrak{t})$ by (quantifier-free) ACF and H_0, then $P(\mathfrak{a}, \mathfrak{x}, \psi^{(1)}) \vee \cdots \vee P(\mathfrak{a}, \mathfrak{x}, \psi^{(k)})$, where the ψ are made up of the field operations and $\rho_1, \cdots, \rho_N$: apply the assumption on P.—*Example.* For P take $1 = t_1 f_1 + \cdots + t_p f_p$ where f_i is the general polynomial in n variables of degree d with coefficients among the $\mathfrak{a}$; for H take $(x_1) \cdots (x_n)(f_1 \neq 0 \vee \cdots \vee f_p \neq 0)$ (meaning that in some, hence every, AC extension of the field generated by $a_1, \cdots, a_m$, the f do not have a common zero). Then the conditions on P apply, yielding Hilbert's Nullstellensatz with a primitive recursive bound in n, p, and d for the degrees of τ_i, independent of the field of coefficients. (Received November 13, 1956.)

174. Georg Kreisel: *Hilbert's 17th problem.* I.

Let $f(a_1, \cdots, a_m, x_1, \cdots, x_n)$ be the general rational function (with numerator and denominator) of degree d in n variables, with coefficients a in a real commutative field P in which every positive element is the sum of $\leq r$ squares. There are rational functions $g_i(a_1, \cdots, a_m, x_1, \cdots, x_n, b_1, \cdots, b_q)$, $i \leq \lambda(n, d, r)$, of degree $\leq \lambda(n, d, r)$ in the variables shown, with the following property: if $f \not< 0$ in some real closed extension of the field generated by $a_1, \cdots, a_m$, then there are $\bar{b}_1, \cdots, \bar{b}_q$ in this field such that $f = \sum_1^\lambda g_i^2$, λ primitive recursive (and independent of P). This strengthens Robinson's result [Math. Ann. vol. 130 (1955) pp. 257–271] where P was assumed to be real closed and λ was shown to be general recursive.—$f \not< 0$, i.e. $(x_1) \cdots (x_n) \cdot (Et)(f = t^2) \leftrightarrow (Eb_1) \cdots (Eb_q) H_0(a_1, \cdots, a_m, b_1, \cdots, b_q)$, H_0 quantifier-free, for real-closed fields, i.e. from (i) field axioms, (ii) $(u_1) \cdots (u_k)(u_1^2 + u_2^2 + \cdots + u_k^2 + 1 \neq 0)$, k depending on n, d, and r, (iii) $(x)(Ey)(x = y^2 \vee -x = y^2)$ and (iv) $(v_1) \cdots (v_{2m+1}) \cdot (Ew_m)(w_m^{2m+1} + v_1 w_m^{2m} + \cdots + v_{2m+1} = 0)$. (iii) and (iv) are made quantifier-free by replacing y by $\zeta(x)$, w_m by $\zeta_m(v_1, \cdots, v_{2m+1})$. Since $(x_1) \cdots (x_n)(Et_1) \cdots (Et_k) \cdot (f = t_1^2 + \cdots + t_k^2)$ follows from (i), (iii) and the negation of (ii), it follows from (i), (iii), (iv) and H_0. By Hilbert's first ϵ-theorem, as in the preceding abstract, $(*) f = \sum_{j=1}^k \phi_{j1}^2 \vee \cdots \vee f = \sum_{j=1}^k \phi_{jN}^2$, with ϕ composed of the field operations, ζ and ζ_m. $(*)$ can be contracted to $f = \sum \Phi_i^2$, and ζ and ζ_m eliminated. (Received November 13, 1956.)

175t. Georg Kreisel: *Hilbert's 17th problem*. II.

A simpler construction of ϕ and λ (of the preceding abstract) is obtained by applying the methods of JSL 16 (1951) 241–267, to Artin's solution of Hilbert's problem [Abh. Math. Seminars Hamburg Univ. vol. 5 (1927) pp. 100–115]. A rough calculation shows that $\lambda(2, d, r)$ is of order $v(7, d)$ where $v(0, n) = n$, $v(m+1, n) = 2^{v(m,n)}$.—f and its Sturm chain $f^{(k)}$ are formally factorized into (at most) quadratic factors, $f^{(k)} = h^{(k)}(x_2, \cdots, x_n)\prod[(x_1-\alpha^{(k)})^2+\beta_i^{(k)}]$ in an extension of P by terms built up of ζ and ζ_p. Elementary arguments show that either f or -1 can be represented as (different) sums of squares in each set of extensions of the following form: $(\pm h^{(k)})^{1/2}$; $(\pm\beta_i^{(k)})^{1/2}$ and, for each adjunction of $(-\beta^{(k)})^{1/2}$, $(-\beta_j^{(k)})^{1/2}$, $\{\pm[(\alpha_i^{(k)}\pm(-\beta_i^{(k)})^{1/2}) -(\alpha_j^{(k)}\pm(-\beta_i^{(k)})^{1/2})]\}^{1/2}$ (all possible orderings of the "real" zeros of $f^{(k)}$). These adjunctions can now be eliminated, as in I. *Sample*: Suppose $s^{1/2}$ and $(-s)^{1/2}$ are both adjoined (separately, cf. ζ in I), $-1 = \sum(a_n+b_ns^{1/2})^2$, and $-1 = \sum(c_n+d_n-(s)^{1/2})^2$; (i) if (N), i.e. $\sum a_nb_n \neq 0$ or $\sum c_nd_n \neq 0$, either $-1 = \sum[a_n-2^{-1}b_n(1+\sum a_n^2+s\sum b_n^2) \cdot(\sum a_nb_n)^{-1}]^2$ or $-1 = \sum[c_n-2^{-1}d_n(1+\sum c_n^2-s\sum d_n^2)(\sum c_nd_n)^{-1}]$ where, without loss of generality, $a^{-1}=0 \leftrightarrow a=0$, (ii) if not N, and $\sum b_n^2+\sum d_n^2 \neq 0(Z)$, then $-1 = (\sum a_n^2\sum d_n^2+\sum c_n^2\sum b_n^2)(\sum b_n^2+\sum d_n^2)^{-1}$, (iii) if not N and not Z, then $\sum b_n^2 = 0$ and $-1 = \sum a_n^2$, or $\sum b_n^2 \neq 0$ and $-1 = (\sum d_n^2)(\sum b_n^2)^{-1}$. Since $-1 = a$ or $-1 = b$ implies $-1 = a+b+ab$, cases (i)–(iii) can be recombined. The elimination of $\zeta_p(a_1, \cdots, a_{2p+1})$ proceeds similarly by cases, according to all possible occurrences of simple and multiple factors of $x^{2p+1}+a_1x^{2p}+ \cdots +a_{2p+1}$. (Received November 13, 1956.)

176. J. B. Kruskal: *Group invariants and the free calculus*.

Suppose G is a group given by a presentation $(x_1, \cdots, x_n/r_1, \cdots, r_m)$. Let X be the free group on the x's, let R be the consequence of the r's, and let ϕ: $X \rightarrow X/R \approx G$ be the natural map, so the sequence $1 \rightarrow R \rightarrow X \rightarrow G \rightarrow 1$ is exact. Let JG be the integral group ring, let o: $JG \rightarrow J$ be the coefficient-sum homomorphism, and let $\mathfrak{G}$, called the fundamental ideal, be the kernel of o. As shown by R. H. Fox, partly in *The free differential calculus* I and II. Ann. of Math. 1953 and 1954, the "Jacobian matrix" $\phi\|D_jr_i\|$ (where D_j indicates differentiation with respect to x_j) is a relation matrix for $\mathfrak{G}$, considered as a left-module over JG, so the equivalence class of this matrix is an an invariant of G. As the relation matrices (or rather, the associated equivalence classes) of the powers $\mathfrak{G}^k$ are also invariants of G, it is of interest to be able to compute them. With the aid of the operator vectors $\nabla = (D_1, \cdots, D_n)$, $\nabla^2 = (D_1D_1, D_2D_1, \cdots, D_nD_n)$, etc., a specific, computable relation matrix for $\mathfrak{G}^k$ has been determined. It is not yet clear how useful these invariants will be. (Received November 16, 1956.)

177. H. T. LaBorde: *Bounds for the real parts of the characteristic roots of a matrix*.

In the Journal of Research of the National Bureau of Standards, vol. 48 (1952) P. Stein published a note giving bounds for the real parts of the characteristic roots of a matrix. Two theorems give these bounds in terms of an arbitrary set of positive constants. Though the method is trivial, it will, by judicious selection of values for the arbitrary constants in terms of the elements of the original matrix, give expressions for bounds which are easily obtained and which give better approximations than other existing theorems in many cases. These results can be strengthened so as to be applicable to a larger class of matrices and so as to give refinements of these bounds. This is done in the following theorem: Let $A = (a_{rs})$ be an arbitrary square matrix of order n, M the maximum real part and m the minimum real part of the characteristic roots of A, and let W_1, W_2, W_{n-1}, W_n be the maximum real part, the largest real part save one, the smallest real part save one, and the minimum real part of the main diagonal elements a_{rr} of A, respectively. Set $\sigma_r = \sum_{s=1, s\neq r}^{n} |a_{rs}+\bar{a}_{sr}|/2$, $\sigma_p = \max_r \sigma_r$, $\sigma_q = \max_{r\neq p} \sigma_r$. Then $M \leqq \max\{W_1+\sigma_q, W_2+\sigma_p\}$ and $m \geqq \min\{W_n-\sigma_q, W_{n-1}-\sigma_p\}$. This theorem, together with its applications to real matrices, is further improved by using a theorem of A. Brauer (Duke Math. J. vol. 14 (1947)). (Received December 17, 1956.)

178t. G. F. Leger, Jr.: *A note on the cohomology of groups.*

The first three cohomology groups $H^n(G, A)$ ($n=1, 2, 3$) of a group G with respect to a G-module A have well known interpretations. If G is a normal subgroup of a group Γ and A is also a Γ-module then G operates on $H^n(G, A)$. Operations of Γ on the interpretations of $H^n(G, A)$ are devised which agree with the operations of Γ on $H^n(G, A)$. The analogue of this for Lie algebras has been done earlier by the author (Bull. Amer. Math. Soc. Abstract 62-4-474). (Received November 13, 1956.)

179. Saunders MacLane: *Obstructions to the extensions of rings.*

The natural homology theory of a ring Λ is given by a certain complex $R(\Lambda)$, as described in Abstract 62-6-629. This homology yields a complete treatment for extensions of rings, parallel to the known homological treatment of the extensions of groups. Explicitly a *multiplication* (Hochschild) of a ring A is a pair of additive mappings $a \to \sigma a$, $a \to a\sigma$ satisfying the associative laws $\sigma(ab) = (\sigma c)b$, $(a\sigma)b = a(\sigma b)$, and $(ab)\sigma = a(b\sigma)$. All multiplications of A form a ring M_A, and this is a natural homomorphism $A \to M_A$ mapping A into "inner" multiplications, and with kernel $K_A \subset A$, and cokernel the ring P_A of multiplication classes. Any extension E of A by a ring Λ yields a homomorphism $\theta: \Lambda \to P_A$. Not every such θ can be realized by an extension; indeed, θ has an "obstruction" a certain three-dimensional cohomology class in $H^3(R(\Lambda), K_A)$, and can be realized by an extension exactly then when this obstruction is zero. When there is such an extension the set of possible (inequivalent) extensions is in one-one correspondence with $H^2(R(\Lambda), K_A)$. Finally, every three dimensional cohomology class which annihilates a certain three-dimensional cycle can be realized as an obstruction to a suitable θ. (Received November 1, 1956.)

180. M. D. Marcus (p), B. N. Moyls, and R. Westwick: *Some extreme value results for indefinite Hermitian matrices.*

Let A be an n-square complex indefinite Hermitian matrix. Let x_j $j=1, \cdots, k \leqq n$ be an orthonormal (o.n.) set and let $\phi = \prod_1^k (Ax_j, x_j)$. The extreme values of ϕ as the x_j run over all o.n. sets are obtained as functions of the eigenvalues. The result for the minimum specializes to the theorem of K. Fan [Amer. Math. Monthly, vol. 60 (1953) pp. 48–50]. Also obtained are the extreme values of $E_2[(Ax_1, x_1), \cdots, (Ax_k, x_k)]$ where E_2 is the second symmetric function and A is again indefinite Hermitian. (Received November 6, 1956.)

181. D. W. Miller: *On a theorem of Hölder.*

A classical result of finite group theory, due to Hölder [Math. Ann. vol. 46 (1895) pp. 343–345] is the following. S_n, the symmetric group on n letters, is a complete group if $n \neq 6$; the group of inner automorphisms of S_6 has index two in the group of all automorphisms of S_6. A new proof, suggested by R. H. Bruck, of the existence of an outer automorphism of S_6 is given here. A mapping θ of S_6 having the property that $(12)\theta = (12)(36)(45)$, $(23456)\theta = (23456)$ is defined and, without extensive computation, is shown to be an (outer) automorphism of S_6. (Received November 13, 1956.)

182. J. M. Osborn: *Commutative diassociative loops.* Preliminary Report.

A loop is diassociative if every two elements generate a subgroup. A commutative diassociative loop L is a p-loop of degree one if every element except the identity has order p, so that every simple commutative diassociative loop with elements of finite order is of this type. Let G be a configuration of points and lines such that every two points determine a unique line, and that every line has exactly $n+1$ points on it for some fixed n, then G is called a finite geometry of order n. There is a 1-1 correspondence between finite geometries of order p and classes of p-loops of degree one, which leads to many results on p-loops, including the construction of several classes of simple ones. This correspondence "coordinitizes" a projective plane of order p with a p-loop of p^2 elements, and a generalization of this process will coordinitize any projective plane. An application is the determination of all commutative Moufang loops on three generators. (Received November 7, 1956.)

183t. Gordon Pall and Olga Taussky: *Application of quaternions to the representation of a binary quadratic form as a sum of four squares.*

Let $r_s(n)$ denote the number of representations of the integer n as a sum of s squares of integers, and let $r_s(\phi)$ denote the number of representations of the binary quadratic form ϕ as a sum of s squares of linear forms with integral coefficients. Write $\phi = e\phi_1$, $\phi_1 = ax^2 + 2t_0xy + by^2$, where $(a, t_0, b) = 1$; set $k = (a, 2t_0, b) = 1$ or 2. Using simple arguments based almost entirely on the arithmetic of quaternions it is proved that $r_4(\phi) = f(e) \cdot 8(2+(-1)^k) \cdot r_3(d)$. Here $d = ab - t_0^2$ and $f(e)$ is a certain factorable function which is explicitly evaluated. (Received November 9, 1956.)

184. E. S. Rapaport: *On free groups and their automorphisms.*

A theorem (Whitehead, J. H. C., Ann. of Math. vol. 37 (1936) p. 789) providing an algorithm for determining whether two given sets of m words each in the free group $F_n = F(a_1, \cdots, a_n)$ are automorphic images of each other, is proved by algebraic means. Let A be the automorphism group of F_n, $T_1, \cdots, T_s = T_1^{-1}, \cdots$, a particular set of generators of A, $L(w)$ the sum of the absolute values of the exponents of generators occurring in $w = w(a_1, \cdots, a_n) \subset F_n$; furthermore, let $\alpha = T_2T_1$, $T_1(w_0) = w_1$, $T_2(w_1) = \alpha(w_0)$ subject only to the conditions (1) $L(T_1w_0) > L(w_0)$ and (2) $L(\alpha w_0) \leq L(T_1w_0)$. A finite set of words $s_i \subset F_n$ are found such that $w = \prod_{s_i \subset w} s_i$ uniquely for every w in F_n and, for $T = T_1$ and $T = \alpha$, $L(Tw_0) - L(w_0) = L(T\pi s_i) - L(\pi s_i) = \sum_{s_i \subset w_0} (L(Ts_i) - L(s_i))$. Let s_i' be the number of times s_i occurs in w_0. The conditions (1) and (2) and relations of the group A then yield inequalities between certain of the s_i'; these are used to prove the existence of an automorphism $B = T_{i_k} \cdots T_{i_1} = \alpha$, such that the lengths of the "intermediate words" $T_{i_1}(w_0)$, $T_{i_2}T_{i_1}(w_0)$, $\cdots$ are monotone decreasing up to i_h, $h \leq k$, are unchanged from i_{h+1} to $i_{h'}$, $h \leq h' \leq k$, and are monotone increasing from $i_{h'+1}$ to i_k. This result is extended to the case where w_0 is a set of m words and α arbitrary. Several incidental results are given; for example: if $L(Tw_0) > L(w_0)$, then $L(T^n w_0) > L(Tw_0)$ for any $T = T_i$. (Received November 9, 1956.)

185. J. B. Roberts: *On binomial coefficient residues.*

The number of binomial coefficients $\binom{u}{\nu}$, $0 \leq \nu \leq u < n$, which are congruent to j, $0 \leq j \leq p-1$, modulo the prime number p is denoted by $\theta_j(n)$. Systems of simultaneous linear difference equations are given whose solutions yield the $\theta_j(n)$. Computations of $\theta_j(n)$ for $p=2$ and of $\theta_j(p^k)$ for $p=3$ or 5, $k \geq 0$, are carried out. Also it is proved that every prime p divides "most" binomial coefficients in the sense that $\theta(n)/\theta_0(n) \to 0$ as $n \to \infty$ where $\theta(n) = \sum_{j=1}^{p-1} \theta_j(n)$. (Received November 23, 1956.)

186. M. P. Schützenberger: *Some properties of semigroups.* I.

Let A be a semigroup and X and Y two subsets of A. Define $XY^{(-1)} = A-(A-X)$: $Y = \{a \in A : aY \cap X \neq \phi\}$; $Y^{(-1)}X = A-(A-X)::Y$; $X/Y = \{(a,\ b) \in (A \times A): aYb \subset X\}$; $a = b(X)$, if and only if $X/a = X/b$ (cf. M. Teissier, Compt. Rend. vol. 232 (1951) p. 1987); ψ_X is the homomorphism attached to $=(X)$. With groups, $=(X)$ is the equivalence associated with the largest normal subgroup contained in X. In any case, if $\overline{X} = \psi_X X$, $\psi_{\overline{X}}$ is the identical application of $\psi_X A$ onto itself. Let P be a subsemigroup of A; let the following conditions be defined: $\mathcal{U}: P^{(-1)}P \cap PP^{(-1)} = P$; $\mathcal{U}_l$: $P^{(-1)}P = P$; $\mathcal{N}: A^{(-1)}PA^{(-1)} = A$; $\mathcal{N}_r: PA^{(-1)} = A$; $\mathcal{A}: P^{(-1)}PP^{(-1)} = A$; $\mathcal{A}_r: PP^{(-1)} = A$. $\mathcal{U}_l(\mathcal{N}_r)$ is the definition of a semisubgroup "unitaire" ("net"), cf. P. Dubreil in Mem. Acad. Sci. vol. 63 (1941) p. 8. Any of these conditions is satisfied simultaneously by P in A and $\psi_P P$ in $\psi_P A$. (Received November 5, 1956.)

187*t*. M. P. Schützenberger: *Some properties of semigroups.* II.

1. If A is free, the subsemigroups satisfying $\mathcal{U}$ form the complete lattice of the free subsemigroups of A. From this a new simple proof of the corresponding Nielsen Schreier-theorem for groups can be obtained by using an adequate ψ_Q. From now on we assume that $\psi_P A$ satisfies the descending chain condition for left and right ideals. 2. If $\mathcal{U}$ and $\mathcal{N}_r$, then $\mathcal{U}_l$; if $\mathcal{N}$ and $\mathcal{U}_l$, then $\mathcal{N}_r$; if $\mathcal{U}_l$ and $\mathcal{A}$ then $\mathcal{A}_r$. 3. If $\mathcal{U}$ and $\mathcal{N}$, then $\mathcal{A}$ if and only if the Suschkevitsch group of $\psi_P A$ (cf. A. H. Clifford, Amer. J. Math. vol. 70 (1948) p. 520) is reduced to its unit element. The proofs are based on the representation indicated in Compt. Rend. vol. 432 (1956) p. 2907. The results, although they are similar, do not coincide with those given in Trans. IRE, vol. IT-2, 47–80 (Sept. 1956), where an interpretation of $\mathcal{A}_r(\mathcal{A})$ as a condition of ergodic indecomposability for (generalized) recurrent events is discussed in detail. (Received November 5, 1956.)

188*t*. B. M. Stewart: *Sums of powers of digits.*

Let $A = \sum a_i B^i$ be a positive integer expressed to the base B. Define $G(A) = \sum a_i^t$ where t is a fixed positive integer. There exists A' such that $G(A') \geq A'$, but $G(A) < A$ for all $A > A'$. When $t=1$, $A' = B-1$ and when $B=2$, $A'=1$ trivially. In the non-trivial cases, $A' < K = (t-1)B^t$. When $B \geq t^t$, $A' = K-1$. In other cases an algorithm is given for determining A'. By the iterative notion of Isaacs (Canadian Journal of Mathematics vol. 2 (1950) pp. 409–416) $G(A)$ divides the positive integers into T classes. T is finite since $T \leq A'$. Further observations about T are given. The case $B=10$, $t=2$, $A'=99$, $T=2$ was discussed by Porges (Amer. Math. Monthly vol. 52 (1945) pp. 379–382). (Received November 13, 1956.)

189*t*. M. H. Stone: *Some algebraic aspects of logic.*

Using a more or less standard formalization of the lower predicate calculus in terms of finite symbol-sequences, this paper shows that in terms of logical equivalence this system is to be identified as a free, locally finite, polyadic Boolean algebra in the sense of Halmos. The principal result of the paper includes as a special case the statement that any locally finite polyadic Boolean algebra is homomorphic to one of these free algebras obtained by a suitable specification of its basic symbols. This special case is the justification for our use of the term "free." In its general form the principal result serves to unify and simplify a series or extension and embedding theorems which occupy a central place in Halmos's theory of polyadic Boolean algebras. Work on this paper was sponsored by the United States Air Force, Office of Scientific Research, Division of Mathematics, under Contract No. AF 18(600)-1125. The paper is to be distributed as a Technical Report under this contract. (Received October 8, 1956.)

190. T. T. Tanimoto: *A theorem concerning doubly stochastic matrices.*

Let $\mathfrak{M}_n$ be the set of all $n\times n$ real matrices, and let Γ_n be a linear operator on $\mathfrak{M}_n$ into the reals such that if $A=(a_{ij})\in\mathfrak{M}_n$, then $\Gamma_n(A)$ is the sum of n elements of A, no two in the same row of column. $\bar{\Gamma}_n$ is said to be maximal (minimal) with respect to A if $\bar{\Gamma}_n$ is a maximum (minimum) over all possible Γ_n's. If c, called the global constant of A, is the sum of all the elements of A, then there exists a unique doubly stochastic matrix $S=(s_{ij})\in\mathfrak{M}_n$ defined by $s_{ij}=(k+b_{ij})/kn+c$, where $b_{ij}=\sum_j a_{ij}+\sum_i a_{ij}-na_{ij}$ and $k=0$ if $\min_{ij} b_{ij}\geqq 0$, and $k=-(\min_{ij} b_{ij})$ if $\min_{ij} b_{ij}<0$, such that if $\bar{\Gamma}_n$ is maximal with respect to A, then $\bar{\Gamma}_n$ is minimal with respect to S. Conversely, given any doubly stochastic matrix $S\in\mathfrak{M}_n$ and a constant c, if $n>2$, there exists a unique matrix $A\in\mathfrak{M}_n$ having c as its global constant such that if $\bar{\Gamma}_n$ is minimal with respect to S, then $\bar{\Gamma}_n$ is maximal with respect to A. The proof depends upon the solution of a linear system of n^2 equations in n^2 unknowns whose matrix D of the coefficients in terms of direct products is $D=R\times I+I\times(\Xi-I)$, where $R\in\mathfrak{M}_n$ is the matrix consisting of ones except for the main diagonal whose elements are all $2-n$, $\Xi\in\mathfrak{M}_n$ is the matrix consisting entirely of ones, and $I\in\mathfrak{M}_n$ is the identity matrix. When $n>2$, D is never singular. (Received November 13, 1956.)

191. E. T. Wong: *On injective rings.*

Let R be a left faithful ring i.e., $x\in R$, $x\neq 0$ then $xR\neq 0$. Consider the additive group R^+ of R as a right R-module. M is any right R-injective module, then $\bar{M}=M/H$, where H is a maximal submodule of M which $H\cap R^+=0$, is a right R-injective module and is the maximal essential extension of R^+. In the ring $Q=\operatorname{Hom}_R(\bar{M},\bar{M})$, if $a\in Q$, $a\neq 0$ then $aR^+\neq 0$, Q is a self right injective ring containing R. Also Q is the unique maximal right quotient ring of R. If S is a self right injective ring containing R and $sR=0$ if $s=0$, $s\in S$, then $S\supset Q$. Q is a regular ring if and only if the singular ideal of R is zero. (Received November 13, 1956.)

ANALYSIS

192. D. G. Aronson: *The fundamental solution of a linear parabolic equation containing a small parameter.* Preliminary report.

Consider the differential equation (*) $L_\epsilon(u)=\epsilon\sum_{i,j=1}^{n} a_{ij}(x,\ y)\partial^2u/\partial x_i\partial x_j+\sum_{i=1}^{n} a_i(x,\ y)\partial u/\partial x_i+a(x,\ y)u-\partial u/\partial y=0$, where $\epsilon>0$ is a parameter, and the matrix $\|a_{ij}(x,\ y)\|$ is positive definite and symmetric for all $x=(x_1,\ \cdots,\ x_n)$ and $y'\leqq y\leqq y''$. Under suitable regularity conditions on the coefficients of (*), it is shown that the fundamental solution $\Gamma(x,\ y;\ \xi,\ \eta;\ \epsilon)$ of (*) can be written in the form (**) $\Gamma(x,\ y;\ \xi,\ \eta;\ \epsilon)=z(x,\ y;\ \xi,\ \eta;\ \epsilon)+\int_\eta^y\int z(x,\ y;\ s,\ \tau;\ \epsilon)g(x,\ \tau;\ \xi,\ \eta;\ \epsilon)ds\,d\tau$, where the parametrix z is given explicitly and g is the solution of a certain integral equation.

The parametrix employed differs from those of Dressel (Duke Math. J. vol. 7, pp. 186–203; vol. 13, pp. 61–70) and Feller (Math. Ann. vol. 113, pp. 113–160) in that it essentially takes into account the full dependence of Γ on ϵ. Moreover, it is shown that for suitable ψ, $\lim_{\epsilon\to 0+} \int\psi(\xi)\Gamma(x, y; \xi, \eta; \epsilon)d\xi = \lim_{\epsilon\to 0+} \int\psi(\xi)z(x, y; \xi, \eta; \epsilon)d\xi = \Psi(x, y; \eta)$, where Ψ is the solution of the reduced equation $L_0(u) = 0$ such that $\Psi(x, \eta; \eta) = \psi(x)$. The proof of the representation (**) follows closely the corresponding proof given by Dressel (loc. cit.). (Received November 13, 1956.)

193. Nachman Aronszajn: *On a problem of Hermann Weyl in the theory of singular Sturm-Liouville equations.*

The problem concerns the changes in spectrum or spectral measure corresponding to a Sturm-Liouville equation $-(px')' + qx = \lambda x$, $0 \leq t \leq \infty$ in the limit point case when the boundary condition $\sin \alpha x(0) - \cos \alpha p(0)x'(0) = 0$ is being changed. The main result is as follows: if ρ_α and ρ_β are corresponding spectral functions with $\alpha \neq \beta$ and $\rho_\alpha = \rho_\alpha^0 + \rho_\alpha'$, $\rho_\beta = \rho_\beta^0 + \rho_\beta'$ are the decompositions of the measures into absolutely continuous and singular parts, then the measures $d\rho_\alpha^0$ and $d\rho_\beta^0$ are equivalent, whereas $d\rho_\alpha'$ and $d\rho_\beta'$ are orthogonal. By using results of Gelfand-Levitan, (Izvestiya Akad. Nauk. USSR Ser. Math. 16 (1952)), two examples of Sturm Liouville equations are constructed such that (1) for one boundary condition the spectrum is pure continuous, whereas for all other boundary conditions it is a pure point spectrum composed of isolated eigenvalues; (2) for one boundary condition the spectrum is a pure point spectrum with eigenvalues dense on the real axis and for all other boundary conditions it is a pure continuous spectrum. (Received November 13, 1956.)

194. Nachman Aronszajn and W. F. Donoghue (p): *On the exponential representation of analytic functions with positive imaginary part in the upper half-plane.*

It is well-known that such functions $\phi(\zeta)$ have canonical representations $\phi(\zeta) = \alpha\zeta + \beta + \int[(\lambda - \zeta)^{-1} - \lambda(\lambda^2 + 1)^{-1}]d\mu(\lambda) = \exp[\sigma + \int[(x - \zeta)^{-1} - x(x^2 + 1)^{-1}]f(x)dx]$ where $\alpha \geq 0$, β real, $d\mu$ is a positive measure satisfying $\int(\lambda^2 + 1)^{-1}d\mu < \infty$, σ real and $0 \leq f(x) \leq 1$ for $-\infty < x < \infty$. The first representation can be simplified to the form $\phi(\zeta) = \alpha\zeta + \beta_\infty + \int(\lambda - \zeta)^{-1}d\mu$ if and only if $\int|\lambda|(\lambda^2 + 1)^{-1}d\mu < \infty$. Our purpose is to give relations between the behavior of μ and f at ∞. For an integer $k \geq 0$ we define $\mathfrak{M}_k = [\mu : \int|\lambda|^k(\lambda^2 + 1)^{-1}d\mu < \infty]$, $\mathfrak{G}_k = [g(x) : \int|x|^k(x^2 + 1)^{-1}|g(x)|dx < \infty]$. For a function $f(x)$ with $0 \leq f(x) \leq 1$ we define $f_1(x) = f(x)$ for $x \geq 0$, $f(x) - 1$ for $x < 0$: $f_2(x) = f(x) - 1$ for $x > 0$, $= f(x)$ for $x < 0$; $f_3(x) = f(x) - 1$. Our results are the following: (I) for $k > 0$: $\alpha > 0$ and $\mu \in \mathfrak{M}_k$ is equivalent to $f_1 \in \mathfrak{G}_{k+1}$; (II) for $k \geq 1$: $\alpha = 0$, $\mu \in \mathfrak{M}_k$ and $\beta_\infty > 0$, is equivalent to $f \in \mathfrak{G}_k$; (III) for $k \geq 1$: $\alpha = 0$, $\mu \in \mathfrak{M}_k$ and $\beta_\infty < 0$ is equivalent to $f_3 \in \mathfrak{G}_k$; (IV) for $k \geq 2$: $\alpha = 0$, $\mu \in \mathfrak{M}_k$ and $\beta_\infty = 0$ is equivalent to $f_2 \in \mathfrak{G}_{k-1}$. (The representations II and III for $k = 2$ coincide with a theorem of Verblunsky and Delange.) (Received November 13, 1956.)

195. M. G. Arsove: *Composite functions and mass distributions.*

Let $w = u - v$, where u and v are locally bounded subharmonic functions on a region Ω, and let ϕ be a C'' function on a closed interval containing the range of w. Then the composite function $\phi \circ w$ is almost δ-subharmonic on Ω, and its mass distribution μ is given in terms of the mass distribution m for w as $\mu(E) = \int_E(\phi' \circ w)dm - (1/2\pi)\int_E(w_x^2 + w_y^2) \cdot (\phi'' \circ w)da$, E being any bounded Borel set with closure in Ω. In particular, absolute continuity of m implies absolute continuity of μ. (The case of w

a continuous subharmonic function and $\phi=\exp$ has been established by M. O. Reade). (Received November 13, 1956.)

196. J. H. Barrett: *Complex differential equations with respect to a real variable.*

Solutions of the second-order differential equation (1) $(y'/q)'+\bar{q}y=0$ where $q=q_1(x)+iq_2(x)\neq 0$ and $q_1(x)$ and $q_2(x)$ are continuous real functions of the real variable x on $a\leq x<\infty$, are investigated with emphasis on oscillation and boundedness. Explicit solutions are given for $q=r(x)\exp(i\theta(x))$, where θ'/r is constant (e.g. q_1 and q_2 linearly dependent; $q=\exp(ix)$). Recall that for q real $\neq 0$, $\sin\int_a^x q$ and $\cos\int_a^x q$ are linearly independent solutions of (1) which oscillate if and only if $\int_a^\infty |q|=\infty$. Define $y=c[a,x;q]$ to be the solution of (1) such that $y(a)=1$ and $y'(a)=0$ and $y=s[a,x;q]$ to be the solution satisfying initial conditions: $y(a)=0$ and $y'(a)=q(a)$. Then $|s|^2+|c|^2=1$ and other "trigonometric" identities are satisfied. Examples are given where the "sine" function, s, is oscillatory but the corresponding "cosine," c, has no zeros. Conditions are given to insure oscillation (or nonoscillation) of solutions of (1). Finally, solutions of the more general complex equation (2) $(py')'+fy=0$ are expressed in terms of solutions of (1) thus effecting a Prüfer-type transformation of (2). Boundedness Theorems for (2) occur as a by-product as in the author's previous work on the real case (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 247–251) and the matrix case (to appear in Proc. Amer. Math. Soc.). (Received November 13, 1956.)

197*t*. Anatole Beck: *On a functional equation in Banach space.*

If $\mathfrak{X}$ is a Banach space and T is a bounded linear operator in $\mathfrak{X}$ with $\|T\|\leq 1$, then there exist a finite measure space (S, Σ, m), an ergodic measure-preserving transformation h in S and a strongly measurable function X from S into $\mathfrak{X}$ for which $T(X(s))=X(h(s))$ almost everywhere in S *if and only if* there is in $\mathfrak{X}$ an element $x\neq 0$ for which the point-set closure $P=\mathrm{Cl}\,\{T^i(x)\}_{i=1}^{\infty}$ of the images of x under repeated applications of T has the two properties: 1° $x\in P$ and 2° P is compact. (Received November 16, 1956.)

198*t*. Stefan Bergman: *Operators generating solutions of certain differential equations in three variables and their properties.* I.

The author constructs solutions of the equation: $\Psi_{xx}+\Psi_{yy}+\Psi_{zz}+\tilde{F}(y,z)\Psi=\psi_{XX}+\psi_{ZZ^*}+F(Z,Z^*)\psi=L(\psi)=0$ where $Z=(z+iy)/2$, $Z^*=(z-iy)/2$ and F is an entire function of Z, Z^*. Let $U=Z\zeta+X-Z^*\zeta^{-1}$, $\gamma=1-\int_0^Z\int_0^{Z^*}F_1\delta_1+\int_0^Z\int_0^{Z^*}F_1\cdot\int_0^{Z_1}\int_0^{Z_1^*}F_2\delta_2-\cdots$, $F_n=F(Z_n,Z_n^*)$, $\delta_n=\prod_{\mu=1}^{n}dZ_\mu dZ_\mu^*$. Then the functions $\psi^{(N,K)}=(2\pi i)^{-1}\gamma\int_C U^N\zeta^{-N+K-1}d\zeta+\sum_{\nu=0}^{[K/2]}\Gamma^{(N,K,K-2\nu)}X^{\nu}$, $K=0,1,\cdots,2N$, $N=0,1,\cdots$, form a set of particular solutions of $L(\psi)=0$. Here $[n/2]$ is the largest integer which is smaller or equal to $[n/2]$. C is a closed curve around $\zeta=0$. Let $P^{(N,K,K-2\nu)}(Z,Z^*)$ be the coefficients of the development $(Z\zeta+X-Z^*\zeta^{-1})^N=\sum_{K=0}^{2N}\left[\sum_{\nu=0}^{K-[(K+1)/2]}P^{(N,K,K-2\nu)}(Z,Z^*)X^{K-2\nu}\right]\zeta^{N-K}$. Then $\Gamma^{(N,K,K-2\nu)}$ are solutions of the equation (A): $N\gamma_{Z^*}P^{(N-1,K,K)}+\Gamma_{ZZ^*}^{(N,K,K)}+F\Gamma^{(N,K,K)}=0$ and the $\Gamma^{(N,K,K-2\nu)}$ are solutions of the system (B): $N\gamma_{Z^*}P^{N(-1,K,\nu)}-N\gamma_Z P^{(N-1,K-2,\nu)}+(N-K-2\nu+2)\cdot(N-K-2\nu+1)\cdot\Gamma^{(N,K,\nu+2)}+\Gamma_{ZZ^*}^{(N,K,\nu)}+F\Gamma^{(N,K,\nu)}=0$, $\nu=K-2, K-4,\cdots, K-2[K/2]$. Further $\Gamma=\Gamma^{(N,K,\nu)}(Z,Z^*)$ are solutions of the form $\Gamma=-\int_0^Z\int_0^{Z^*}T_1\delta_1+\int_0^Z\int_0^{Z^*}F_1\int_0^{Z_1}\int_0^{Z_1^*}T_2\delta_2-\cdots$, $T_K=T(Z_K,Z_K^*)$, of the equation $T+F\Gamma+\Gamma_{ZZ^*}=0$. (Received December 27, 1956.)

199t. Stefan Bergman: *Operators generating solutions of certain differential equations in three variables and their properties.* II.

Using the previous results, author defines operators $T_m(X, Z, Z^*; f) = \gamma \sum_{K=0}^{[m/2]} f^{(m-K)}(Z) X^{m-2K}(-Z^*)^K [(m-2k)!k!]^{-1} + \sum_{K=0}^{[m/2]} X^{m-2K}\Gamma^{(m-2K)}(Z, Z^*; f)$ transforming analytic functions $f = f(\zeta)$ of a complex variable ζ into solutions ψ of $L(\psi) = 0$. $\Gamma^{(m-2K)}$ are solutions of a system analogous to (A) and (B), see I. Further real operators $R_m(X, Z, Z^*; f) = (T_m + \overline{T}_m)/2$ are defined transforming analytic functions f into solutions Ψ of $L(\Psi) = 0$ which solutions for X real and $Z^* = \overline{Z}$ (conjugate to Z) become real functions. For the R_m holds: $R_m(X, Z, 0; f) = 2^{-1}[(1/m!)f^{(m)}(Z)X^m + [(m-2K)!K!]^{-1}f^{(m-K)}(0)X^{m-2K}(-Z)^K]$. Using the above relation and classical results, it is possible to determine the location and some properties of singularities of $\Psi(X, Z, Z^*)$ from the coefficients $\{A_{mn0}\}$, $m = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots$ of the series development $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} A_{mnp} X^m Z^n Z^{*p}$ of Ψ at the origin. Many of these relations are independent of the coefficient F of the equation L or depend only on some properties of F. For instance: Let $\{A_{mn0}, n = 0, 1, 2 \cdots\}$ for $m = 0, 1, \cdots, P$, satisfy the Hadamard conditions, insuring that $f_m(Z) = \sum_{n=0}^{\infty} A_{mn0}Z^n$ has l_m poles at the points $Z = Z_{ms}$, and let $A_{mn0} = 0$ for $m > P$. Then Ψ has as it's only singularities the lines $[Z = Z_{ms}, |X| < \infty]$, $s = 0, 1, \cdots l_{m-1}$, $m = 0, 1, \cdots, P$. (Received December 27, 1956.)

200t. H. D. Block. *Discrete isoperimetric-type inequalities.*

The classical isoperimetric inequality bounds the area A enclosed by a curve C in terms of its length L. For a gridwork, an analogue of A is T, the total number of lattice points inside C; an analogue of L is either N, the number of boundary lattice points, or M, the sum of their multiplicities (the number of missing neighbors). Other analogous quantities are $\overline{T^{1/2}}, \overline{N}, \overline{M}$ which are $T^{1/2}$, N, M averaged over rotations of the gridwork. Inequalities relating these quantities are obtained for a gridwork with sufficiently small mesh width δ. Typical results: $|L - \pi\overline{M}\delta/4| \leq \pi\delta$, $M^2 \geq 16T$, $L \geq \pi\delta(\overline{T^{1/2}} - 1)$. (Received November 13, 1956.)

201t. R. P. Boas and R. C. Buck: *Notes on polynomials defined by generating relations.* I.

Consider the generalized Appell polynomials defined by $A(w)\psi(zg(w)) = \sum_{n=0}^{\infty} p_n(z)w^n$ where A, ψ, and g are power series, where $A(0) \neq 0$, ψ has no zero coefficients, and $g(0) = 0$, $g'(0) \neq 0$. If ψ is entire, and the generating series converges in a region containing a zero, $w = \alpha$, of A, then it is obvious that there is a non-null representation of zero by the set $\{p_n(z)\}$, namely $0 = \sum_{n=0}^{\infty} p_n(z)\alpha^n$. Further admissible zeros of A contribute further representations of zero. It is shown that the only representations of zero in the form $0 = \sum_{n=0}^{\infty} c_n p_n(z)$ with $c_n = O(R^n)$ are linear combinations of representations obtained in this manner from zeros of A, provided that A is regular and g is univalent in the disc $|w| \leq R$. If g is not univalent, there evidently are infinitely many additional representations of zero; for example, if $g(w_1) = g(w_2)$ but $A(w_1)A(w_2) \neq 0$, then $0 = \sum p_n(z)\{(w_1)^n/A(w_1) - (w_2)^n/A(w_2)\}$. This situation arises, for example, with the choice $g(w) = 1 - e^w$, $\psi(t) = e^t$ and $A(w) = 1$; when z is chosen as 1, there result an infinite number of identities relating the exponential numbers considered by G. T. Williams [Amer. Math. Monthly vol. 52 (1945) pp. 323–327] and Touchard [Canadian Journal of Mathematics vol. 8 (1956) pp. 305–320]. (Received October 22, 1956.)

202t. R. P. Boas and R. C. Buck: *Notes on polynomials defined by generating relations.* II.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are generalized Appell polynomials (cf. the preceding abstract) if and only if $\alpha-\beta$ is 0, 1, or -1 [R. C. T. Smith, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 636–641]. However, $P_n^{(\alpha,\beta)}(1+x)$ are generalized Appell polynomials [Erdelyi et al., *Higher transcendental functions*, vol. 3, 1955, p. 264]. The authors use this fact to give a short proof of the known result about the convergence of the Jacobi expansion of a function analytic in an ellipse. (Received October 22, 1956.)

203t. R. P. Boas and R. C. Buck: *Notes on polynomials defined by generating relations.* III.

Let $\{p_n(z)\}$ be Humbert's polynomials, defined by $(1-3zw+w^3)^{-\nu}=\sum p_n(z)w^n$. These may be obtained as generalized Appell polynomials, with $g(w)=3w/(1+w^3)$ [cf. Abstract 202]. The function g is univalent in the disc $|w|\leqq\rho$ for $\rho<(1/2)^{1/3}$. It is shown that every function which is regular inside the hypotrochoid $3x=\rho^{-1}\cos\theta+\rho^2\cos 2\theta$, $3y=-\rho^{-1}\sin\theta+\rho^2\sin 2\theta$ has an expansion in Humbert polynomials, convergent in the same region. This is analogous to the elliptical expansion regions for the Gegenbauer polynomials. In particular, if f is regular in a disc $|z|<R$ with $R>(1/4)^{1/3}$, then its Humbert expansion converges at least in the disc $|z|<r$ where $(R-r)(R+r)^2=8/27$; for large R, $r=R-O(R^{-2})$. Since g is not univalent in the unit disc, there are non-null representations of zero which converge in a region containing the origin and bounded by part of a prolate hypotrochoid. Accordingly, the representation of analytic functions by series of Humbert polynomials is not unique in sufficiently small neighborhoods of the origin. (Received October 22, 1956.)

204t. R. P. Boas and R. C. Buck: *Notes on polynomials defined by generating relations.* IV.

A class of polynomials can be characterized by the recursion relation $p_{n-1}(z)=\{p_n(z)-p_n(qz)\}/(z-qz)$ (suggested to the authors by T. S. Motzkin); they can be shown to be generalized Appell polynomials [cf. the third preceding abstract] with $g(w)=w$, $A(w)$ arbitrary, and $\psi(t)=\sum_0^\infty (q-1)^n t^n/\{(q-1)(q^2-1)\cdots(q^n-1)\}$. Their formal properties have been investigated independently by Sharma and Chak [Riv. Mat. Univ. Parma vol. 5 (1954) pp. 325–337]. They reduce to classical Appell polynomials when $q\rightarrow 1$. The general expansion theory for generalized Appell polynomials leads, in this particular case, to the following results. When $q>1$, the expansion class is a certain space of entire functions of zero order. When $0<q<1$, its consists of certain functions analytic at the origin. If $A(w)$ is analytic in $|w|<\rho$, then any function analytic in the disc $|z|<\{(1-q)\rho\}^{-1}$ can be expanded in this disc. The second statement depends upon the fact that the sequence $\mu_n=(1-q)(1-q^2)\cdots(1-q^n)$ is completely monotonic for $0<q<1$. (Received October 22, 1956.)

205t. R. P. Boas and R. C. Buck: *Notes on polynomials defined by generating relations.* V.

The Boole polynomials [Jordan, *Calculus of finite differences*, 2d ed., New York, 1947, pp. 317 ff.] may be shown to be generated by $[1+(1+w)^h]^{-1}(1+w)^z$, where the principal values of the powers are to be taken. The formal expansion of a function in a series of Boole polynomials is given by Jordan but its convergence seems not to have been investigated. The authors show that if $0<h\leqq 3$ the expansion of an entire

function of exponential type has the same convergence properties as its expansion in a series of Newton polynomials [Buck, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 283–298], and if $0<k\leq 1$ the properties under Mittag-Leffler summation are the same as for the Newton series. In particular, if $0<k\leq 3$, the Boole expansion converges for functions of exponential type less than log 2. If $k>3$ it converges for functions of exponential type less than $\log\,[1+2\sin\pi/(2k)]$. Similar results hold for the generalized Boole polynomials of Peters [Bull. Amer. Math. Soc. Abstract 62-1-24]. (Received October 22, 1956.)

206t. R. P. Boas and A. C. Schaeffer: *Variational methods in entire functions.*

If $f(x)$ is an entire function of exponential type τ and $|f(x)|\leq M$ then, according to S. Bernstein, $|f'(x)|\leq M\tau$. The converse is naturally false. The authors establish some theorems which sharpen the conclusion and do have valid converses. One of these is: If $\tau\leq\pi$ then (*) $\left|\sum_{k=1}^{n} f'(k)\right|\leq M\pi$ for all integers n; if $\tau<\pi$ and (*) holds there are numbers f_0 and $K(\tau)$, depending respectively only on f and τ, such that $|f(x)-f_0|\leq K(\tau)M$. They use these results to answer a question of Gaier's [Proc. Amer. Math. Soc. vol. 6 (1955) p. 879] on the relationship between $\lim n^{-1}\sum_{k=1}^{n} f(k)$ and $\lim x^{-1}\int_1^x f(t)dt$. When n is odd the constant π in (*) is sharp, but when n is even it is not. The precise bound is determined explicitly in some cases, but in general it depends on certain hyperelliptic integrals. (Received October 25, 1956.)

207. F. G. Brauer: *Eigenfunction expansions for a class of singular boundary value problems.*

Let L and M be linear ordinary differential operators, with L having higher order than M. Suppose L and M are self-adjoint with respect to some given boundary conditions on a finite interval δ, and satisfy certain positivity conditions. Let (u, v) denote the usual L^2 inner product on δ and define a new inner product $[u, v]=(Mu, v)$ for functions satisfying the boundary conditions. Let H_δ be the Hilbert space completion of this set of functions in the new inner product. The boundary value problem $Lx=\lambda Mx$ with the given boundary conditions has a countable number of eigenfunctions which form a complete set in H_δ. There exist a Green's function and a spectral matrix for the equation $Lx=\lambda Mx$ on an infinite interval. Let H be the Hilbert space completion in the inner product $[u, v]=(Mu, v)$ of the set of differentiable functions vanishing outside a finite sub-interval. The Parseval equality and expansion theorem hold for functions in H. (Received November 9, 1956.)

208. C. C. Braunschweiger: *A geometric construction of the M-space conjugate to an L-space.*

The following theorem is proved: *Let X be an abstract L-space with F-unit e and positive cone C. Define* $S=(C-e)\cap(e-C)$, $p(x)=\inf\ \{a\colon (1/a)x\in S,\ a>0\}$ *and* $Y=\{aS\colon a\geq 0\}$. *Then Y is an abstract M-space with unit element e, norm $p(x)$ and unit sphere S and is linearly isomorphic and isometric to the space $L^\infty(\Omega, m)$ conjugate to the concrete representation $L(\Omega, m)$ of X.* This geometric construction of the conjugate to a given L-space is based upon Clarkson's *A geometric characterization of C-spaces* (Annals of Math. vol. 48 (1947) pp. 845–850). The proofs of the theorem and some related results rely heavily upon the classic papers of S. Kakutani (Annals of Math. vol. 42 (1941) pp. 523–537 and 994–1024). The above theorem is slightly generalized to include the case where X has no F-unit by using the fact that then X is a direct sum of L-spaces, each with an F-unit. (Received October 5, 1956.)

209. H. J. Bremermann: *Solution of the Dirichlet problem for pseudo-convex domains by plurisubharmonic functions.* Preliminary report.

It is well known that to continuous real boundary values prescribed on the boundary of a domain D in the complex number space C^n, $n>1$, there does not exist in general a pluriharmonic function in (real part of a holomorphic function) that assumes the boundary values, not even if these are prescribed only on the (real) n-dimensional "distinguished boundary surface." This latter boundary value problem is solved by functions of S. Bergman's "extended class." It is shown that the extended class can be replaced by the class of plurisubharmonic functions. This class solves also the Dirichlet problem for continuous values $v(z)$ prescribed on the whole boundary of a totally pseudo-convex domain D. The proof of this result consists in determining the envelope of holomorphy of the set $\{(z, w) | z \in \bar{D};\ |w| < e^{-v(z)} \text{ for } z \in \dot{D},\ w=0 \text{ for } z \in D\}$. It is conjectured that in general the plurisubharmonic functions solve the Dirichlet problem for boundary values prescribed on the Šilov boundary of a pseudo-convex domain. (Received November 14, 1956.)

210. F. E. Browder: *Mixed boundary-value problems.*

Let G_t for each $t \geqq 0$ be a domain E^n varying smoothly and monotonically with t, while for each t, $A(t)$, $B(t)$, and $C(t)$ are strongly elliptic systems of differential operators in the x-variables of orders $2m$, $2r$, and $2s$ respectively with coefficients defined on $G_t(A(t)$ positive). Let $u(x, t)$ be drawn from the class of functions defined for $t>0$ and x in G_t and satisfying one of a certain class of null elliptic boundary conditions on the boundary of G_t for all t. Under certain smoothness and boundedness hypotheses on the coefficients of the operators A, B, and C, existence, uniqueness, and regularity theorems are established for the solution of each of the following problems: (1) $(1)^m A(t)(\partial^2 u/\partial t^2)+(-1)^r B(t)(\partial u/\partial t)+C(t)u=f(x, t)$, with u and $\partial u/\partial t$ prescribed at $t=0$; (2) $(-1)^m A(t)(\partial u/\partial t)+(-1)^r B(t)u=f(x, t)$ with u prescribed at $t=0$. Regularity properties on the boundary are established, and in particular it is shown that, with smooth data, the solutions and their appropriate derivatives assume their prescribed boundary values. The theorems established extend previous results of Ladizenskaya, Visik, Yosida, and Lions. (Received November 13, 1956.)

211. Leon Brown: *On Erhardt Schmidt branching theory.*

We are interested in generalizing the Schmidt branching theory, and take the functional analysis point of view. Let X be a complex B space and $f(x)$ be any given function with domain and range in X, which is analytic and bounded for $\|x\| \leqq 1$. We assume $f(0)=0$ and $f'(0)$, a linear function on X to X, is a compact transformation. We consider the functional equation $x-f(x)=y$; where y is given. The situation where $(I-f'(0))^{-1}$ exists is well known. A more interesting problem presents itself when the inverse of $I'-f'(0)$ does not exist. In this case, we find that there exists: (1) a finite dimsional subspace $M \subset X$, (2) a subspace $N \subset X$, (3) $f_i(y, z)$, $i=1, \cdots, k$, analytic in a neighborhood of the origin of $X \times M$ with range contained in the complex numbers, and (4) $x_1(y, z)$ analytic in a neighborhood of the origin of $X \times M$ with range contained in N. These are related to our problem in the following ways: (1) $x \in X$, then $(I-f'(0))x=0$ iff $x \in M$, and (2) $x \in X$, then $x-f(x)=y$ iff $x=x_1(y, z)$ where $x_1 \in N$ and $z \in M$, and $f_i(y, z)=0$, $i=1, \cdots, k$. We observe that the set of solutions is contained in a finite dimensional analytic manifold of X. The sizes of all the above mentioned neighborhoods are estimated in this paper. (Received November 14, 1956.)

212t. T. S. Chihara: *On co-recursive orthogonal polynomials*. Preliminary report.

Let $\{P_n(x)\}$ be the orthogonal polynomials defined by the recursion formula (1) $P_n(x)=(x+b_n)P_{n-1}(x)-\lambda_n P_{n-2}(x)$ $(n\geqq 2)$; $P_0(x)=1$, $P_1(x)=x+b_1$ $(\lambda_n>0$, b_n real). The "co-recursive" polynomials $\{P_n^*(x)\}$ are defined as the polynomials which satisfy (1) subject to the "initial conditions": $P_0^*(x)=1$, $P_1^*(x)=P_1(x)-c$ (c real). The $P_n^*(x)$ are orthogonal polynomials related to the $P_n(x)$ by: $P_n^*(x)=P_n(x)-cQ_{n-1}(x)$, where $Q_n(x)$ satisfies: $Q_n(x)=(x+b_{n+1})Q_{n-1}(x)-\lambda_{n+1}Q_{n-2}(x)$ $(n\geqq 1)$; $Q_0(x)=1$, $Q_{-1}(x)=0$. The zeros of $P_n^*(x)$, $P_n(x)$ and $Q_{n-1}(x)$ are mutually separated, and various relations among the distribution functions for the three systems of orthogonal polynomials are easily deduced. The $Q_n(x)$ have been studied by Shohat and Sherman [Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1920) pp. 282–287] and Sherman [Trans. Amer. Math. Soc. vol. 25(1), pp. 64–87], and some of their methods and results are applied to the study of the $P_n^*(x)$. The co-recursive polynomials for some special cases give rise to simple examples of orthogonal polynomials whose distributions are of "mixed" type; i.e., the distribution is a step function (with a single jump) on part of the orthogonality interval and a weight function on the remainder of the interval. (Received November 13, 1956.)

213. Joshua Chover (p) and J. Feldman: *On positive-definite integral kernels and a related quadratic form.*

Let $C(J)$ be the Banach space of continuous complex functions on an interval J, and $B(J)$ the dual space of Radon measures. Let ρ be defined on $J\times J$ and strictly positive-definite. Let $\sigma=\{t_i\}$ be a finite collection of distinct points of J, and for any $x\in C(J)$, form $\mu_\sigma(x)=\sum_{i,j} r_{ij}x(t_i)x(t_j)$, where $[r_{ij}]$ is the matrix inverse to $[\rho(t_i, t_j)]$. Theorem: The collection of those $x\in C(J)$ for which $\mu_\sigma(x)$ remains bounded under refinements of σ is identical with the range of $S^{1/2}$, where S is the continuous operator from $L_2(J)$ to $C(J)$ defined by $\int_J\rho(t, u)h(u)du$. *A corollary*: If J' is a subinterval of J and S' the corresponding integral operator from $L_2(J')$ to $C(J')$, then range $(S'^{1/2})$ contains range $(S^{1/2})$ with functions restricted to J'. If the operator R from $B(J)$ to $C(J)$ is given by $Rm=\int_J\rho dm$, one can define $R^{1/2}$ from $B(J)$ to $L_2(J)$. *Theorem*: $(h, R^{1/2}m)=\int_J S^{1/2}h(u)d\overline{m(u)}$ for all $h\in L_2(J)$ and $m\in B(J)$; and $S^{1/2}R^{1/2}=R$, range (R) being a proper subset of range $(S^{1/2})$. For suitable ρ, $C(J)$ may be considered as a probability space, and $X_t=x(t)$ a (Gaussian) stochastic process with ρ as covariance function. Then $\mu_\sigma(x)$ may be interpreted as a component of a finite-dimensional approximation to a maximum likelihood ratio, based on "sampling" $x(t)$. (Received November 13, 1956.)

214. Albert Edrei (p) and W. H. J. Fuchs: *Deficient values which are also asymptotic.*

The authors prove the following inequality for the Nevanlinna characteristic $T(r)$ of a meromorphic function $f(z)$ $[f(0)=1]$. Let q be a positive integer and let $\sigma\geqq\lambda>1$. Then $T(r)\leqq A\sigma^{q-1}T(r/\sigma)+B\sigma^{-q}T(\sigma r)+Cq\sigma^{q-1}\{N(\sigma r, 0)+N(\sigma r, \infty)\}$ where the positive constants A, B, C depend only on λ and $N(r, a)$ is the usual counting function of the Nevanlinna theory. If lim sup $\{N(r, 0)+N(r, \infty)\}/T(r)$ is small, the above inequality implies that the growth of $T(r)$ is fairly regular. Several applications of this remark are made. In particular it is shown that if $f(z)$ is entire, of finite order ρ, and if the sum Δ of the finite deficient values of $f(z)$ is sufficiently close to 1, then there are deficient values whose deficiencies exceed $1/32\rho$ and these values are also asymptotic.

The special case $\Delta=1$ has been investigated by A. Pfluger who proved [Commentarii Mathematici Helvetici vol. 19, pp. 91–104] that the order of $f(z)$ is necessarily an integer p and that all deficiencies are multiples of $1/p$. The authors now add that all deficient values are asymptotic. (Received November 6, 1956.)

215. M. B. A. Foos: *On the values of certain sets of modules.*

Let a be a fixed point in $(0, 1)$. Let G' be any simply-connected domain in $|w|<1$, such that G' contains the origin. Let G be any doubly-connected domain contained in $|w|<1$, disjoint from G' and separating G' and a from $|w|=1$. Suppose neither G nor G' contains the point $w=a$. Let M' be the reduced module of G', and M the module of G for the class of curves separating the boundaries. Let b be a parameter between 0 and 1. Then by using the quadratic differential $Q(w)dw^2=-(w-b)\cdot(w-b^{-1})[w^2(w-a)(w-a^{-1})]^{-1}dw^2$, and the method of extremal metrics the exact range of values of the pairs (M, M') is found to be a simply-connected domain in the fourth quadrant of the M, M'-plane, the boundaries of which are given explicitly. Similarly, we can find the exact range of values of the pair (M', M''), the reduced modules of G', G'', which are disjoint simply-connected domains containing zero and the point at infinity, respectively, and such that neither contains the point 1. Let a and b be two positive, finite, fixed points. Let G' be a simply-connected domain containing $w=0$ and let G'' be a simply-connected domain disjoint from G' and containing the point at infinity. Let G be a doubly-connected domain disjoint from G' and G'' and separating G' and a from G'' and b, and such that any curve separating the boundaries of G is homotopic to a curve which separates G' and a from G'' and b and is symmetric with respect to the real axis. Suppose neither G', G'', nor G contains a or b and let M', M'' be the reduced modules of G', G'', respectively, and M the module of G for the class of curves separating the boundaries. Then using quadratic differentials and extremal metrics, we find the exact range of values of the triples (M'', M, M'). (Received November 13, 1956.)

216. M. B. A. Foos: *On a theorem of Kubo and one of Tsuji.*

By using the method of extremal metrics, considerably shorter proofs can be given for the following theorems. I. (Kubo) Let D and D_1 be two bounded domains of connectivity n, both containing the origin and such that D_1 contains D and is obtained from D only by a variation of the outer boundary component of D, the inner boundary components remaining fixed. Let the outer boundary components of D and D_1 bound a doubly-connected domain of which the module for the class of curves separating the boundaries is $(1/2\pi)\log M$. Denote by $w=F(z)$, $(F(0)=0)$, the analytic function which maps D conformally onto the unit circle slit along radial segments, such that the outer boundary component of D maps into $|w|=1$. Denote by $w=F_1(z)$, $(F_1(0)=0)$, the same type of mapping for D_1. Then $|F'(0)|\geq M|F_1'(0)|$. II. Let D and D_1 be defined as in I. Let $F(z)$ and $F_1(z)$ be the corresponding circular slit mappings. Then $|F'(0)|\geq M|F_1'(0)|$. III. (Tsuji). Let D be a simply-connected domain in the z-plane which contains the origin and is contained in $|z|<m$. Let E be a continuum which contains the origin and is contained in D, such that a disc of radius b about any point of E is contained in D. If D is mapped conformally onto $|w|<1$ by the function $w=w(z)$, $z=z(w)$, with $w(0)=0$, then the image of E is contained in $|w|\leq t<1$, where $t=t(b/m)$ depends on b/m only. The bound we obtain is better than that given by Tsuji. (Received November 13, 1956.)

217. Avner Friedman: *Classes of solutions of linear nonhyperbolic differential equations.*

Denote by $C\{M_n\}$ $(M_n>0)$ the class of all indefinitely differentiable functions $f(x)$ defined in a given m-dimensional domain D and possessing the following property: if $x\in D_1$ $(\overline{D_1}\subset D)$, $|\partial^n f(x)/\partial x_1^{j_1}\cdots\partial x_m^{j_m}|\leqq H_0H^nM_n$ $(n=1, 2, \cdots)$, where H_0, H depend on f and D_1. Let $Au(x)=g(x)$ be a linear elliptic system of differential equations of order s. We can prove: If $g(x)$ and the coefficients of A belong to $C\{M_n\}$, so do the solutions provided that the M_n satisfy: $\binom{n-s+2}{i}M_iM_{n+s-i}\leqq KnM_{n+1}$ $(i=1, \cdots, n-s+2;\ n=1, 2, \cdots;\ K>0)$. $M_n=n!$ is the analytic case. A similar result holds for the parabolic equation $\partial u(x, t)/\partial t=Lu(x, t)$ where L is an elliptic operator and the coefficients of its essential part are functions of t only. The differentiability properties are considered with respect to x. (Received October 29, 1956.)

218. I. S. Gál: *On the continuity and limiting values of functions.*

Let A be an index set and let $\mathfrak{J}$ be a topology on A such that every noncountable subset B of A has a point of accumulation which belongs to B. Let $\phi^i=\{\mathfrak{F}_\alpha^i\}$ $(i=1, 2)$ be a family of filters $\mathfrak{F}_\alpha^i$ $(\alpha\in A)$ defined on a set X. We say that ϕ^1 has property (P) with respect to ϕ^2 and $\mathfrak{J}$ if for every $F_{\alpha_0}^1\in\mathfrak{F}_{\alpha_0}^1$ there is a neighborhood N_{α_0} of α_0 such that every $\mathfrak{F}_\alpha^2$ with $\alpha\in N_{\alpha_0}$ contains a set $F_\alpha^2\subseteq F_{\alpha_0}^1$. Theorem: *Suppose that* ϕ^1 *has property* (P) *with respect to* ϕ^2 *and* $\mathfrak{J}$. *Let* f *be a function from* $X\subseteq A$ *into a metric space* Y. *Then the set of those points* $x\in X$ *at which* f *is continuous with respect to* $\mathfrak{F}_x^1$ *and discontinuous with respect to* $\mathfrak{F}_x^2$ *is countable.*—A similar result can be proved concerning the set $f[\mathfrak{F}]$ of cluster points of f with respect to $\mathfrak{F}$. (Received November 13, 1956.)

219. T. M. Gallie, Jr.: *Mandelbrojt's inequality and Dirichlet series with complex exponents.*

Let $\sum_{n=1}^\infty c_n\exp(-\lambda_n s)$ converge to $f(s)$ in a disc of radius greater than $\pi\tau$ where $\tau=\limsup n/|\lambda_n|<\infty$. Suppose $\liminf(|\lambda_{n+1}|-|\lambda_n|)>0$ but $\arg\lambda_n$ is arbitrary. Mandelbrojt's inequality estimates $|c_n|$ in terms of $\max|f(s)|$ on a disc of radius $\pi\tau$ onto which $f(s)$ is continued analytically "by moving a disc" of radius $\pi\tau$. It is proved in this paper that the disc can be replaced by the generally much smaller conjugate diagram C of $\sum_{n=1}^\infty(1-s^2/\lambda_n^2)$. Applications follow. Every boundary point of the region of convergence of $f(s)$ is a limit point of zeros of its partial sums. If $f(s)$ is bounded in the vertical angles, however small, between any two intersecting lines, then $f(s)\equiv 0$. If C is "put on top of" a boundary point of the region of convergence, then $f(s)$ has a singular point on C. (This contains Pólya's theorem: For λ_n positive, every segment of length $2\pi\tau$ of the axis of convergence contains a singular point of $f(s)$.) If $\tau=0$, the region of existence of $f(s)$ is the convex region of convergence. (This fact, undoubtedly known to Pólya, contains the Fabry gap theorem for Taylor series.) (Received November 13, 1956.)

220. R. P. Goblirsch: *An area for simple surfaces.*

Let Σ be the boundary of a tetrahedron and let h be a homeomorphism of Σ into Euclidean 3-space, E^3. Let $A(h)=\inf_P\liminf_n A(h_n)$ where the infimum is taken over all sequences $P=[h_n]$ of piecewise linear homeomorphisms of Σ into E^3 which converge uniformly to h and where $A(h_n)$ is the elementary area of h_n. Then, under a certain restriction on $h(\Sigma)$, there is a positive number k, independent of h, such that $A(h)\leqq k\cdot L(h)$ where $L(h)$ is the Lebesgue area of h. The techniques of proof are modelled after those used by R. H. Bing (see Bull. Amer. Math. Soc. Abstract 61-6-824) in establishing the existence of a sequence P. (Received November 13, 1956.)

221. Michael Golomb: *The Dirichlet-initial value problem for partial differential equations in several time-like variables.*

Let Ω_s be an open set of R^n with boundary Γ_s, Ω_t the quadrant $t_1>0, t_2>0$ in R^2, D_s a linear differential operator on Ω_s and $D_t = a_{12}\partial^2/\partial t_1\partial t_2 + a_1\partial/\partial t_1 + a_2\partial/\partial t_2 + a_0$, $a_i \in L^\infty(\Omega)$, $a_{12}(x) \geqq a > 0$ for almost all $x \in \Omega$. Given the functions f, g, g_i on $\Omega_s \times \Omega_t$, a solution u of the equation $(D_t + D_s)u = f$ in $\Omega_s \times \Omega_t$ is to be found subject to the conditions $u = g$ on $\Gamma_s \times \Omega_t$ and $u = g_i$ on that part of the boundary of $\Omega_s \times \Omega_t$ where $t_i = 0$. That this is a correctly set problem is proved for the case where D_s is a general elliptic operator, say of second order. Let $\mathfrak{D}$ be the space of (Schwartz) testing functions on Ω and let $\mathcal{E}$ be the Hilbert space defined by functions u, v which together with their first derivatives are in $L^2(\Omega)$, the scalar product being defined as $(u, v)_1 = (u, v) + \sum(u_{x_r}, v_{x_r})$, (u, v) being the usual scalar product in $L^2(\Omega)$. Let $\mathfrak{D}_1$ be the closure of $\mathfrak{D}$ in $\mathcal{E}$, $\mathfrak{D}_1'$ its strong dual, $\mathfrak{D}'(\Omega_t, \mathfrak{D}_1)$ the space of distributions on Ω_t with values in $\mathfrak{D}_1$. Then $D_t + D_s$ maps $\mathfrak{D}'(\Omega_t, \mathfrak{D}_1)$ into $\mathfrak{D}'(\Omega_t, \mathfrak{D}_1')$. The result is that this is an isomorphism onto. (Received December 17, 1956.)

222. A. W. Goodman: *Variation of the branch points for an analytic function.*

Let $w = f(z)$ be regular in $|z| < 1$, and map that domain onto a Riemann surface R having at least one simple branch point B, and let $f(0) = 0$, $f'(0) > 0$. Let R^* be the new Riemann surface formed from R by moving the branch point B to $B^* = B + \lambda$, where λ is small, leaving all other branch points fixed. If $f^*(z)$ is the function mapping $|z| < 1$ onto R^* taking $z = 0$ into the same place on R^* that $f(z)$ does, and having $f^{*\prime}(0) > 0$, then $f^*(z) = f(z) + \lambda P(z) + \bar{\lambda} Q(z) + O(\lambda^2)$. The functions $P(z)$ and $Q(z)$ are determined explicitly. Generalizations and applications are discussed. (Received November 8, 1956.)

223t. Donald Greenspan: *On a "best" 9-point difference equation analogue of Laplace's equation.*

It is shown that the well known "9-point" difference equation: $-20u_0 + 4(u_1 + u_2 + u_3 + u_4) + u_5 + u_6 + u_7 + u_8 = 0$, is a best "9-point" approximation for the Laplace equation by showing that there exist no higher order approximations and that there exist no other approximations of the same order. (Received November 1, 1956.)

224. D. S. Greenstein: *Derivative manifolds in L^2 and the Hamburger moment problem.*

Given $f(x) \in C^\infty(-\infty, \infty)$ such that $f^{(n)}(x) \in L^2(-\infty, \infty)$ $(n \geqq 0)$, it can be shown that the closed linear manifold D_f spanned by $f(x), f'(x), f''(x), \cdots$ is contained in the closed linear manifold T_f spanned by the translates $f(x+h)$. It is of interest to determine when $D_f = T_f$. This is so if and only if the absolutely continuous mass distribution whose derivative is almost everywhere equal to the square of the modulus of the Fourier transform of $f(x)$ is uniquely determined by its moments. It is further shown that if $D_f = T_f$, then $D_f(n) = T_f$ $(n \geqq 0)$ which implies, via the Hamburger moment problem, a new result on L^2 completeness of polynomials. (Received November 13, 1956.)

225. Kenneth Hoffman: *Boundary behavior of generalized analytic functions.*

Recently, Arens and Singer (*Generalized analytic functions*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 379–393) have studied a generalization of part of the theory of analytic functions in the unit disc, stemming from summable functions on partially ordered, locally compact groups. The paper treats of the generalization in this context. Let G be a discretely topologized subgroup of the additive group of real numbers, and let A be the Banach algebra of those summable functions on G which vanish on the negative portion of G. The Gelfand representation defines an isomorphism of A with an algebra of continuous functions on the space Δ, of maximal ideals of A. The Silov boundary of Δ is the character group Γ of G. A function on Δ is called "analytic" if it can be uniformly approximated on compact subsets of $\Delta - \Gamma$ by representing functions of elements of A. For these abstract analytic functions, generalizations are established of Fatou's theorem on bounded analytic functions, Riesz's theorem on functions of Hardy's class H_p, Fatou's general Dirichlet principle, and Priwaloff's theorem on the measure of the set of boundary zeros of an analytic function. (Received November 13, 1956.)

226. J. A. Hummel: *Doubly orthogonal functions.*

Given a bounded linear mapping J of the Hilbert space H_2 into the Hilbert space H_1, a set $\{\phi_n\}$ in H_2 is called doubly orthogonal with respect to J if $\{\phi_n\}$ is complete and orthonormal in H_2 and $\{J\phi_n\}$ is orthogonal in H_1 (cf., Bergman, *The kernel function*, Mathematical Surveys, no. 5, 1950). It is easily shown that the bounded, positive, self-adjoint transformation A defined by $(Jf, Jg)_1 = (Af, g)_2$ has the property that $\phi \in H_2$ is one of a doubly orthogonal set if and only if ϕ is a characteristic element of A. From this, it is easily shown that a doubly orthogonal set exists whenever J is completely continuous. This result can be applied to a great variety of examples. (Received November 13, 1956.)

227. R. C. James: *Characterizations of reflexivity.*

A reflexive Banach space has the property that each linear functional attains its sup on the unit sphere. The converse of this is true for separable spaces, while a nonseparable Banach space is reflexive if each linear functional attains its sup on the unit sphere of each separable subspace [Bull. Amer. Math. Soc. 62-4-496]. The following theorems are valid for Banach spaces B in general. If B has a complemented, nonreflexive, separable subspace, then there is a linear functional which does not attain its sup on the unit sphere of B. If B has an unconditional basis (not necessarily denumerable), then B is reflexive if each linear functional attains its sup on the unit sphere. If S is the unit sphere of B, then each of the following is necessary and sufficient for B to be reflexive: If W is a bounded closed convex subset of B with $W \cap S = 0$, then W and S can be separated by a hyperplane which does not intersect W; if W is a bounded closed convex subset of B, then $S + W$ is closed; if W is a bounded closed convex subset of B, then the convex hull of $S \cup W$ is closed. (Received November 13, 1956.)

228. W. B. Jurkat: *On the converse of Borel's limit theorem by complex methods.*

The paper contains a direct proof of the well known statement that (I) $f(s) = e^{-s} \sum s_n s^n / n! = o(1)$ as $s \to \infty$, $|s| - \Re s = O(1)$ implies $s_n = o(1)$, if the Tauberian condition (ii) $a_n = s_n - s_{n-1} = O(n^{-1/2})$ is satisfied. The method of proof consists of evaluating s_n by a contour integral which can be estimated by proper choice and subdivision of the path of integration. There are several extensions: In condition (i) the variable z may be restricted to real values. The condition (ii) may be replaced by $\sum k|a_k|^2 = O(n^{1/2})$ summing from n to $n+n^{1/2}$. The results and methods correspond to those of a previous paper concerning Abel's limit theorem. (Received November 19, 1956.)

229. S. N. Karp: *Relations between diffracting cylinders and their diffraction patterns.*

The problem of diffraction of the plane wave $u^{inc}(r, \theta: \theta_0) = \exp [ikr \cos (\theta - \theta_0)]$ by a conducting cylinder C involves the determination of a function $u(r, \theta)$, regular outside C and such that (1) $\Delta u + k^2 u = 0$, (2) $u = 0$ on C; (3) at infinity; $u \to u^{inc}(r, \theta: \theta_0) + r^{-1/2} f(\theta, \theta_0) \exp ikr$. The function $f(\theta, \theta_0)$ is the diffraction pattern. We show that if $\det \{f(\theta_m, \theta_n)\} = 0$, where $\theta_1 \cdots \theta_N$ are N distinct angles, then the equation of C is $\sum_1^N A_m u^{inc}(r, \theta: \theta_m) = 0$, and conversely. It follows that no determinant of order three can ever vanish. Similar results hold for the Neumann Boundary Condition. It is also shown that if $f(\theta, \theta_0) = F(\theta - \theta_0)$ then C is a circular cylinder. (Received December 28, 1956.)

230t. N. D. Kazarinoff: *Asymptotic solution with respect to a parameter of ordinary linear second order differential equations in a domain containing a regular singular point.*

The differential equations considered are of the type (*): $d^2u/dz^2 + \lambda^2 Q(z, \lambda)u = 0$, wherein $Q(z, \lambda) = \sum_0^\infty q_j(z)\lambda^{-j}$ and z is confined to a simply connected domain containing the origin and in which zq_0, zq_1, and $z^2 q_j$, $j \geq 2$, are analytic. $Q(z, \lambda)$ is assumed to be analytic in λ for $|\lambda| > N$. Ziebur [Thesis, University of Wisconsin, 1950] and Olver [Philos. Trans. Roy. Soc. London, Ser. A vol. 249 (1956) pp. 65–97] have investigated the asymptotic behavior of solutions of classes of equations which are subclasses of those of type (*). The present investigation, as well as those cited, is based upon methods introduced by Langer [Trans. Amer. Math. Soc. vol. 67 (1949) pp. 461–490]. The asymptotic behavior of solutions of (*) is obtained in a full neighborhood of $z = 0$ which may be unbounded provided $Q(z, \lambda)$ is suitable bounded in z. (Received November 13, 1956.)

231t. J. H. B. Kemperman: *On a class of singular integral equations.*

Let $k(s)$ be defined on $[-\infty, +\infty]$, $k(s)e^{us}$ in $L^1(-\infty, +\infty)$ for $\sigma_1 < u < \sigma_2$. Let $\sigma_1 < \alpha < \sigma_2$, and let B_α be the Banach space of measurable functions on $[0, +\infty]$ with norm $\sup |e^{-\alpha s} y(s)| < \infty$. Results are obtained concerning the integral equation $(\lambda - T)y = f$, $(y, f \in B_\alpha)$, where $Ty = \int_0^\infty k(t-s)y(t)dt$; let S_α be the corresponding spectrum. (i) If $\sigma_1 \leq \beta < \alpha$, $f - \sum_1^m a_j e^{c_j s} \in B_\beta$, then, for each $\epsilon > 0$, $y = \sum_1^m b_j e^{c_j s} + \sum_1^n d_k e^{\rho_k s} + O^{(\beta+\epsilon)s}$ with $\phi(\rho_k) = \lambda$; $(a_j, b_k, d_k$ = polynomials, c_j, ρ_k = constants). Here, $\phi(u) = \int_{-\infty}^{+\infty} k(s)e^{us}ds$. This implies a result of C. O. Segerdahl, (Thesis, Uppsala, 1939), concerning a homogeneous random process. (ii) Assume $k(s) \geq 0$. Let λ, β be real, $\phi(\beta) < \lambda \leq \phi(\alpha)$. Then $\lambda \in S_\alpha$; λ is an eigenvalue if and only if $\beta < \alpha$. Related results hold when $\phi'(\alpha) = 0$. The proof of (ii) uses a generalization to Banach spaces of a result due to R. Jentzsch (J. Math. (1912)) concerning the spectrum of a "positive" operator. (iii) Further results, a.o. exact solutions when $k(t-s)$ is degenerate, either for $t > s$ or for $t < s$. (Received November 13, 1956.)

232t. W. J. Klimczak: *The convergence of Airy series.*

Let $a_n(s)$, $n=1, 2, 3, \cdots$, denote the characteristic functions of the Airy differential operator $A_s=d^2/ds^2+s$ corresponding to the characteristic values λ_n, $n=1, 2, 3, \cdots$. Bounds for $|\lambda_n|$ and for $|a_n(s)|$ when s is real and nonreal are obtained as $n\rightarrow\infty$, and the following theorem proved: If the series (1) $\sum_{n=1}^{\infty} c_n a_n(s)$ converges absolutely for any nonreal value of s, then it converges absolutely everywhere in the complex plane. It is also shown that the series (1) and the associated series $\sum_{n=1}^{\infty} (c_n k_n/\rho) \sin \rho s$, where the k_n are suitably defined constants and $\rho=(-\lambda_n)^{1/2}$, are absolutely equiconvergent. This is a consequence of a convergent expansion found for $a_n(s)$ of the form, $a_n(s)=k_n P(\rho s) \cos \rho s+k_n Q(\rho s) \sin \rho s$, where $P(u)$ and $Q(u)$ are series of polynomials in u with real, rational coefficients. Properties of $a_n(s)$ for s real are derived, from which it is proved that the sequence of functions $\{a_n(x)\}$ is orthonormal on the interval $(-\infty, 0)$. (Received November 13, 1956.)

233. G. L. Krabbe: *On the spectra of certain Laurent matrices.*

As usual, l_p denotes the space of all sequences $c=\{c_n\}$ such that $\sum_n |c_n|^p<\infty$ $(-\infty<n<\infty)$. Let a be a fixed sequence in l_1, and let A be the function defined by $A(\theta)=\sum_n a_n e^{in\theta}$ $(-\infty<n<\infty)$. If $a*c$ denotes the convolution of sequences (see p. 44 in [L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953]), then the "Laurent matrix" (a_{n-m}) represents a linear transformation $c\rightarrow c*a$ of l_p into itself, which is denoted T_p. THEOREM: For any p with $1\leq p<\infty$, the spectrum of T_p is the image of $[-\pi, \pi]$ by A. Let $\hat{G}$ denote the character-group of a compact abelian group G; the theorem is easily extended to the case $a\in L^1(\hat{G})$, if A is the Fourier transform of a [loc. cit., Chapter VII], and if T_p is the mapping $c\rightarrow c*a$ of $L^p(\hat{G})$ into itself. (Received December 5, 1956.)

234. H. P. Kramer: *A generalized sampling theorem.*

A statement to the effect that a function $f(t)$ can be expanded in a series: (1) $f(t)=\lim_{n\rightarrow\infty} \sum_{k=-n}^{n} f(k/2w)(\sin 2\pi w[t-k/2w])/(2\pi w[t-k/2w])$ is known as a sampling theorem. A proof of (1) is given under the hypothesis that $f(t)=\int_{-w}^{w} e^{2\pi i\omega t} ds(\omega)$ where $s(\omega)$ is a complex valued function of bounded variation and has equal jumps at the end points $-w$, w of the spectral interval. Equation (1) is false if the jumps are not equal. For $|\lambda_k-k|<1/4$, equation (1) is generalized to the following formula: $f(t)=\lim_{n\rightarrow\infty} \sum_{k=-n}^{n} f(\lambda_k/2w) h_k(t)$ where the family $\{h_k\}$ depends only on $\{\lambda_k\}$ but not on f. (Received November 15, 1956.)

235. P. D. Lax: *The largest eigenvalue as a convex matrix function.*

Notation: Let X be a square matrix all of whose eigenvalues are real, denote by $\lambda(x)$ the largest eigenvalue of X. *Theorem*: Let A and B be a pair of matrices such that all linear combinations $aA+bB$ of them with real coefficients a and b have real eigenvalues. Then $\lambda(A+B)<\lambda(A)+\lambda(B)$. *Proof*: Let D denote the domain of determinacy of the point (0, 0, 1) on the plane $t=0$ for the partial differential equation $u_t+Au_x+Bu_y=0$. The support function of the convex hull of D is $\lambda(aA+bB)$. The result then follows from the convexity of support functions. For A and B symmetric, the result follows from the maximum property of the largest eigenvalue. The theorem has applications to the study of stability of difference operators. (Received November 14, 1956.)

236*t*. Walter Leighton and Zeev Nehari: *On the oscillation of the solutions of self-adjoint linear fourth-order differential equations.*

A systematic investigation is made of the oscillatory behavior of the solutions of $(ry'')''+(qy')'+py=0$ $(r>0)$. It is found that these equations fall into two distinct classes, typified by $y^{\text{iv}}+py=0$ and $y^{\text{iv}}-py=0$ $(p>0)$ in the simplest case. The first case shows many analogies to the second-order equation $y''+py=0$, while the second case presents a number of novel features. (Received November 14, 1956.)

237. Karel deLeeuw: *A duality theorem for Banach spaces with compact operator group.*

Let A be a Banach space with a compact abelian group G of isometries which is such that for each x in A the map $\sigma\rightarrow T_\sigma(x)$ of G into A is continuous. Assume that for each character χ of G the subspace $\{x: T_\sigma(x)=\chi(\sigma)x, \text{ all } \sigma \text{ in } G\}$ is finite dimensional. Let A^* be the dual space of A and $A^\sharp$ the closed G-invariant subspace of A^* consisting of all F which are such that the map $\sigma\rightarrow T_\sigma^*(F)$ of G into A^* is continuous. Repeat the process to obtain spaces $(A^\sharp)^*$ and $(A^\sharp)^\sharp$. *THEOREM:* A and $(A^\sharp)^\sharp$ are canonically isomorphic. More particularly, the canonical map of A into $(A^\sharp)^*$ is 1-1, an isometry and onto $(A^\sharp)^\sharp$. Similar results hold for compact nonabelian groups G and some locally convex spaces A. (Received November 14, 1956.)

238. A. J. Lohwater (p) and G. Piranian: *Sets of ambiguous points of functions of bounded characteristic.*

Let $f(z)$ be a complex-valued function defined in $|z|<1$, and let Γ be a simple, closed Jordan curve lying in $|z|<1$ except for one point $P=e^{i\theta}$. If $f(z)$ approaches two different limits as $z\rightarrow P$ along Γ from opposite sides, the point P is called an *ambiguous point* of $f(z)$. It is known [Bagemihl, Proc. Nat. Acad. Sci. USA vol. 41 (1955) pp. 379–382] that the set of ambiguous points for any complex-valued function is at most denumerable. It is shown that if E is an arbitrary denumerable set on $|z|=1$, there exists in $|z|<1$ an analytic function of bounded characteristic whose set of ambiguous points coincides with E. (Received December 27, 1956.)

239*t*. R. G. Long: *A T-system which is not a Bernstein system.*

A fundamental linearly independent sequence $\{u_i\}_1^\infty$ of points of a real Banach space E is called a T-system for E if for each x in E and for each non-negative integer n, x has a unique nearest point, $P_n(x)=\sum_{i=1}^n x_{i,n}u_i$, in $L_n=\text{lin.ext.}[x_1, \cdots, x_n]$. $(L_0=\{\phi\}.)$ The deviation sequence, $\{\delta_i(x)\}_0^\infty$, of x relative to $\{u_i\}_1^\infty$ is defined by: (1) $|\delta_i(x)|=\|x-P_i(x)\|$ and (2) for $\delta_i(x)\neq 0$, sgn $\delta_i(x)=$ sgn $x_{i',i'}$, where i' is the least integer greater than i for which $x_{i',i'}\neq 0$. If $\{\delta_i(x)\}_0^\infty=\{\delta_i(y)\}_0^\infty$ implies that $x=y$, then the T-system $\{u_i\}_1^\infty$ is called a Bernstein system. Kadeč [Doklady Akad. Nauk N.S. vol. 92 (1953) pp. 465–468] has shown that any two infinite dimensional Banach spaces with Bernstein systems are homeomorphic and has stated as an open question: Is every T-system a Bernstein system? An affirmative answer to this would have shown that all separable infinite dimensional Banach spaces are homeomorphic. This paper exhibits a T-system which is not a Bernstein system, using c_0 with its usual basis and the T-norm of Nikol'skiĭ [Doklady Akad. Nauk N.S. vol. 59 (1948) pp. 639–642]. (Received November 13, 1956.)

240. D. B. Lowdenslager: *A Poisson formula and an extended class of functions for symmetric bounded complex domains.*

Let D be a Hausdorff space, and $B\subset D$. Suppose G is a compact transformation group on D, transitive on B, and leaving some point $z\in D$ fixed. Let F be a linear set of real continuous functions on D, containing the constants and invariant under G. If $|f(z)| \leq \max\{|f(b)| : b \in B\}$ for $f \in F$, then $f(z) = \int_B f(b)\,dm(b)$, where m is the G-invariant normalized measure on B. A bounded domain in complex n-space is called symmetric if for each point z there is an automorphism of the domain with z as sole fixed point. The formula above applies to the real parts of analytic functions on D, the closure of the domain, for the nonexceptional domains at least. More generally, it applies to the functions satisfying $\Delta f = 0$, where Δ is an invariant Laplacian on the domain, which is always continuous on D. The set B is the distinguished boundary of the domain in either case, and is a proper subset of the topological boundary in general. (Received October 31, 1956.)

241. Werner Meyer-Koenig: *Absolute summability by circle methods.*

Absolute summation by the circle method (or Taylor's method) T_α and similar methods (S_α, T'_α) are studied. These methods prove to be absolutely regular for the same orders for which they are regular. There are immediate consequences, for example (with regard to $|S_\alpha|$, $0<\alpha<1$): If $f(z) = \sum a_n z^n$ and $\sum |a_n| < \infty$, then the Taylor series expansion $\sum b_n(z-z_0)^n$ of $f(1/z)$ about $z_0 = 1/\alpha$ is absolutely convergent for $z=1$. Taking into account also $|E_\alpha|$ and $|B|$ there are numerous inclusion relations such as $|T'_\alpha| \subset |T'_\beta|$, $|T_\alpha| \subset |S_\alpha|$, $|B| \subset |S_\alpha|$ (holding under suitable conditions). Among the results about absolute translativeness the case $|T_\alpha|$ for $\alpha = 1/2$ (in one direction) is of interest. Since not only T_α but each of the above mentioned methods is a circle method in some sense, we shall refer to all of these methods as circle methods. (Received November 15, 1956.)

242. W. R. Mann: *Improved approximations to ordinary and partial differential equations. A comparative study.*

L. Fox (Proc. Roy. Soc. A 190 (1947) pp. 31–59) has introduced a method for improving the finite difference approximations to differential boundary value problems. His improvements are achieved by solving a sequence of problems which successively take into account the higher order differences usually neglected when constructing the approximating finite difference equations. An alternative procedure is to make use of the differential equation in constructing an improved difference equation of low order. This second approach not only reduces truncation error, but sometimes brings about important improvement in stability. (Cf. Jim Douglas, Jr., Jour. of Math. and Phys. XXXV (1956) pp. 145–151). This paper is a comparative study of the two methods. The results show that neither is universally superior to the other, but that each has its peculiar advantages which make it much to be preferred in particular cases. (Received November 7, 1956.)

243t. C. B. Morrey, Jr.: *On the analyticity of the solutions of analytic, nonlinear elliptic systems of partial differential equations.*

It is proved that if the vector $u^1, \cdots, u^N$ is a solution of an analytic nonlinear elliptic system of the type considered by Douglis and Nirenberg (Comm. Pure Appl. Math. vol. 8 (1955) p. 532) and if each u^j has Holder continuous derivatives of order t_j (in the notation of the cited paper) in a domain D, then the u^j are analytic in D. If the system is strongly elliptic in the sense that the equations of variation are strongly elliptic as defined by Nirenberg (Comm. Pure Appl. Math. vol. 8 (1955) pp. 649–675), the components u^j of a solution have continuous derivatives of order $s+s_j$ ($s = \max s_j$) in the neighborhood of and up to an analytic portion Γ of the boundary of D, and if the derivatives of u^j of order p, $0 \leq p \leq s_j - 1$, are analytic along Γ, then the u^j can be continued analytically across Γ. The proof is much simpler than the proof of the slightly less general theorem of Petrowsky. The proof involves a simple implicit function theorem in Banach spaces (a la Hildebrandt-Graves) and the extension of certain generalized potentials to the complex domain (see E. E. Levi, 1907, E. Hopf, 1932). (Received November 13, 1956.)

244*t*. T. S. Motzkin and J. L. Walsh: *Location of zeros of infrapolynomials.*

If there exists a function $f(z)=(z^m+\cdots)/(z^n+\cdots)\not\equiv 1$ with $|f(z)|\leqq 1$ on a closed set E, E is called a *substar of degree n*. If E is perfect and F a sum of components of the complement of E such that $E+F$ is not a substar of degree n, then $z^n+\cdots$ with no zero on F is a strong infrapolynomial on E. If E is a noncollinear closed set containing the nonempty boundary of its convex hull H, then the infra-polynomials on E and H are identical. There exist sets E for which the set of infrapolynomials is not closed. Let E be $0\geqq y\geqq A$ $(0\geqq A\geqq -\infty)$ plus $z=i$; a boundedly generated infrapolynomial on E has its zeros in $1\geqq y\geqq A$ with sum of ordinates of zeros in $1\geqq y>0$ less than one unless i is the only such zero, and conversely. Let C_t be the disc $|z-t\alpha|\leqq tr$, $0<a\leqq t\leqq b$, $|\alpha|>r$; if E lies in C_a and C_b, all zeros ζ_j of an arbitrary infrapolynomial on E lie in C_a, C_b, and some C_t; if no C_t containing ζ_1 cuts C_a or C_b, all other ζ_j lie in C_a and C_b. (Received November 20, 1956.)

245. J. C. C. Nitsche: *A bound for the Gaussian curvature of minimal surfaces $z=z(x, y)$.*

Let $z=z(x, y)$ be a minimal surface defined over the circle $C[(x-x_0)^2+(y-y_0)^2<R^2]$. Consider the transformation (*) $\xi=x+\int_{(x_0,y_0)}^{(x,y)}W^{-1}[(1+p^2)dx+pqdy]$, $\eta=y+\int_{(x_0,y_0)}^{(x,y)}W^{-1}[pqdx+(1+q^2)dy]$. Here $W=(1+p^2+q^2)^{1/2}$. (*) defines a one-to-one mapping of C onto a domain Γ in the ξ,η-plane conformal with respect to the metrics induced by the minimal surface. Γ contains at least the circle $(\xi-x_0)^2+(\eta-y_0)^2<R^2$, and the function $F(\zeta)=(p-iq)(1+W)^{-1}$ is an analytic function of the variable $\zeta=\xi+i\eta$ in Γ (see a preceding abstract by the author). For the Gaussian curvature $K=(rt-s^2)\cdot W^{-4}$ of the surface one obtains the expression $K=K(x, y)=-|dF/d\zeta|^2\cdot(1+1/W)^4$. As $|F|<1$ in Γ one has in $\zeta_0=x_0+iy_0$: $|F'|\leqq[1+((W-1)/(W+1))^{1/2}]R^{-1}$. Therefore in ζ_0 the following inequality holds: $|K|\leqq[1+((W-1)/(W+1))^{1/2}]^2\cdot(1+1/W)^4\cdot R^{-2}<20\cdot R^{-2}$. In particular, if the x, y-plane is tangential in (x_0, y_0), $|K|<16\cdot R^{-2}$. For similar inequalities see E. Heinz (Goett. Nachr. Math.-Phys. Kl. IIa, 1952) and E. Hopf (J. Rat. Mech. Anal. vol. 2). (Received December 10, 1956.)

246. Robert Osserman: *On a nonlinear differential inequality.*

The inequality considered is (1) $\Delta u\geqq f(u)$, where Δu is the Laplacian in n dimensions. It is proved that if $f(u)$ is a differentiable function with $f(u)>0$ and $f'(u)>0$ for all u, then necessary and sufficient that there exist a solution of (1) throughout space is that $\int^\infty(\int^t f(u)du)^{-1/2}dt=\infty$. Somewhat weaker results were proved in 2 dimensions by Wittich (*Ganze Lösungen der Differentialgleichung* $\Delta u=e^u$, Math. Zeit. vol. 49 (1944)) and in 3 dimensions by Haviland (*A note on unrestricted solutions of the differential equation* $\Delta u=f(u)$, J. London Math. Soc. vol. 26 (1951)). The proof uses the maximum principle for subharmonic functions, and has several advantages over the method of integral means used by Wittich and Haviland. Namely, it is equally applicable in any number of dimensions, it eliminates the requirement that $f(u)$ be convex needed by Wittich and Haviland, and it gives best possible bounds on the largest circle in which a function can satisfy (1), as well as pointwise bounds on a function known to satisfy (1) throughout a given circle. (Received November 9, 1956.)

247. C. Y. Pauc: *Generalised Jacobians as Radon-Nikodym integrands.*

The setting is Cesari's *Surface area*, Annals of Mathematics Studies vol. 35, Princeton, 1956. The fundamental cell A is a compact polygonal region in E_2, the mapping T of A into E_3 of finite area, μ the Borel measure in E_2. The cells are the closed polygonal regions in A. To each whole Borel set B_0 of A there corresponds a vector $\phi_0=\phi_0(B_0)$ whose components are the algebraic area-measures of the (T_i, B_0). J_0 denotes the Radon-Nikodym integrand of $\phi_0|\mathfrak{B}_0$ with respect to $\mu_0=\mu|\mathfrak{B}_0$ (restriction of μ to $\mathfrak{B}_0$), $J(w)$ the Jacobian vector in $w \in A$. (I) *Theorem* 1: If there exists a thin net (monotone sequence of cell partitions of A whose boundaries are thin), for which the sequence of the euclidean norms in E_2 tends to 0, then $J_0=E\{J|\mathfrak{B}_0\}$. (II) If (T, A) is an A.C. representation of a surface S of finite area, the cell function $\gamma(I)$ whose components are the algebraic areas of the (T_i, I), can be extended in a unique manner to a measure ϕ defined on $\mathfrak{B}$ and $\phi(B)=\mu$-integral of $J(w)$ over B, $\phi_0=\phi|\mathfrak{B}_0$. If M is any Borel set in E_3, $\Omega(M)=\phi(T^{-1}(M \cap [S])$, the Radon-Nikodym theorem applied to Ω and $\|\Omega\|$ yields a L. C. Young field. *Theorem* 2: If $F(p, d)$ is a Calculus of Variations integrand, $\mathfrak{P}$ a finite partition of A into Borel sets $B_1, \cdots, B_n$, w_i a point of B_i, $i=1, \cdots, n$, $\sum F(T(w_i), \phi(B_i))$ tends to the Cesari-Weierstrass integral of F over S when the norm of $\mathfrak{P}$ tends to 0. (Received November 5, 1956.)

248. L. E. Payne and H. F. Weinberger (p): *A generalized Rellich identity.*

The authors' upper and lower bounds for solutions of Laplace's equation and quadratic functionals associated with them [J. of Math. and Phys. vol. 33 (1955) pp. 291–307] depend upon an integral identity due to F. Rellich. In consequence, they apply only to domains with star-shaped boundaries. Rellich's identity is generalized by applying Gauss's theorem to the identity $[(a^k g^{ij}-a^j g^{ik}-a^i g^{jk}) u_{,i} u_{,j}]_{,k} = -2a^j u_{,j} \Delta u+(a^k g^{ij}-a^j g^{ik}-a^i g^{jk})_{,k} u_{,i} u_{,j}$ where , denotes covariant differentiation and Δ the Beltrami operator with respect to a metric g_{ij}, while a^k is an arbitrary vector field. If, in particular, a_k is a unit normal on the boundary, the boundary integral obtained from the left-hand side is the integral of $|\text{grad } u|^2-2(\partial u/\partial n)^2$, while the right-hand side is easily bounded above and below by multiples of Dirichlet's integral. Thus the bounds given in the above-mentioned paper can be generalized to general boundaries and Beltrami operators. The Rellich identity follows from the above when g_{ij} is Euclidean and $a^k=x^k$. (Received November 13, 1956.)

249. R. N. Pederson: *A unique continuation theorem for solutions of a quasi-linear biharmonic partial differential equation.*

Let $\Delta^2 u=f(x, u, p_i, p_{ij}, p_{ijk})$ be defined in a domain D in Euclidean n-space, $x \in D(\Delta=\sum \partial^2/\partial x_i^2, p_i=\partial u/\partial x_i$, etc.). We assume the function f to be uniformly Lipschitz continuous in the arguments u, p_i, p_{ij}, p_{ijk}. The Unique Continuation Theorem for solutions of the above equation is proved by using the following inequality. Let $r=(\sum x_i^2)^{1/2}$. If $u \in C^4$ vanishes near $r=0$ and for $r \geqq R$, then $\int \exp(\alpha/r) \cdot(\Delta^2 u)^2 dV \geqq (c/R^2) \int \exp(\alpha/r)\left|\sum_{ijk} \partial^3 u/\partial x_i \partial x_j \partial x_k\right|^2 dV$ for all sufficiently large α. The proof of this inequality makes use of an inequality previously proved by the author (see Bull. Amer. Math. Soc. 62-6-699). (Received November 13, 1956.)

250t. Pasquale Porcelli: *On the existence of the Stieltjes-Mean integral.*

All integrals in this paper are Stieltjes-Mean integrals obtained by the finer subdivision process, and all functions considered are bounded. *Definition*: Let f be a function on $[a, b]$, g nondecreasing on $[a, b]$, and $k>0$, then $M(f, k+) \subset [a, b]$ such that $x \in M(f, k+) \Leftrightarrow x \neq b$ and for each (x, y) there exist points t and t' in (x, y) such that $|f(t)-f(t')| \geq k$. $M(f, k-)$ is defined in a similar manner and $M(f, k)$ is the logical sum $M(f, k+)+M(f, k-)$. $M(f, k)$ is said to have directed g-content zero (hereafter denoted by $C_g M(f, k)=0$) if for each $\epsilon>0$, there exists a finite collection $\{[a_p, b_p]\}$ of nonoverlapping intervals covering $M(f, k)$ such that (i) $\sum [g(b_p)-g(a_p)]<\epsilon$, (ii) $b_p \notin M(f, k+)$, and (iii) $a_p \notin M(f, k-)$. *Theorem* 1. $\int_a^b f dg$ exists if, and only if, $C_g M(f, k)=0$ for every $k>0$. *Theorem* 2. If g is of bounded variation, then $\int_a^b f dg$ exists if, and only if, $\int_a^b f dg'$ exists where g' is the variation function of g. *Theorem* 3. If $\int_a^b f dg$ exists, then $\int_a^b |f| dg$ exists. In the case where g is semi-regular (i.e. $g(x) \neq g(x-)$ if, and only if, $g(x) \neq g(x+)$) $C_g M(f, k)=0$ can be replaced by $1_g M(f, k)=0$, where $1_g M(f, k)$ denotes the outer g-length of $M(f, k)$. (Received November 8, 1956.)

251. Carlo Pucci: *Bounds of solutions of boundary problems for elliptic equations.* Preliminary report.

Let A be an open bounded set of R^m, $f(x, u, u_i, u_{ij})$ a real function for $x \in \bar{A}$, u_i, $u_{ij} \in R$ $(i, j=1, 2, \cdots, m)$. Let f and the first derivatives of f with respect to u, u_i, u_{ij}, be continuous in $\bar{A}$ and let $f'_u \leq 0$, $\sum f'_{u_{ij}} \lambda_i \lambda_j \geq \rho |\lambda|^2$ $(\rho>0)$. Bounds are established for the maximum of the absolute value of solutions of the Dirichlet, Neumann and mixed problems for the differential equation $f(x, u, u_i, u_{ij})=0$, $x \in A$. In the case of the Neumann problem it is supposed also that $f'_u<0$ at some points of A. The uniqueness of solution is also proved. The results are an extension of Hopf's theorem regarding the Dirichlet problem (Sitzungb. Preuss. Akad. Wiss. vol. 19 (1927) pp. 147–152) and of previous results of the author for linear equations (Rend. Acc. Lincei 8, 13, (1952) pp. 360–366). As a particular case one obtains the following theorem: Let D and N be two sets, $D \cap N=\phi$, $D \cup N=\dot{A}$, $D \neq \phi$. The assumption is made that, in every point x of N, $\dot{A}$ has a tangent hyper-plane and a sphere S exists with $x \in S$, $S \in \bar{A}$. Let u be a function of class $C^{(2)}$ in A satisfying the considered differential equation, continuous in $\bar{A}$ and with normal derivative equal zero on N. If $f(x, 0, 0, 0)=0$ the maximum (minimum) value of u is assumed on D or is zero. (Received November 14, 1956.)

252t. P. V. Reichelderfer: *A study of the essential Jacobian in transformation theory.*

Let T be a continuous transformation from a bounded domain D in Euclidean n space R^n into a bounded portion of R^n. For a function of domains D contained in D the concept of a T derivative at a maximal model continuum C for T is introduced, wherein the domains D containing C follow the contour of the transformation as their images TD close down on the point TC in a regular manner. Assume that T is essentially of bounded variation in D and consider the function $\mathfrak{A}_e$ of domains whose n dimensional derivative is the essential Jacobian (T. Rado and P. Reichelderfer, *Continuous transformations in analysis, with an introduction to algebraic topology*, Die Grundlehren der Mathematischen Wissenschaften vol. 75 (1955) Springer-Verlag). If C is a nonessential maximal model continuum for T then $\mathfrak{A}_e$ has a T derivative at C whose value is zero. If C is an essential maximal model continuum for T not belonging to a certain exceptional set then $\mathfrak{A}_s$ has a T derivative at C whose nonzero value is the product of the local topological index for T at C and the local magnification ratio for T at C. (Received November 2, 1956.)

253. Walter Rudin: *The closed ideals in an algebra of analytic functions.*

Let K and C be the closure and boundary, respectively, of the open unit disc U. Let A be the Banach algebra whose elements are those continuous functions on K which are analytic in U, with norm $\|f\| = \max |f(z)|$ $(z \in K)$. A function of the form $M(z) = B(z) \exp \{ -\int_C (w+z)(w-z) d\lambda(w) \}$, where B is a Blaschke product and λ is a non-negative measure on C, is said to be associated with the subset E of C if all limit points of the zeros of B are in E and if λ is concentrated on E. *Theorem*: Suppose E is a closed subset of C, of Lebesgue measure zero, M is associated with E, and $I(E, M)$ is the set of all $f \in A$ such that f/M is bounded in U and such that $f(z) = 0$ on E. Then $I(E, M)$ is a closed ideal of A, and every closed ideal of A (with the exception of the null ideal) is obtained in this way. *Consequences*: (1) Every closed ideal of A is principal. (2) $I(E, M)$ is the intersection of maximal ideals of A if and only if M is a Blaschke product without multiple zeros (in particular, λ must be zero). (3) $I(E, M)$ is the intersection of primary ideals of A if and only if λ is a discrete measure. (Received November 13, 1956.)

254t. Diran Sarafyan: *Existence of integral curves for first order ordinary differential equations.*

In Cauchy's and Picard's proofs of the existence of a solution to $dy/dx = f(x, y)$ there is, besides the continuity and single-valuedness of $f(x, y)$ another condition. Peano's proof is without this additional restriction but is somewhat involved. This paper is a proof which like Peano makes use of polygonal lines but it is a great deal simpler. Let m and M be the minimum and maximum values of $f(x, y)$ on the closed simply-connected region R. From the given point $P_0(x_0, y_0)$ belonging to R two straight lines with slope m and M are drawn. These lines intersect the vertical through x in N_1 and N_2. To a polygonal line Γ_j there corresponds on the segment N_1N_2 a terminal point T_j. By Bolzano-Weierstrass Theorem it is known that the bounded infinite set of terminal points $\{T_n\}$ has at least one limit point. The author shows that if $\{T_n\}$ has j limit points then through P_0 and for the interval $[x_0, x]$ there are j integral curves. (Received November 27, 1956.)

255t. Diran Sarafyan: *One-parameter family of integral curves.*

If $f(x, y)$ is a single-valued continuous function of x and y on a simply-connected closed region R, then it is known that $dy/dx = f(x, y)$ offers at least one integral curve through any point $P_0 \in R$. The aim of this paper is to show that from all the integral curves through the region R, a one-parameter family of integral curves can be extracted. The approach is three dimensional since z is substituted to dy/dx. Then $dy/dx = f(x, y)$ becomes $z = f(x, y)$. An integral curve through P_0 is projected first on the surface $z = f(x, y)$ and then projected on the x, z-plane. Through the use of the series $y = y_0 + z_0(x_1 - x_0) + z_1(x_2 - x_1) + \cdots + z_{n-1}(x - x_{n-1})$ and considering in the x, z-plane the area under the projected curve, corresponding to the interval $[x_0, x]$. the author shows $y = \phi(x, k)$ where k is a parameter. (Received November 27, 1956.)

256t. Diran Sarafyan: *The singular points of first order ordinary differential equations.*

The aim of this paper is the determination of the singular points of $dy/dx=f(x, y)$ particularly if the equation is of the variables separable, homogeneous or linear form. A broad criterion of uniqueness for solutions is given: the necessary and sufficient condition under which through any point of a closed simply-connected region R, there exists a unique integral curve is that $\exp \int_a^x \partial f dx/\partial y \neq 0$ where a is an arbitrary constant which in most cases can be taken as equal to zero. The class of differential equations which satisfy this criterion contains as a subclass the equations which satisfy the well known Lipschitz condition. Then, two following theorems are proved. Theorem: The singular points of the differential equations of the variables separable form $dy/dx=h(y)/g(x)$ are represented by the intersection points of the straight lines, $x=\alpha_n$, $n=1, 2, 3, \cdots$; $y=\beta_m$, $m=1, 2, 3, \cdots$ where α's and β's are the roots, if any of $g(x)=0$ and $h(y)=0$. Theorem: The integral curves of the homogeneous equation $Mdx+Ndy=0$ admit singular points whenever $xM+yN=0$ admits finite or infinite number of linear solutions $y=\gamma x$. It is on these lines which are also integral curves that the singular points are located. (Received November 27, 1956.)

257. Morris Schreiber: *Representation of non-normal operators.*

The results reported in [Bull. Amer. Math. Soc. Abstract 62-3-333; see also 61-2-290] are extended as follows (terminology is that of the references). If A is a proper contraction (that is, $\|A\|<1$) or a generalized nilpotent, then for any $f\in L_2(|z|=1, d\theta)$ the operator $\int_C f(z)dF(z)$, where F is the strong operator measure of A, is bounded, with norm $\leq\|f\|_2$, so that the map $f\rightarrow\int_C f(z)dF(z)$ is a continuous * vector space homomorphism of L_2 into the uniformly closed ring generated by A. Restricted to those f whose Fourier coefficients c_n vanish for $n<0$ the map also preserves products. It is also shown that, for fixed $f\in L_2$ and A varying over operators of the aforementioned types, the map $(A, A^*)\rightarrow\int_C f(z)dF(z)$ is continuous in the uniform, strong, and weak topologies (see Kaplansky, Pacific J. Math. vol. 1 (1951) p. 227). (Received November 14, 1956.)

258. Edward Silverman: *Morrey's representation theorem.*

Morrey showed that a nondegenerate Frechet surface of finite Lebesgue area has a quasi-conformal representation upon the unit circle. Let p_i be a dense sequence of points in the point set covered by the surface. Let x be a representation of the surface and define $x_i(w)=\|p_i-x(w)\|$. A formulation of quasi-conformality which is valid for surfaces not in Euclidean space and can be used to find a "good" representation of the surface is $\sup (x_{iu}^2+x_{iv}^2)=\sup (x_{iu}x_{kv}-x_{iv}x_{ku})$ a.e. The definition of area is that given in *Definitions of Lebesgue area for surfaces in metric spaces*, Rivista di Matematica della Universita di Parma vol. 2 (1951). This definition agrees with that of Lebesgue for surfaces in Euclidean space. For a "good" representation, the classical integrand is replaced by the right hand side of the equation given. (Received November 15, 1956.)

259. K. T. Smith: *An inequality for the Dirichlet integral.*

If $y=(y_1, y_2 \cdots, y_n)$, let $\tilde{y}=(y_1, y_2, \cdots, -y_n)$; and let R^n_+ denote the half space $y_n>0$. The following inequality for the Dirichlet is proved: there is a constant c such that for $0<\alpha\leq n$, $\int_{R^n_+}\int_{R^n_+}|u(x)-u(y)|^2|x-y|^{\alpha-n}|x-\tilde{y}|^{-\alpha-2}dxdy \leq(c/\alpha)\int_{R^n_+}\sum|\partial u/\partial x_k|^2dx$. The constant in the inequality has the right order of magnitude as $\alpha \to 0$. In the inequality, R_+^* can be replaced by any bounded domain $D \subseteq R_+^*$ whose boundary is of class $C^{(0,1)}$. (Received November 13, 1956.)

260. T. H. Southard: *A bivariate analogue of the Aitken-Neville interpolation algorithm*. Preliminary report.

A. C. Aitken (Proc. Edinburgh Math. Soc. vol. 3 (1932) pp. 56–76) and E. H. Neville (Journal, Indian Math. Soc. vol. 20 (1933) pp. 87–120) have defined a process for construction of a univariate polynomial interpolant by means of a set of linear interpolations. A bivariate analogue is defined here and its applications are considered. (Received December 6, 1956.)

261*t*. E. J. Specht and H. T. Jones: *The complete continuity of the Neumann-Poincaré integral operator*.

Warschawski (Applied Mathematics Series 42, National Bureau of Standards) has introduced a Hilbert space H in connection with the numerical solution of a Neumann-Poincare integral equation. The present paper gives a new proof that the integral occurring in this equation gives rise to a completely continuous operator in H. This proof is carried through for multiply connected regions in the plane, uses only the existence of the solution of the exterior Dirichlet problem with constant boundary values and hence generalizes to 3-space. (Received November 14, 1956.)

262. W. F. Stinespring: *Integration theorems for gages and duality for unimodular groups*.

The author proves analogues of ordinary integration theorems not investigated in the paper of I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. vol. 57 (1953) pp. 401–457. Among these are a dominated convergence theorem, a Fatou's lemma, a Fubini theorem, and a theorem giving conditions under which a positive central linear functional on a self-adjoint operator algebra is the integral with respect to a gage on the weak closure. Using gage spaces, a global kind of harmonic analysis is developed for unimodular groups. Many of the familiar relations between functions on an abelian group and functions on the character group hold here between functions on a unimodular group G and operators in the "dual gage space" E. A new binary operation and involution are defined in $L_1(E)$ so that it becomes a Banach algebra with an involution, analogous to L_1 of the character group under convolution and the corresponding involution in the abelian case. In terms of this $*$-algebra $L_1(E)$, a duality theorem analogous to the Tannaka theorem is proved: there is a 1-1 correspondence between elements of G and nonzero $*$-homomorphisms of $L_1(E)$ into the complex numbers. (Received November 9, 1956.)

263*t*. C. T. Taam: *On the solutions of nonlinear differential equations*. III.

Consider (I): $x''+px+qx^3=f$, where $p(t)$, $q(t)$ and $f(t)$ are real-valued, periodic, bounded and Lebesgue-measurable functions with a common least period L. It is further assumed that $p(t)$ and $q(t)$ are even functions with positive lower bounds and $f(t)$ an odd function with a non-negative lower bound on $(0, L/2)$. Several criteria for the existence of periodic solutions, harmonic and subharmonic of order n, are obtained and their oscillatory behaviour is described. These criteria are expressed in terms of L and of the bounds of p, q and f and become very simple when q or f are small perturbations. The method of proof is to show that under suitable conditions there exists a solution of (I) which vanishes at two given points. (This research was supported by the office of Ordnance Research, U. S. Army, Contract No. DA-36-034-ORD-1774.) (Received November 13, 1956.)

264. S. E. Warschawski: *A perturbation method in conformal mapping.*

The mapping function $w=f(z)$ of the interior of a smooth curve $C: z=z(t)$, $0\leqq t\leqq 1$, onto the circle $|w|<1$ may be obtained as a solution of the Lichtenstein-Gershgorin integral equation. If $\theta(t)=\arg f(z(t))$, then $\theta(t)=\int_0^1 K(t,\tau)\theta(\tau)d\tau+g(t)$ where $K(t,\tau)=\pi^{-1}\arg[z(\tau)-z(t)]/\partial t$ and $g(t)$ is known. Suppose $C_\epsilon: z=z(t,\epsilon)$, $0\leqq t\leqq 1$, $0\leqq\epsilon\leqq 1$, denotes a family of contours such that for some integer n, $z(t,\epsilon)=\sum_{k=0}^{n} z_k(t)\epsilon^k+o(\epsilon^n)$ and $z_t(t,\epsilon)=\sum_{k=0}^{n}\dot z_k(t)\epsilon^k+o(\epsilon^n)$, where $z_t(t,\epsilon)$ and $\dot z_k(t)$ are Lip (α), for some α, $0<\alpha\leqq 1$. If $f(z,\epsilon)$ denotes a properly normalized mapping function for C_ϵ and $\theta(t,\epsilon)=\arg f(z(t,\epsilon),\epsilon)$, then there exist functions $\theta_k(t)$, $k=0, 1, \cdots, n$, such that $\theta(t,\epsilon)=\sum_{k=0}^{n}\theta_k(t)\epsilon^k+o(\epsilon^n)$ where the $\theta_k(t)$ is a solution of $\theta_k(t)=\int_0^1 K_0(t,\tau)\theta_k(\tau)d\tau+g_k(t)$. Here $K_0(t,\tau)$ is the kernel of C_0 and $g_k(t)$ is known if $\theta_0, \cdots, \theta_{k-1}$ are determined. The error term is estimated in terms of the errors in $z(t,\epsilon)$ and $z_t(t,\epsilon)$. This result entails: an extension of Hadamard's variation formula for the Green's function in which the higher order terms, up to any order n, and the error term are calculated and "Taylor expansions" of $f(z,\epsilon)$ and of its inverse $\phi(w,\epsilon)$ in powers of ϵ with estimate of the remainder term. (Received November 13, 1956.)

265*t*. Alexander Weinstein: *Subharmonic functions and Tricomi equations.*

Let $u^k(x_1, x_2, \cdots, x_m, r)=u^k(x, r)$ be the solution of the singular Cauchy problem for the Euler-Poisson-Darboux equation with $k\geqq 0$, corresponding to the initial values $u^k(x, 0)=f(x)$. The following results have been established by the use of the formulas for u^k and by the fact that u^k is also a solution of a generalized Tricomi equation (A. Weinstein, Communications on Pure and Applied Mathematics, Vol. VII, pp. 105–116, 1954). Let $M(x, r, f)$ be the mean value of f. For $k\geqq m-1$ the maximum and the minimum of u^k are attained for $r=0$. Assuming the conditions $\partial^i M/\partial r^i\geqq 0$, $i=1, 2, \cdots, n$; $n\geqq(m-1-k)/2$ (which are satisfied if the iterated Laplacians $\Delta^i f\geqq 0$) for $0\leqq k<m-1$, then the minimum of u^k is attained for $r=0$. Moreover if $\partial M/\partial r\geqq 0$ for $k\geqq m-1$ and if $\partial^i M/\partial r^i\geqq 0$, $i=1, 2, \cdots, n+1$ for $0\leqq k<m-1$ then $\partial u^k/\partial r\geqq 0$. Under the same conditions u^k is a convex function of r^{1-k}, $k\neq 1$, and a convex function of $\log r$ for $k=1$. These results include the classical theorems on subharmonic functions. (Received November 5, 1956.)

266*t*. G. L. Weiss: *On certain spaces of analytic functions.*

Let $\Phi\not\equiv 0$ be a non-negative, nondecreasing, convex function on the entire real line such that $\Phi(x)\rightarrow 0$ as $x\rightarrow-\infty$. The space H_Φ is then defined to be the class of all functions $F(z)$, analytic in the circle $|z|<1$, such that $\mu_\Phi(r; F)=(1/2\pi)\cdot\int_0^{2\pi}\Phi(\log|F(re^{ix})|)dx$ is bounded for $0\leqq r<1$. These function spaces generalize H_p spaces in the same direction that Orlicz spaces generalize L_p spaces. Using $\|F\|_\Phi=\inf[|\lambda|; \mu_\Phi(r; (1/\lambda)F)\leqq 1 \text{ for } 0\leqq r<1]$ as a generalization of the H_p norm, one can show that many of the well known properties of H_p spaces, $p>0$, are satisfied by H_Φ spaces. (Received November 5, 1956.)

267. John Wermer: *Function rings on the circle.*

Let C be the normed ring of all continuous complex-valued functions on the unit circle S_0 and let A be a closed subring of C which separates points on S_0 and contains all constants. A is said to be of *type* R provided there exists a homeomorphism χ of S_0 on a simple closed curve γ which lies on a Riemann surface F such that γ bounds a region D on F with $D \cup \gamma$ compact and for each g in A the function $g(\chi^{-1})$ defined on γ is the boundary function of a function analytic in D and continuous in $D \cup \gamma$. A ring of type R cannot equal all of C. Let now ϕ and f be a pair of functions in C which together separate points on S_0 and denote by $A(\phi, f)$ the smallest closed subring of C containing ϕ, f and 1. Assume that ϕ and f are both analytic at each point of S_0, that $\phi' \neq 0$ everywhere on S_0, and that ϕ does not map distinct arcs on S_0 onto the same arc. *Theorem*: Under these hypotheses, either $(A\phi, f) = C$ or $A(\phi, f)$ is of type R. (Received November 13, 1956.)

268. Harold Widom: *Extreme eigenvalues of Toeplitz matrices.*

Let $f(\theta) \sim \sum c_n e^{in\theta}$ be real and continuous, and assume $f(\theta)$ attains its maximum M at a unique point (mod 2π), say at $\theta = 0$. Assume further that $f(\theta)$ is even, has a continuous fourth derivative in a neighborhood of $\theta = 0$, and that $f''(0) \neq 0$. Let the eigenvalues of the Toeplitz matrix (c_{i-j}) $(i, j = 0, \cdots, n)$ be $\lambda_{1,n} \geq \lambda_{2,n} \geq \cdots \geq \lambda_{n+1,n}$. Then for fixed k, $\lambda_{k,n} = f(k\pi(n+2)^{-1} + k\pi(\alpha + \epsilon_n)(n+2)^{-1})$, where $\epsilon_n \to 0$ as $n \to \infty$ and $\alpha = 1 + (2\pi)^{-1} \int_0^\pi \csc^2 \theta/2 \cdot \log\left[(2f''(0))^{-1}(f(\theta) - M) \cot^2 \theta/2\right] d\theta$. (Received October 26, 1956.)

269*t*. J. C. Wilson: *A note concerning an nth order perturbed equation.*

A sufficient condition is obtained whereby one is assured that there exists, for each value of μ, a unique solution, through n points, of the equation $x^{(n)} + \sum_{i=1}^{n} a_i(t) x^{(i-1)} = \mu f(t, x, \cdots, x^{(n-1)})$, μ small. The sufficiency is shown to be equivalent to the uniqueness of a solution of the associated homogeneous equation through these same points. The method of obtaining the result employs standard procedures met with in the study of perturbed systems of differential equations. (Received October 25, 1956.)

270. Mishael Zedek: *Strong infra-(n, s)-polynomials*. Preliminary report.

Given a compact point set E in the z-plane, integers n and s $(0 \leq s < n)$ and complex numbers $A_1, A_2, \cdots, A_s$, the polynomial $S_n^s(z) = z^n + A_1 z^{n-1} + \cdots + A_s z^{n-s} + a_{s+1} z^{n-s-1} + \cdots + a_n$ is called a *strong infra-(n, s)-polynomial* on E, if it has no *weak underpolynomial* on E, that is if there is no other polynomial of the form $P_n^s(z) = z^n + A_1 z^{n-1} + \cdots + A_s z^{n-s} + c_{s+1} z^{n-s-1} + \cdots + c_n$, such that $|P_n^s(z)| \leq |S_n^s(z)|$ for z on E. Generalized Tchebycheff polynomials $T_n^s(z)$ (compare: Walsh and Zedek, Proc. Nat. Acad. Sci. vol. 42 (1956) pp. 99–104) are strong infra-(n, s)-polynomials. If E is a real pointset, containing at least $n-s$ points and if $A_1, \cdots, A_s$, are real, then all the other coefficients of $S_n^s(z)$ are also real and at least $n-s$ of its zeros lie on the smallest segment containing E. This is a generalization of a theorem by Professor Walsh (same reference). If E is symmetric with respect to the real axis and the prescribed coefficients of the generalized Tchebycheff polynomial $T_n^s(z)$ are real, then the rest of its coefficients are real. The nonreal zeros of $T_n^1(z)$ in this case are on the union of all ("Jensen") discs having as diameter the segment joining a nonreal point of E with its conjugate (compare: Fekete and von Neumann, Jber. Deut. Math. Ver. vol. 31 (1922) pp. 125–138). (Received November 13, 1956.)

271*t*. David Zeitlin: *Plane parabolic functions*. Preliminary report.

Given $y^2 = 2px$, one defines the signed parabolic sector area, $A = \int_0^x y\,dx - 2^{-1}(x - p/2)y$, the parabolic radian, $u = 8A/p^2 = 2(y/p) + 2(y/p)^3/3$ and the six parabolic functions: $\operatorname{sinp}(u)=y/p$, $\operatorname{cosp}(u)=2x/p$, $\operatorname{tanp}(u)=\operatorname{sinp} u/\operatorname{cosp} u=p/y$, $\operatorname{cotp}(u)=y/p$, $\operatorname{secp}(u)=p/2x$ $\operatorname{cscp}(u)=p/y$. Then $u=2\operatorname{sinp} u+2\operatorname{sinp}^3 u/3$ $\operatorname{sinp}(-u)=-\operatorname{sinp}(u)$, $\operatorname{cosp}(-u)=\operatorname{cosp}(u)$, $\operatorname{sinp}^2 u=\operatorname{cosp} u$, and for $-\infty<u<\infty$, $-\infty<\operatorname{sinp} u$, $\tan \mathrm{p}\, u$, $\operatorname{cotp} u$, $\operatorname{cscp} u<\infty$ and $0\leqq\operatorname{cosp} u$, $\operatorname{secp} u<\infty$. If $u=(4\tan\theta)/3$ $-90°\leqq\theta\leqq 90°$, then $\operatorname{sinp}((4\tan\theta)/3)=(\tan\theta+\sec\theta)^{1/3}+(\tan\theta-\sec\theta)^{1/3}$. One finds that $\operatorname{sinp} u\sim(3u/2)^{1/3}$ as $|u|\to\infty$ and that $\lim_{u\to 0}\operatorname{sinp} u/u=1/2$. The single valued inverse functions are $\operatorname{arcsinp} u=\operatorname{arccotp} u=2u+2u^3/3$ $\operatorname{arccscp} u=\operatorname{arctanp} u=2/u+2/3u^3$, and the double valued inverses are $\operatorname{arccosp} u=\pm(2u^{1/2}+2uu^{1/2}/3)$ and $\operatorname{arcsecp} u=\pm(2/u^{1/2}+2/3uu^{1/2})$. If $\operatorname{sinp} u=\sum_{n=0}^{\infty}a_n u^n$, then $a_{2n}=0$ for all n, $a_1=1/2$, and for odd n, $a_n+3^{-1}\sum_{m=0}^{n}\sum_{p=0}^{m}a_p a_{m-p}a_{n-m}=0$. The solution of the eq. $2(1+y^2)dy=du$, $(0, 0)$, is $y=\operatorname{sinp} u$. One finds that $(\operatorname{sinp} u_1)(\operatorname{sinp} u_2)=\pm 1$, where 1 is for any family of straight lines through $(p/2, 0)$, and -1 for any family of straight lines through $(-p/2, 0)$. The derivatives and integrals of parabolic functions are found. A useful trigonometry is not apparent. (Received November 2, 1956.)

272. A. D. Ziebur: *Separation and comparison theorems for differential equations of higher order.*

Theorems analogous to the familiar results of Sturm on the oscillations of solutions of self-adjoint differential equations of the second order are developed for equations of higher order. They are proved with the aid of certain properties of the eigenvalues of an associated differential problem. There is a close connection with the work of Leighton [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 325–328] on the second order case. (Received October 15, 1956.)

Applied Mathematics

273. M. I. Aissen: *Set functions of convex domains.*

In *Isoperimetric inequalities in mathematical physics*, Polya and Szego use a geometric property of plane convex domains called the "Inclusion Lemma" to derive bounds and approximations for various set functions of plane convex domains. In the discussion, the radii of the least circumscribing and the greatest inscribed circles play a useful role. In this paper the Inclusion lemma is proven for n-dimensional convex domains and is applied to their set functions. The geometric quantities which play the role of the inradius and circumradius are also discussed. (Received November 13, 1956.)

274. R. J. Arms: *Truncation errors involved in using special difference equations.* Preliminary report.

Partial differential equations under consideration were of the elliptic type. A study was made of the error (i.e., the difference between the true solution and the difference solution) induced by the use of somewhat inaccurate difference equations along interior curves across which there was a change in "mesh size." Also the use of inaccurate difference equations along irregular boundaries was considered. By extending theorems of Gerschgorin (Z. Angew. Math. Mech. vol. 4 (1930) pp. 373–382) it was found that the above mentioned types of inaccuracies did not, in many usual situations, change the order of the error. The theorem proven, as that of Gerschgorin, contains assumptions on the differentiability of the true solution. (Received November 8, 1956.)

275. E. H. Bareiss: *Comparison of the theoretical accuracy of the DTMB transport theory codes with the orthogonal polynomial approximations and S_n-approximations.*

A survey is given of three general methods for the actual numerical solution of the Boltzmann Transport Equation. It is shown under which conditions the method of expanding in orthogonal polynomials can be made equivalent to the method of discrete ordinates. A generalization of the S_n-method is suggested in order to obtain the same theoretical accuracy as for the methods mentioned above.

The results of actual numerical computations are exhibited in graphical form. More details, also concerning the space integration, can be found in the report: *Some theoretical aspects of the numerical solution of the one-dimensional Boltzmann transport equation*, AML technical report Number 20, December 1956. (Received November 13, 1956.)

276*t*. A. T. Bharucha-Reid: *Semi-groups as solutions of the Schrödinger equation.* Preliminary report.

Recently semi-group theory has been used rather extensively in the study of Markov processes. While the probability structure of quantum theory differs from that of Markov processes, the quantum mechanical kernel function $K(t, x; y)$ satisfies the Chapman-Kolmogorov equation $K(s+t, x; y)=\int K(s, y; d\xi)K(t, x; \xi)$ which expresses the semi-group property; and the probability amplitude function $\Phi(t, x)$ satisfies $\Phi(s+t, x)=\int\Phi(s, \xi)K(t, x; d\xi)$. Expansion of $\Phi(s+t, x)$ yields the Schrödinger equation $\Phi_t=\Omega\Phi$, where $\Omega\cdot=(ih/2m)(\partial^2\ /\partial x^2)+(i/h)(E-V(x))\cdot$. In this paper we investigate the semi-groups which arise as solutions of the Schrödinger equation when various potential functions $V(x)$ which are of physical interest are introduced in the operator Ω. (Received November 13, 1956.)

277. R. A. Clark (p) and W. E. Gibson: *Asymptotic solution of a non-homogeneous differential equation.* Preliminary report.

The problem is to determine an asymptotic solution of a differential equation of the form $u''(t)+[\lambda^2\phi(t)+\lambda\chi(t, \lambda)]u=\psi(t, \lambda)$ where λ is a large complex parameter, ϕ has a single zero of first order, and $\chi(t, \lambda)=\sum_{n=0}\lambda^{-n/2}\chi_n(t)$, $\psi(t, \lambda)=\sum_{n=0}\lambda^{-n/2}\psi_n(t)$ with χ_n and ψ_n regular functions of a real variable t over a closed interval including the zero of ϕ. A formal solution is derived by two different methods. One method leads to an expansion of the form $u=\lambda^{-4/3}\sum_{n=0}\lambda^{-n/3}\sum_{k=0}^{n}A_{n,k}(t)S^{(k)}(z)$ where $z=(2\lambda\omega/3)^{2/3}$ with $\omega=\int_0^t\phi^{1/2}\,dt$, S is a solution of $S''(z)+zS=1$, and $S^{(k)}$ is the kth derivative of S with respect to z. The other results in a solution analogous to that obtained by R. E. Langer [Trans. Amer. Math. Soc. vol. 67 (1949) p. 461] for the corresponding homogeneous equation. When λ is restricted to appropriate sectors of the complex λ-plane, S may be chosen so that the formal solution is an asymptotic expansion in λ of an exact solution. (Received November 13, 1956.)

278. H. J. Ettlinger (p) and J. L. Cornette. *A reduced linear system from organic chemistry.*

Let $C(t)$ be an $(n+1)\times(n+1)$ matrix of real integrable functions on $0\leq t\leq T$, $T>0$, and $x(t)$ an $(n+1)\times 1$ matrix of real functions $x_i(t)$, absolutely continuous on $0\leq t\leq T$. Consider $x(t)=x_0+\int_0^t Cx$ where x_0 is a column of $n+1$ real numbers, x_{i0}, such that $0 \leq x_{i0} \leq 1$. Further, $c_{ii}(t) \leq 0$, $c_{ij}(t) \geq 0$, $i \neq j$, sum $c_{i\nu}(t) \leq -k$, k a positive number, on $t \geq 0$ and sum $x_\nu(t) = 1$ on $t \geq 0$. This system is reduced by elimination of $x_{n+1}(t)$ to $x(t) = x_0 + \int_0^t [Ax + b]$ where A and b are readily identified The asymptotic and growth properties of $x_\nu(t)$ are discussed. (Received November 9, 1956.)

279. C. L. Dolph (p) and D. C. Lewis: *On the application of infinite systems of ordinary differential equations to perturbations of plane Poiseuille flow.*

In this paper the authors present an outline of their adaptation and application to the problem of the title of the method used by E. Hopf (Mat. Nach., 4, vol. 1–6, 1951, pp. 213–231) to establish the existence of a weak solution to the Navier-Stokes equations and a resume of the numerical results obtained from it to date on the Ramo-Wooldridge 1103 Computer. A solution of the linearized stream equation is sought in the form $\psi(x, y, t) = \exp(i\alpha x) \sum_1^\infty a_n(t)\phi_n(y)$ where the functions $\{\phi_n(y)\}$ are the known eigensolutions of a prescribed free-free beam problem and where the functions $\{a_n(t)\}$ satisfy the infinite system of ordinary linear differential equations with constant coefficients which results when the above expansion is inserted into the Navier-Stokes equations and sampled by the complete other normal family $\{\phi_n(y)\}$. The eigenvalues of this infinite system are shown to be proportional to those of the Orr-Sommerfield equation. Numerical computations indicate instability is present in a truncated six by six system and the truncated eight by eight system yields a curve of neutral stability with minimum critical Reynolds number of the right order of magnitude. Preliminary calculations with a twenty by twenty system indicate good agreement with the results of Lin. (Received November 13, 1956.)

280. Avron Douglis: *An ordering principle and weak solutions of a quasi-linear partial differential equation.*

A "weak" solution in the half-plane $t \geq 0$ of a quasi-linear partial differential equation in two independent variables of the form $u_t + F(u)_x = 0$ is a solution in an integral sense satisfying the "entropy" condition $u(x-0, t) \geq u(x+0, t)$ for $t > 0$. Given any step function $\phi(x)$, a weak solution is easily constructed such that $u(x, 0) = \phi(x)$. Such a weak solution we call a "reference" solution. Under the hypothesis that $F'(u)$ is a bounded, nondecreasing function, reference solutions are partially ordered like their initial data, or, more precisely, if u_1 and u_2 are two reference solutions such that the inequality $u_1 \geq u_2$ holds for $t = 0$, then the inequality holds for $t > 0$ as well. This ordering property makes it possible by simple monotonic processes to construct weak solutions for integrable initial data. The ordering property, moreover, prevails in a much larger class of weak solutions than is constituted by the reference solutions and their limits, and, therefore, uniqueness theorems can be proved for weak solutions of certain types. In this way, in particular, it is seen that smooth solution, when it exists, can be obtained as the monotonic limit of reference solutions. The existence theorems obtained by this procedure are comparable with those of P. D. Lax and the uniqueness theorem with that of O. Oleinik. The ordering principle is believed to be new. (Received November 13, 1956.)

281. G. E. Forsythe (p) and Peter Henrici: *Proof of convergence of the cyclic Jacobi iterative process for computing eigenvalues of a real symmetric matrix.*

Let $A^{(0)} = (a_{ij}^{(0)})$ be a real symmetric matrix of order n whose eigenvalues λ_α are desired. For $k = 0, 1, \cdots$, let $A^{(k+1)} = U_{ij}^T A^{(k)} U_{ij}$, where $U_{ij} = U_{ij}^{(k)}$ is a rotation of the (i, j) coordinate 2-plane through an angle $\theta_{ij}=\theta_{ij}^{(k)}$ chosen so that $a_{ij}^{(k+1)}=0$. C. G. J. Jacobi [J. Reine Angew. Math. vol. 30 (1846) pp. 51–95] proved that one can compute λ_α iteratively by the relation $(\lambda_\alpha\delta_{\alpha\beta})=\lim_{k\to\infty}A^{(k)}$, where at each stage the couple (i, j) is chosen so that $|a_{ij}|=\max_{p\neq q}|a_{pq}|$. But in current automatic digital computer practice [R. T. Gregory, Math. Tables Aids Comput. vol. 7 (1953) pp. 215–220] one often selects the (i, j) in the cyclic order $(1, 2), (1, 3), \cdots, (n-1, n)$, and repeat. Convergence of such a "cyclic Jacobi process" is now proved, apparently for the first time. The key to the proof is the choice of θ_{ij} so that $-\pi/4\leq\theta_{ij}\leq\pi/4$. If one chooses $0\leq\theta_{ij}<\pi/2$, an example $(n=3)$ is given for which the cyclic Jacobi process fails—both with exact and rounded arithmetic operations. The result for hermitian $A^{(0)}$ is proved analogously. Proofs are elementary. (Sponsored by Office of Naval Research and Office of Ordnance Research.) (Received November 5, 1956.)

282. W. T. Guy, Jr.: *On the Mathieu transform.* II.

An integral transform involving Mathieu functions in the kernel is defined. Several properties are developed and used for working a certain class of differential equations. (Received November 13, 1956.)

283. M. A. Hyman: *Concerning the numerical solution of boundary-value problems.*

Attention is focused on the solution of elliptic partial differential problems, such as occur in elasticity, fluid mechanics, and nuclear reactor design. Usually finite-difference equations are introduced, which are then solved by some iterative technique. This paper proposes to use Fox's method (Proc. R. S. London vol. A190 (1947) p. 31), for setting-up the difference equations. Advantages are (1) a relatively coarse mesh can be used (thus speeding convergence of the iterative solution) associated with a small truncation error; (2) irregular boundaries, the use of finer meshes in certain regions, etc. can be handled without using complicated difference approximations or introducing sizeable truncation errors. It is also shown how, by a certain weighted averaging process, a starting approximate solution is obtained which is largely free of the "fundamental" component of the error, which decays most slowly during iteration and determines the rate of convergence. It is pointed out that the "over-relaxation factor" used during iteration, and usually difficult to estimate, is a domain functional for the partial difference operators employed. One can therefore expect to get good estimates using analogs of recently developed methods for partial differential operators. The paper concludes with certain remarks about extrapolation techniques for speeding convergence. (Received September 10, 1956.)

284t. M. A. Hyman: *Line over-relaxation for solving partial difference equations.* Preliminary report.

Independently Young and Frankel have discovered the effectiveness for solving elliptic partial difference equations of what is here called (modifying Young's terminology) "successive point over-relaxation" (SPOR). Recently Peaceman, Rachford and Keller, Rose have considered two different forms of "successive line over-relaxation" (SLOR). Regarding an iteration (during which each field-point is "swept" once) as a "time-step," relaxation processes such as those above are seen as the orderly "step-wise" solution of certain partial difference equations, first-order (usually, though higher-order equations are possible) in the added variable "time." Stability analysis then yields certain restrictions on the over-relaxation factors and indicates how various components of the error change under relaxation. In particular, the Young-Frankel and Keller-Rose techniques are so studied. The Peaceman-Rachford technique is always stable provided that directions are alternated in traversing the domain; this restriction is usually inconvenient when using automatic computing machines. Certain modifications of the Peaceman-Rachford technique are suggested which tend to avoid this difficulty without causing significant "instability." Finally, it is shown that the Peaceman-Rachford technique, so modified, can be used to solve "transport" problems—in particular those of neutron-transport, usually attacked now with multigroup methods and point-relaxation techniques. (Received September 20, 1956.)

285t. M. A. Hyman: *Finite-difference solution of the heat conduction equation* (*Variable time increments*).

The simplest implicit finite-difference approximant to the partial differential equation $u_t = u_{xx}$ is (*) $w(x, t) - w(x, t-\Delta t) = (\Delta t/\Delta x^2)[w(x-\Delta x, t) - 2w(x, t) + w(x+\Delta x, t)]$, proposed by the author [J. Math. Phys. vol. 23 (1951) p. 223]. (*) is stable for all $r = \Delta t/\Delta x^2$. Douglas, Gallie [Proc. Amer. Math. Soc. vol. 6 (1955) p. 787] have considered numerical solution of $u_t = u_{xx}$ using (*) with certain simple initial and boundary conditions; Δx is constant, but Δt may vary with t. For $r = \alpha + \beta t$ and $r = e^{\gamma t}$ ($\alpha > 0, \beta \geqq 0, 0 \leqq \gamma \leqq \pi^2/2$), they have shown that the maximum truncation error $E = u - w$ in R (the bounded rectangular region within which the solution is sought) is $O(\Delta x^2)$; this implies convergence of w to u in R as $\Delta x \to 0$. In the present paper, the above results are extended as follows: $E = O(\Delta x^2)$ provided that the mesh-ratio r is bounded in R; the variation of r with t in R can be arbitrary. $E \to 0$ (convergence) as $\Delta x \to 0$ if $\max_R \Delta t \to 0$; this is the best possible result. It appears that the methods used can be applied to give similar strong results for other "stepping-ahead" type problems, including other differential equations (not necessarily parabolic) and more complicated initial and boundary conditions. (Received September 20, 1956.)

286. A. B. Lehman: *Lattice order of two-terminal resistor networks.*

Let $P(a_1, \cdots, a_n)$, $Q(a_1, \cdots, a_n)$, $R_1(a_1, \cdots, a_n), \cdots, R_m(a_1, \cdots, a_n)$ be two-terminal network configurations whose branches are $a_1, \cdots; a_n$. Let $\cup$ denote a series configuration and $\cap$ denote a parallel configuration. Let swfn (P) denote the two-terminal switching function of P, and $Z(P)$ denote its two-terminal resistance. *Theorem* 1: swfn $(P(a_1, \cdots, a_n)) \supset$ swfn $(Q(a_1, \cdots, a_n))$ if and only if $Z(P(a_1, \cdots, a_n)) \geqq Z(Q(a_1, \cdots, a_n))$ for all positive values of $Z(a_1), \cdots, Z(a_n)$. *Corollary*: $Z(P(a_1 \cup b_1, \cdots, a_n \cup b_n)) \geqq Z(P(a_1, \cdots a_n) \cup P(b_1, \cdots, b_n))$. *Theorem* 2: swfn $(P(a_1, \cdots, a_n)) \supset$ swfn $(\bigcap_k R_k(a_1, \cdots, a_n))$ if and only if $Z(P(a_1, \cdots, a_n)) \geqq Z(\bigcap_k R_k(a_1, \cdots, a_n))$ for all positive values of $Z(a_1), \cdots, Z(a_n)$. *Theorem* 3: swfn $(\bigcup_k R_k(a_1, \cdots, a_n)) \supset$ swfn $(Q(a_1, \cdots a_n))$ if and only if $Z(\bigcup_k R_k(a_1, \cdots, a_n)) \geqq Z(Q(a_1, \cdots, a_n))$ for all positive values of $Z(a_1), \cdots, Z(a_n)$. Given real numbers $p \geqq r \geqq s \geqq q \geqq 0$ such that $p \leqq r + s + t$ and $1/q \leqq 1/r + 1/s + 1/t$ for some $t \geqq 0$, then a bridge type two-terminal two-switch (four-state) resistor network will realize the four driving point resistances p, r, s, q. Furthermore, the four driving point resistances $p \geqq r \geqq s \geqq q$ of any two-terminal two-switch resistor network must be such that $p \leqq r + s + 2t$ and $1/q \leqq 1/r + 1/s + 2/t$ for some $t \geqq 0$. The proof uses theorems (2) and (3). (Received November 13, 1956.)

287. Emanuel Parzen: *A simple proof and some extensions of the sampling theorem.*

The sampling theorem states essentially that if the frequency spectrum, or Fourier transform, $g(\omega)$ of a time function $f(t)$ vanishes for ω outside some interval I, then

$f(t)$ is completely determined by its values at certain discrete sampling points, whose density is proportional to the length of the interval I. This note gives a method of proof of the sampling theorem, both for the case where the interval I is centered at the origin and where it is not, which is somewhat simpler, than the previously given proofs, and at the same time is more rigorous, and yields several useful generalizations to functions of several variables and random functions. (Received November 14, 1956.)

288. W. H. Peirce: *Numerical integration over the spherical shell.*

In this paper is given a formula for numerical integration over the arbitrary spherical shell. The type of formula developed is a finite weighted sum of integrand values, and the formula is exact whenever the integrand function is a polynomial of arbitrarily high degree. The points at which the integrand function is evaluated are inside the region of integration. Several specific formulas are given, and a brief error analysis is included. (Received November 5, 1956.)

289. W. P. Reid: *Heat flow in a cylinder.*

A formal, analytical solution is obtained to the problem of finding the temperature in a cylinder which is gaining or losing heat by radiation exchange with a concentric, thin-walled metal tube. The tube in turn is radiating to, or receiving radiation from, its surroundings. Because of the boundary condition, the eigenfunctions for this problem are not orthogonal. The modified orthogonality relation which they satisfy is given. The application of Duhamel's theorem is lengthy in this problem, and so a procedure is used which avoids the need for it. (Received November 14, 1956.)

290. George Seifert: *Pendulum-type systems with periodic forcing.*

Consider the system $\dot{\theta} = y$, $\dot{y} = g(\theta) + p(t) - \alpha f(\theta)$, where f, g, and p are of class C_1, $f(\theta) > 0$, $f(\theta + 2\pi) = f(\theta)$, $g(\theta + 2\pi) = g(\theta)$, $p(t + T) = p(t)$, for all θ and t, and if $A = \max |p(t)|$, each function $g(\theta) \pm A$ has at least one simple zero. By means of a procedure similar to one used by Lillo and Seifert, (cf. ZAMP vol. 7 (1956) pp. 239–243), bounded regions R_n in the (θ, y) plane are constructed such that for α sufficiently large, each such region contains the trajectory of a solution of period T, and that each solution is ultimately bounded in either this region, which is of width less than 2π, or another explicit region of the same type. It is also found that for α sufficiently large, a sub-region of each R_n containing a unique periodic trajectory which is stable with respect to all trajectories in this subregion may be constructed. (Received November 13, 1956.)

291*t*. Deonisie Trifan: *On a complementary energy principle.*

A minimal principle in terms of the differential of stress deviation for the so-called "stress theory of plastic flow" is established. The stress-strain relation for this theory is applicable to homogeneous, isotropic, incompressible, strain-hardening materials exhibiting a gradual transition from the elastic to the plastic state. It is proved that the actual differential of stress and corresponding strain deviations occurring in a body under external forces make a volume integral involving admissible differential stress deviations an absolute minimum. (Received November 13, 1956.)

292. Herschel Weil: *On the convergence of the Mie series for scattering spheres.*

G. Mie's classical solution for the electromagnetic field scattered by a sphere irradiated by a plane wave consists of expansions in spherical vector wave functions of integral order, n. These series do not begin to converge until somewhat more than a terms, where a is the sphere radius in units of wavelength/2π, and considerable numerical error can and has been made by not including sufficient terms beyond $n=a$. With the help of bounds on the combinations of cylinder functions involved, a function $N(a, p)$ is obtained with the property that if N terms in the series are computed for a given a, the percentage error in the result will be less than p. $N(a, p)$ is independent of the coordinates of the field point (receiver) and is a good estimate of the smallest permissable number of terms for a given error p. (Received November 14, 1956.)

GEOMETRY

293. Smbat Abian (p) and I. A. Barnett: *Semi-covariants in projective differential geometry.*

Consider the system of differential equations (1) $y_i^{(m)}+\sum_{q=1}^{m}\sum_{j=1}^{n}A_{qij}(x)y_j^{(m-q)}=0$, and the continuous group of transformations (2) $y_i=\sum_{j=1}^{n}K_{ij}(x)\bar{y}_j$ and (3) $\tilde{x}=\xi(x)$. If a function $f(y_i^{(k)},\cdots,A_{1ij}^{(m+r-1)},\cdots,A_{qij}^{(r)},\cdots)$, $(k=0,1,\cdots,H\leqq m-1$, $q=2,\cdots,Q\leqq m$, $r=0,\cdots,R)$ remains unchanged by both (2) and (3), it is said to be a covariant; if unchanged by (2) only, a semi-covariant of height H, class Q and order R of (1). This paper deals with the representation theory of semi-covariants. It is proved that the matrices $F_{q,r}$ of dimension n with elements $F_{q,r;ij}$ and their covariant derivatives obtained from the semicanonical form $z_i^{(m)}+\sum_{q=2}^{r}\sum_{k,j,r=1}^{n}U_{ik}F_{q;kj}V_{jr}z_r^{(m-q)}=0$ of (1) undergo a similarity transformation and the vectors Y_k with their elements y_i and their covariant derivatives undergo the transformation (2). Any semi-covariant f is a solution of the complete system of n^2 equations $S_{pk}(H, Q, R)=\sum_{q=2}^{Q}\sum_{r=0}^{R}\sum_{i,j=1}^{n}(\partial f/\partial f_{q,r;ij})(F_{q,r;ip}\delta_j^k-F_{q,r;kj}\delta_p^i)-\sum_{h=0}^{H}(\partial f/\partial Y_{h;p})Y_{h;k}=0$. The number of independent semi-covariants of height H, class Q and order R of (1) is found to be equal to the expression $(QR+Q-R-2)n^2+(H+1)n$. The explicit expressions for the semi-covariants are obtained as functions of elements of matrices of the form $(Y_0,\cdots,Y_H,\ F_{q,r}Y_0,\cdots)^{-1}F_{q,r}(Y_0,\cdots,Y_H,\ F_{q,r}Y_0,\cdots)$. (Received November 8, 1956.)

294. Iacopo Barsotti: *Repartitions on abelian varieties.*

Let A be an n-dimensional normal variety over the algebraically closed field k; a (*repartition*) *class on* A is a mapping $\mathfrak{b}$ such that (1) for each $(n-1)$-dimensional irreducible subvariety X of A, $\mathfrak{b}X=x(X)+Q(X/A)$, $x(X)\in k(A)$, and (2) $\mathfrak{b}X=Q(X/A)$ for all, but finitely many, X's. The class $\mathfrak{b}$ is *exact* if there is an $x\in k(A)$ such that $\mathfrak{b}X=x+Q(X/A)$ for each X; it is *closed* if for each $P\in A$ there is an exact class $\mathfrak{a}$ such that $(\mathfrak{b}+\mathfrak{a})X=Q(X/A)$ for all $X\ni P$. If A is an abelian variety, let $\mathfrak{B}$, $\mathfrak{B}_0$ be the k-modules of, respectively, the closed, and the exact classes; typical results are: (1) $\mathfrak{B}/\mathfrak{B}_0$ is a k-module of order n; (2) there is an isomorphism α of the k-module $\mathfrak{D}$ of the invariant derivations on A, onto $\mathfrak{B}/\mathfrak{B}_0$, such that, if $p\neq 0$, $\alpha(D^p)=\alpha(D^p)$ for $D\in\mathfrak{D}$; (3) there is a homomorphism β of $\mathfrak{B}/\mathfrak{B}_0$ onto $\mathfrak{D}_2/\mathfrak{D}_1+\mathfrak{D}_e$ (where $\mathfrak{D}_i$, $\mathfrak{D}_e$ are respectively the k-modules of the differentials of the ith kind, or exact), whose kernel is 0 if $p=0$, or $(\mathfrak{B}/\mathfrak{B}_0)^p$ if $p\neq 0$; (4) each $\omega\in\mathfrak{D}_1$ is closed. If A is the jacobian of a curve C, relations are easily established between $\mathfrak{B}/\mathfrak{B}_0$ and the operation $\mathfrak{b}\rightarrow\mathfrak{b}^p$ for repartitions on C, or the dual operation $\omega\rightarrow\omega^{1/p}$ for differentials on C. (Received November 13, 1956.)

295. Eugenio Calabi: *Closed, locally euclidean, 4-dimensional manifolds.*

Let M^q be a closed, flat, q-dimensional manifold (q-fold), G the (Compact Lie) group of all isometries of M^q, r its first Betti number ($=\dim(G)=$ dimension of space of autoparallel, global vector fields), Δ a finite, Abelian subgroup of G with finite centralizer (i.e. leaving no autoparallel vector field in M^q invariant) and of rank $\leqq p$. Then Δ has a faithful representation $\phi: \Delta \rightarrow \Delta'$ into the translation group of T^p ($=p$-dimensional, flat torus). Let Δ'' be the graph of ϕ in $\Delta' \times \Delta$, considered as a group of isometries of $T^p \times M^q$. Then the quotient space $M^{p+q}=(T^p \times M^q)/\Delta''$ is a closed, flat $(p+q)$-fold with first Betti number $=p$. *Theorem* 1: *Every closed, flat n-fold M^n with first Betti number $p>0$ can be constructed by the method just outlined; the manifolds T^p, M^q $(q=n-p)$ and the groups Δ, Δ' are uniquely determined within affine equivalence by the topological invariants of M^n.* From the known classification of closed, flat n-folds $(n \leqq 3)$ it is possible to list all closed, flat fourfolds. *Theorem* 2: *There exist exactly* 31 *topologically inequivalent closed, flat, orientable fourfolds* (*all with first Betti number* $\geqq 1$) *and, not counting those with first Betti number* $=0$, 40 *nonorientable ones.* (Received November 14, 1956.)

296. Michael Goldberg: *Rotors tangent to n fixed circles.*

Every closed curve (a rotor) which remains tangent to n fixed straight lines in a plane during a complete rotation of the curve is a special case of a family derived analytically by Meissner [*Vierteljahrschrift der naturforschenden Gesellschaft*, Zürich, vol. 54 (1909) pp. 309–329]. This paper considers the replacement of the straight lines by circles. This is a generalization of a problem of Lusternik [Uspekhi Matematicheskikh Nauk, vol. 1, parts 3–4 (13–14)(1946) pp. 194–195]. He posed the problem of the determination of all the rotating closed curves which remain tangent to three fixed circles. A partial solution (possibly complete) of this generalized problem is derived from the known rotors in regular polygons. A direct kinematic derivation is shown for these general rotors. Several special extremal cases are shown. (Received October 30, 1956.)

297t. Donald Greenspan: *On vertices in euclidean 3-space.*

A space curve C: $\mathbf{x}=[x^1(s), x^2(s), x^3(s)]$, is called a curve of type I if and only if: (a) $x^i(s)$ is analytic for each $i=1, 2, 3$, (b) parameter s represents arc length, (c) neither $K=0$ nor $\tau=0$, at any point of C, and (d) the curve is not spherical. A point P of such a curve C is defined to be a vertex if and only if the osculating sphere at P has at least five consecutive points in common with C at P. A stationary point Q of C is a point where $dr/ds=0$, where r is the radius of the osculating sphere. It is then shown that every vertex point is a stationary point. Consideration of the helix $K=\tau=1+s^2$, at $s=0$, reveals that a stationary point need not be a vertex. (Received November 1, 1956.)

298. Frank Harary: *On arbitrarily traversable graphs and directed graphs.*

A new proof of Bäbler's unique prime decomposition theorem for arbitrarily traversable graphs is given. This proof serves to characterize a prime arbitrarily traversable graph as a maximal connected subgraph of an arbitrarily traversable graph, with no cut points of itself. In addition, the theorems of Ore and Bäbler are extended to directed graphs. (Received November 14, 1956.)

299. M. Herzberger (p) and E. W. Marchand: *Elementary linear and differential geometry in four-space.*

Since the physical laws can be considered as geometry in space-time, the authors feel that the transformation and variation of well known three-dimensional laws and methods to four-space merit interest. Problems of spherical trigonometry, elementary geometry, and differential geometry are discussed in four-space. *Literature*: H. Grassman, *Die lineale Ausdehnungslehre*, O. Wigand, Leipzig, 1878. H. Forder, *The calculus of extension*, Cambridge University Press. E. W. Marchand, *Spherical trigonometry in n-space*, Dissertation, Rochester, 1952. M. Herzberger, *Einige über lineare transformationen und Vektoranalysis in n-Dimensionen*, Z. Angew. Math. Mech. vol. 11 (1931) pp. 431–433. (Received November 14, 1956.)

300. C. C. Hsiung: *A uniqueness theorem on two-dimensional Riemannian manifolds.*

Let M^2 and M^{*2} be two orientable two-dimensional Riemannian manifolds of class C_3 imbedded in a Euclidean space E^m of dimension $m>2$ with positive Gaussian curvature in every normal direction and closed boundaries C and C^* respectively. Suppose that there is a homeomorphism of M^2 and M^{*2} such that at corresponding points M^2 and M^{*2} have the same normal spaces of dimension $m-2$ and equal sums of the principal radii of curvature in every common normal direction, and such that at corresponding points C and C^* have the same tangent vectors and equal linear elements. Then M^2 and M^{*2} are transformed into each other by a translation. When $m=3$, this result is due to the author (Math. Zeit. vol. 64 (1955) pp. 41–46) and to E. B. Christoffel for closed M^2 and M^{*2}. (Received October 30, 1956.)

301. Wilhelm Klingenberg: *Projective geometry with homomorphism.*

A projective space with homomorphism (abbr.: w.h.) P_3 is a set of points, in which two kinds of subsets, called lines and planes, are distinguished. Furthermore there is given a mapping $P_3 \to \bar{P}_3$, called homomorphism, of the points, lines and planes of P_3 onto the points, lines and planes of a projective space (in the usual sense) $\bar{P}_3$, and the usual incidence properties between points, lines and planes of P_3 hold, whenever they hold between the images of these points, lines and planes in $\bar{P}_3$. It is well known, that $\bar{P}_3$ may be represented by the submodules $\neq 0$ of the left vector space $\bar{K}_s^4$ of rank 4 over a field $\bar{K}$. Conversely, the set of submodules $\neq 0$ of $\bar{K}_s^4$, with arbitrary field $\bar{K}$, represents a projective space. Therefore fields will be called the characteristic coordinate structure of a projective space. As a generalization of this it will be proved, that the characteristic coordinate structure of a projective space w.h. is a (general) local ring A (that is a ring A with unit, which is not necessarily commutative, and there exists a greatest ideal $I \neq A$, which contains any left and any right ideal $\neq A$). A/I is a field, isomorphic with the coordinate field $\bar{K}$ of $\bar{P}_3$. The onto homomorphism $A \to \bar{K}$ induces an onto homomorphism $A_s^4 \to \bar{K}_s^4$, where A_s^4 is the left A-module of rank 4. P_3 is represented by those submodules of A_s^4, which are not mapped into 0 under $A_s^4 \to \bar{K}_s^4$. (Received October 9, 1956.)

302. Lawrence Markus: *Complete and convex affine connections.*

Several results are noted on the geometry of the geodesics for various complete and convex affine connections. (Received November 13, 1956.)

303. W. O. J. Moser: *On the number of ordinary lines determined by n points in the plane.*

Let n points, not all collinear, in the real projective plane be joined in all possible ways by straight lines. A connecting line which contains exactly 2 of the given points is called an ordinary line. Let $m = m(n)$ be the minimum number of ordinary lines determined by any set of n noncollinear points. A famous conjecture of Sylvester (Educational Times, 59, 1893: 98) may be expressed in the form $m \geqq 1$. This was proved by Gallai in 1933 (see de Bruin and Erdos, Nederl. Akad. Wetensch., Proc. 51: 1277–1279) and Steinberg (Problem 4065, Amer. Math. Monthly vol. 51 (1944) p. 169). Motzkin (Trans. Amer. Math. Soc. vol. 70 (1951) p. 451) showed that $(m+1)(m+2) \geqq 2n$ and the author (Minutes Proc. Royal Soc. Canada, 1955, Appendix C, abstract 59: 23) showed $2m(m+5) \geqq 5n$. In this paper it is shown that $m > (n+11)/6$ when n is even. (Received October 22, 1956.)

304. T. K. Pan: *Isometric mapping of two surfaces.*

Let S and $\bar{S}$ be isometric surfaces associated with unit vector fields v and $\bar{v}$, respectively. Let P and $\bar{P}$ be their corresponding points. This paper is to investigate: (1) the existence of S and $\bar{S}$ such that the curves and the lines of curvature of v and $\bar{v}$ at P and $\bar{P}$ are corresponding; (2) the existence of $\bar{S}$ which can be mapped isometrically onto a given S such that the curves and the lines of curvature of v and $\bar{v}$ at P and $\bar{P}$ are preserved. It includes as special cases some problems studied by Bonnet and Cartan. Exterior differential calculus and method of moving trihedral are employed for this investigation. It is found that surfaces associated with unit vector fields are mapped isometrically with the curves and lines of curvature of the vector fields in correspondence if and only if they are mapped isometrically with the curves and asymptotic lines of the vector fields in correspondence. The general solution of (1) depends on one arbitrary function of two variables and that of (2) on two arbitrary functions of one variable. The nature of singular solution, the characteristic, and the problem of Cauchy are studied. (Received November 13, 1956.)

305. E. J. Schweppe: *A projective generalisation of convexity.*

Consider a projective space P which satisfies the Postulate of Fano and the property of the complete quadrangle. A set Q of points in P is called *quasi convex* if at least one of the points C, D is in Q whenever $[A, B; C, D]$ is a harmonic quadruple and A, B are in Q. If no points of Q lie on some fixed hyperplane H, then Q is a quasi convex subset of the affine space $P-H$. It is easily verified that classical convex subsets of affine spaces are also quasi convex subsets; furthermore, examples may be constructed of quasi convex subsets in projective spaces over fields which cannot be ordered. In particular, the points in the projective plane over the complex numbers such that $x_1\bar{x}_1+x_2\bar{x}_2-x_3\bar{x}_3<0$ form a quasi convex set. When the dimension of P is greater than two, it is shown that the intrinsic properties of a quasi convex set Q, which spans P, completely determine the structure of P. This result generalizes the work of H. Kneser [*Konvexe Räume*, Archiv der Math. vol. 3 (1952) pp. 198–206] and M. H. Stone [*Postulates for the barycentric calculus*, Annali di Matematica vol. 29 (1949) pp. 25–30]. (Received November 13, 1956.)

306. Ernst Snapper: *Cohomology groups and genera of higher dimensional fields.*

Let $k \subset E$ be fields, where $E = k(e_1, \cdots, e_n)$, for $e_1, \cdots, e_n \in E$. The Riemann manifold S of E/k is the limit of the inverse system $\{V_i; >\}$ of its projective models

V_i (O. Zariski, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 683–691). We replaced each V_i by its sheaf of local rings (V_i, L_i) (J.-P. Serre, Ann. of Math. vol. 61 (1955) pp. 197–278). The system $\{(V_i, L_i); >\}$ has as limit a sheaf of rings (S, L) over S as base space. For $x \in S$, the stalk L_x is isomorphic with the valuation ring corresponding to x. We called (S, L) *the sheaf of valuation rings of* E/K. We defined the qth cohomology group $H^q(E/k)$ of E/k as the qth cohomology group of (S, L). Let $h^q(E/k)$ denote the dimension of the k-vectorspace $H^q(E/k)$. We proved that $h^q(E/k) = 0$ when q exceeds the degree of transcendency r of E/k; that $h^0(E/k)$ is the field degree of k^*k, where k^* is the algebraic closure of k in E; and that $h^r(E/k)$ is finite. Geometric genus and irregularity of E/k are defined as respectively $h^r(E/k)$ and $h^1(E/k)$. If all $h^q(E/k) < \infty$, the arithmetic genus of E/k is defined as $\sum (-1)^q h^q(E/k)$, where $q = 0, 1, \cdots, r$. These *purely field theoretic* genera and allied notions are investigated in this paper. (Received November 13, 1956.)

307. Andrew Sobczyk: *Spatial families of lines.*

A family F of lines in three-dimensional Euclidean space E_3 is *representative* if it contains exactly one line in each (unsensed) direction of E_3. A representative family, which has the further property that every pair of its lines intersect, is the pencil of all lines through a point. A representative family F determines a mapping $V \to P$ on the unit sphere $S_2 \subset E_3$ to E_3; for V in S_2 the image P is the point in which the line L of F parallel to vector V pierces the tangent plane of S_2 at V. The family is *continuous* in case this mapping is continuous. The following theorems are proved concerning any continuous representative family F: If L is any line of F and Ω is any plane oblique to L, then there exists a line $N \in F$ which intersects L and is parallel to Ω. For any pair of lines L, M of F, either L and M intersect, or there is a third line N of F which intersects both L and M. (Received November 13, 1956.)

308. S. R. Struik: *Two Euclidean theorems, a by-product of their affine counterpart.*

An affine reflection is the same as the commonly known one, but the angle, under which the connection of the point P and its image point P' intersects the mirror, is of course an arbitrary angle (and not usually a right angle). PP' can lie in a line or a plane (or a three-space in 4-space) intersecting the mirror s. In affine two-space two (in three-space three) reflections suffice to transform two equal triangles (tetrahedrons) into each other. The Euclidean counterpart of the theorem is simpler: One (resp. two) reflections suffice in two- (resp. three-) space, because the mutual position of the compared triangles (tetrahedrons) can be conveniently selected. The same construction yields the ratio of their areas (volumes), purely synthetically, in the event of inequality, a parallel vector pair. Hilbert's arithmetization of the problem of area in the "Grundlagen" is scrutinized as is Süss' ad hoc solution of the difficulties. The writer believes axiomatics and definition had to be modified along affine lines. Moreover, the dualistic nature of the affine reflection leads to additional, worthwhile theorems in two three, and higher dimensions. (Received November 14, 1956.)

309. J. S. Taylor: *The use of projective coordinates in real representations of spaces of n complex dimensions.*

Among the real representations for a space of n complex dimensions is that of using the real points of a flat space of $2n$ dimensions. A weakness of such representations has been the lack of one-to-one correspondence with respect to the infinite elements. With the use of projective coordinates of the complex space the correspondence becomes one-to-one throughout. Further light is thus thrown on the nature of the Argand plane representation of the complex space of one dimension, since the point at infinity is exhibited as an arbitrary point of the plane; in particular, it is observed that the coordinate lines of the Argand plane are to be considered as two families of mutually orthogonal circles, each family having a common tangent at the infinite point. Of more importance, however, is the application of projective coordinates to the 4-space representation of the plane of two complex variables, where the line at infinity of the complex plane is represented by an arbitrary "regular" plane of the 4-space. This makes the representation completely one-to-one and resolves many difficulties previously existing. The generalization for spaces of n complex dimensions is given in conclusion. (Received November 13, 1956.)

LOGIC AND FOUNDATIONS

310. M. D. Davis: *An intrinsic definition of recursive functional.*

A definition of recursive functional of arbitrary type is given, in terms of partial recursiveness. For functionals of type ((1)1) this definition was known to G. Kreisel as a formalization of a definition of Brouwer. x_1, g_1, etc. range over the (natural) numbers; $f_{(\alpha_1\alpha_2\cdots\alpha_n)}$ ranges over functionals from n-tuples $(g_{\alpha_1}, \ldots, g_{\alpha_n})$ to numbers. Numbers (including the "undefined" number) are *of finite domain*; f is *of finite domain* if its domain consists of a finite set of n-tuples of functionals of finite domain. $f \subset h$ if the domain of f is included in that of h and they agree where defined. As w takes on integral values, w^α ranges over all functionals of finite domain which are of type α. If f_α is of finite domain, $\langle f_\alpha\rangle$ is the number such that $\langle f_\alpha\rangle_x = f_\alpha \cdot x_1^{(w)} = x_1$ for $x \leq y$ and is undefined otherwise; $f^{(y)} = f$ for argument $g_\alpha = g_\alpha^{(w)}$ if $f \leq y$, otherwise $f^{(y)}$ is undefined. $f_{(\alpha_1\alpha_2\cdots\alpha_n)}$ is *recursive* if there is a partial recursive function $q(x_1, \cdots, x_n)$ such that if $q(w_1, \cdots, w_n)$ is defined, then $w_i^{\alpha_i} \subset v_i^{\alpha_i}$ implies $q(v_1, \cdots, v_n) = q(w_1, \cdots, w_n)$ and $w_i^{\alpha_i} \subset g_{\alpha_i}$ implies $f(g_{\alpha_1}, \cdots, g_{\alpha_n}) = q(w_1, \cdots, w_n)$. We have $f_{(\alpha_1\alpha_2\cdots\alpha_n)}(g_{\alpha_1}, \cdots, g_{\alpha_n}) = U(\mu y \mathfrak{T}_n(e, \langle g_{\alpha_1}^{(y)}\rangle, \cdots, \langle g_{\alpha_n}^{(y)}\rangle, y))$, for primitive recursive $\mathfrak{T}_n$, an extension of Kleene's extended normal theorem. (Received December 27, 1956.)

311. Joachim Lambek: *Calculus of syntactic types.*

The methods of K. Ajdukiewicz [Studia philosophica vol. 1 (1935) pp. 1–27] and Y. Bar-Hillel [Language vol. 29 (1953) pp. 47–58] are further developed to obtain the types of English pronouns and articles. Pronouns are regarded as functors which transform predicates into sentences, articles as functors which transform nouns into pronouns. Subject and object pronouns have different types, because juxtaposition is noncommutative. A calculus of types was suggested by algebraic techniques due to G. D. Findlay and the present author. Basic types are n (noun) and s (sentence); types x and y give rise to compound types $x\cdot y$, x/y, $x\backslash y$. There is a transitive and reflexive relation $\to$ between types, satisfying two further postulates: (1) $(x\cdot y)\cdot z \to x\cdot(y\cdot z)$, (2) $x\cdot y \to z$ if and only if $x \to z/y$, and their symmetric duals. Consider for example the sentences "John likes fresh milk" and "he likes it." Their types are $n\cdot((n\backslash s)/n)\cdot(n/n)\cdot n$ and $(s/(n\backslash s))\cdot((n\backslash s)/n)\cdot((s/n)\backslash s)$ respectively. Both types reduce to s by the arrow. The relationship between noun and subject pronoun is expressed by the formula $n \to s/(n\backslash s)$. (Received November 13, 1956.)

312. Hartley Rogers, Jr.: *Provable recursive functions*. Preliminary report.

P is a standard axiomatization of elementary number theory, assumed sound. $A(i, x, y)$ is a well-known formula-schema such that $A(i, x, y) \Leftrightarrow \phi_i(x) = y$. ($\phi_i$ is partial recursive function of index i.) f is a *provable function* (*p-function*) if $(\exists j)[f = \phi_j$ and "$(\forall x)(\exists y)A(j, x, y)$" is provable in $P]$. *p-one-one functions* and *p-permutations* are similarly defined. (i) The p-permutations form a noninvariant subgroup of the recursive permutations. *p-isomorphism types*, *p-one-one reducibility* ($\leq_1^p$) and *p-many-one reducibility* ($\leq_m^p$) are defined. (ii) p-isomorphism types coincide with $\leq_1^p$ degrees of unsolvability. (iii) Degrees under $\leq_m^p$ form a semi-lattice; recursive sets fall into infinitely many degrees (a semi-lattice) with a minimum nontrivial element and no maximum element; creative sets form a semi-lattice of more than one degree (infinitely many p-isomorphism types) possessing a maximum element (also the maximum p-isomorphism type, and consisting of sets whose productive functions are p-functions). (iv) Completions of two sets of same Turing degree are p-isomorphic. (v) Theorems of any two sound systems of number theory of minimal strength are p-isomorphic (analogue to theorem of Myhill). (A system of number theory is *sound* if no false formula is provable.) (Received November 13, 1956.)

Statistics and Probability

313. H. D. Block: *On Stochastic approximation.*

Dvoretzky (Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability) proved convergence of stochastic approximation methods in a setting general enough to include all previous results on convergence. The proof was simplified by Wolfowitz (Annals Math. Statist., December, 1956). Extending these ideas, the present paper proves convergence with probability one under weaker conditions and where the process takes place in any normed vector space. (Received November 13, 1956.)

314*t*. H. D. Block: *Estimates of error for two modifications of the Robbins-Monro stochastic approximation process.*

Dvoretzky [Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability] found optimal bounds on the mean square error after n steps, V_n, and the optimal choice of coefficients a_k for the Robbins-Monro process when (*) the regression line is bounded between two straight lines with positive finite slopes. We modify the process by taking the average of n_k observations at the kth step and, assuming (*), find optimal values for V_n, a_k, n_k; similarly for the modification arising from the requirement $a_k = a$ we find optimal values of V_n, a, n_k. For a given total number of observations the optimal estimate of V_n is the same for all three processes. A similar procedure is valid for certain types of mappings of a normed vector space. (Received November 13, 1956.)

315*t*. A. B. Clarke: *On statistics independent of sufficient statistics.*

Let $\{f(x; \theta): \theta \in \Theta\}$ be a family of probability density functions of exponential type, Θ being a connected, open subset of R^k. Let $s = s(x_1, \cdots, x_n)$ be a nontrival k-dimensional sufficient statistic for θ. *THEOREM*: *If* $(x_1, \cdots, x_n)$ *is a sample from a population having frequency function* $f(x; \theta)$, $\theta \in \Theta$, *and if* $t = t(x_1, \cdots, x_n)$ *is any statistic on this sample, then* t *is independent of* s *if and only if the distribution of* t *is in-*

dependent of θ. This generalizes a result by R. G. Laha (Proc. Amer. Math. Soc. vol. 1 (1956) pp. 172–174), for normal and gamma populations. (Received November 13, 1956.)

316. Gopinath Kallianpur: *Optimum filtering with finite data.*

The paper considers the following problem treated in a heuristic manner by Laning and Battin (*Random process in automatic control*): Let $Z(t)$ $(0 \leqq t \leqq T)$ be the corrupted message which is the sum of the message $Vg(t)$ and the Gaussian noise $y(t)$ with continuous covariance function $R(t, s)$. It is assumed that V is a random variable independent of $y(t)$ and that $g(t)$ is a known, continuous function of t. Under the hypothesis that the integral equation of the first kind (*) $\int_0^T R(t, s)p(s)ds = g(t)$ has a continuous solution $p(t)$ it is proved that the optimum filter which minimizes the expected squared error is of the form $F(\int_0^T Z(t)p(t)dt)$, where the function F is easily calculated knowing the distribution of V. When the message function is of the form $\sum_{i=1}^{m} V_i g_i(t)$, the V_i's being independent r.V.'s and the g_i's continuous functions of t, it is shown that the optimum filter is a function of the statistics $\int_0^T Z(t)p_i(t)dt$ $(i=1, \cdots, m)$ where each $p_i(t)$ is a solution of an integral equation of the form (*). (Received November 15, 1956.)

317*t*. D. G. Kendall: *A note on Doeblin's central limit theorem.*

The proof of Doeblin's central limit theorem for Markov chains is modified so as to yield a stronger result: the assumption of finite second moments for the recurrence times can now be dropped. (Received Octobr 30, 1956.)

318. B. O. Koopman: *Uniqueness conditions for a class of stochastic differential equations.*

The initial value problem is studied for the general stationary-transitional "birth-death" process whose time differential equation is $P_n'(t) = R_{n-1}P_{n-1}(t) - (R_n + L_n)P_n(t) + L_{n+1}P_{n+1}(t)$ ($P_n(t)$ is the probability of state n, $P_n'(t)$ its time derivative; the right and left transition probability coefficients R_n and L_n are perfectly general positive time-independent quantities; $n \geqq 0$ for the unilateral case; n a general integer in the bilateral one.) Also, for its adjoint $Q_n'(t) = L_nQ_{n-1}(t) - (L_n + R_n)Q_n(t) + R_nQ_{n+1}(t)$. In the unilateral case, it is proved that the necessary and sufficient condition for the uniqueness of the initial value problem is the divergence of the first of the two infinite triangular series $\sum_{n,m=1,0}^{\infty} (R_nR_{n+1} \cdots R_{n+m})/(L_nL_{n+1} \cdots L_{n+m}L_{n+m+1})$, $\sum_{m,n=0}^{\infty} (L_{n+1} \cdots L_{n+m})/(R_nR_{n+1} \cdots R_{n+m})$ while the adjoint will be unique if and only if the second series diverges. Equally simple necessary and sufficient conditions are given in the bilateral case; in addition to these series, two corresponding ones with indefinitely decreasing subscripts are used. The method applies analytic continued fraction theory to the Laplace transforms of these equations. (Work supported by the Office of Scientific Research of the U. S. Air Force.) (Received November 30, 1956.)

319. S. C. Moy: *Integral operators and Markov chain.*

Let K be a linear operator on a Banach space into itself and K^* be its adjoint. If $\|K^n\| \leqq c$ for all n then the existence of a nonzero fixed point of K implies the existence of a nonzero fixed point for K^*. Consider a transition function of a Markov chain as the adjoint of a linear operator on the space of bounded measurable functions, then the hypothesis of the theorem stated is satisfied. Therefore the following statement is concluded. There always exist absolute stationary probability distributions for any transition function of a Markov chain if finitely additive probability distributions are allowed. (Received November 13, 1956.)

320t. R. P. Pakshirajan: *On the maximum partial sums of sequences of independent random variables.*

Let X_ν, $\nu=1, 2, \cdots$ be a sequence of independent random variables with $EX_\nu=0$, $EX_\nu^2=\sigma_\nu^2$, and with distribution function $F_\nu(x)$. Set $S_n=X_1+X_2+\cdots+X_n$, $S_n^*=\max_{1\leq\nu\leq n}|S_\nu|$, and $s_n^2=\sum_{\nu=1}^n \sigma_\nu^2$. Assume $s_n\to\infty$ as $n\to\infty$. Erdős and Kac (*Bull. Amer. Math. Soc.* vol. 52 (1946) p. 292) found the limiting distribution of S_n^* under the sole assumption that the X_ν satisfy the Central Limit Theorem. Chung (*Trans. Amer. Math. Soc.* vol. 64 (1948) pp. 205–233) found the rate of this convergence and proved some strong limit theorems for S_n^* under assumptions stronger than existence of finite third moments for the X_ν. In this paper we establish all of Chung's results under the condition $(\log s_n)^{-1/2}s_n^{-(3-2\alpha)}\sum_{\nu=1}^n \int_{|x|>s_n^{1-\alpha}} x^2dF_\nu(x)=O(1)$. It is easily shown that Chung's conditions always imply ours. It may be observed that if our condition holds for some $\alpha=\alpha_1$, then it holds for all $\alpha\leqq\alpha_1$. If our condition holds for a maximum α in $3/4<\alpha<1$ then our conditions and Chung's conditions are equivalent. If our condition holds for a maximum α in $0<\alpha\leqq 3/4$, then it is much weaker than Chung's, as there exist random variables X_ν whose third absolute moments do not exist but which satisfy our condition. If the X_ν have a common distribution function our condition is satisfied if absolute moments of order $2+\delta$ $(0<\delta<1)$ exist. Thus even in the identically distributed case our condition is much weaker than Chung's. The method of proof is the same as Chung's, the point of departure being the use of Berry's estimates for convergence to Normal involving only second moments in place of Esseen's estimates involving third moments. (Received November 13, 1956.)

321t. M. P. Schützenberger: *A generalization of the Fréchet-Cramér inequality to the case of Bayes estimation.*

Let $f(x)$ be the a priori density function of x; $g(y|x)$ the conditional density function of y. For fixed x, the set of n independent y-variates is represented by s. The density function of s is $f'(s)$ and $g'(x|s)$ is the a posteriori density function of x, for given s. The a posteriori variance of the Bayes estimate is $v_s^2=\int(x-\bar{x})^2g'(x|s)dx$ and $v^2=E_sv_s^2=\int v_s^2f'(s)ds$ is its average over s. $F=\int(\partial f(x)/\partial x)^2\,(f(x))^{-1}dx$; $G=E_xG_x$ with $G_x=\int((\partial/\partial x)g(y|x))^2(g(y|x))^{-1}dy$; $G'=E_sG_s'$ with $G_s'=\int((\partial/\partial x)g'(x|s))^2(g(x|s))^{-1}dx$. The usual assumptions on f and g, which insure that F, G_x, G_s' are finite are made. Since $O=F'=\int((\partial/\partial x)f'(s))^2(f'(s))^{-1}ds$, it is easily seen that $F+nG=G'$ (Third London Symposium on Information Theory, 1955, p. 18). Furthermore, it is a classical result that $v_s^2G_s'\geqq 1$. Thus $v^2=E_sv_s^2\geqq(E_s1/v_s^2)^{-1}\geqq(E_sG_s')^{-1}=(F+nG)^{-1}$, which is the desired inequality that tends to the usual form when n goes to infinity. It reduces to an equality if and only if $v^2=v_s^2=(G_s')^{-1}$ for all s, that is, if and only if $g'(x|s)$ is gaussian with variance independent of s. If, furthermore, $y-x=t$ has a distribution $h(t)$ independent of x, this implies that $f(x)$ and $h(t)$ are also gaussian. (This work was supported in part by the Army (Signal Corps), the Air Force (Office of Scientific Research, Air Research and Development Command), and the Navy (Office of Naval Research).) (Received November 5, 1956.)

322. Esther Seiden: *On a mathematical model for a problem of epidemiology.*

The following problem was suggested by D. Blackwell: denote the number of susceptibles and infected individuals in a population by s and i, respectively. Consider a Markovian stochastic model of the problem characterized by the following transitions within the time interval $(t, t+dt)$: 1. Immigration $I \to I+1$ with probability $adt+0(dt)$, 2. Immigration $S \to S+1$ with probability $bdt+0(dt)$, 3. Emigration $I \to I-1$ with probability $cIdt+0(dt)$, 4. Emigration $S \to S-1$ with probability $dSdt+0(dt)$, 5. Infection $I \to I+1$, $S \to S-1$ with probability $eISdt+0(dt)$, where a, b, c, d, e are non-negative constants. It is shown that in case the r.v. I and S are uncorrelated one can describe the population in terms of EI and ES as follows: The population will tend to a state of equilibrium independently of its initial state. The shape of the path will depend on the relative rates of passing from one state to another. If in addition it is assumed that b is equal to c, then it is possible to describe the state of the population at any given time t. The system of differential equations for the deterministic model reduces to the above by replacing $E(IS)$ by $EIES$. Hence in this case the conclusions drawn are in fact valid. (Received November 16, 1956.)

TOPOLOGY

323. R. D. Anderson: *The group of all homeomorphisms of the universal curve.*

Let G be the group of all homeomorphisms of the universal curve M. It is known that G may be given a metric as follows with respect to which G is a complete metric space. For $g_1, g_2 \in G$, $d(g_1, g_2)$ is the least upper bound of the set of numbers $\rho(g_1(x), g_2(x))+\rho(g_1^{-1}(x), g_2^{-1}(x))$ for $x \in M$ and ρ a distance function on M. It is known that M is homogeneous. It is shown in this paper (by a fairly easy argument) that G is zero-dimensional. (Received November 13, 1956.)

324. B. J. Ball: *On the normality of the product of two ordered spaces.*

First it is shown that for every nonparacompact ordered space X, there exists a compact ordered space Y such that $X \times Y$ is not normal. Hence an ordered space is paracompact if and only if its product with every compact space is normal. Secondly, if X is a nonparacompact ordered space in which every closed, bounded interval is compact, and Y is a compact ordered space, then $X \times Y$ is normal if and only if no point of Y has a left or right character equal to the character of either end-gap of X. This is then used to give a necessary and sufficient condition for the normality of $X \times Y$, where X is a locally compact ordered space and Y is a compact ordered space. (Received November 14, 1956.)

325. R. H. Bing: *Conditions under which monotone decompositions of E^3 are simply connected.*

An example is given of a monotone upper semicontinuous decomposition G of E^3 such that the decomposition space is neither simply connected nor locally simply connected. The only nondegenerate element of G is a solenoid. It is shown that if H is a monotone upper semicontinuous decomposition of E^3, then the decomposition space is simply connected provided either each element of H is locally connected or lies in a plane. If the decomposition space of H is not simply connected, there is an element h of H such that the decomposition of E^3 whose only nondegenerate element is h is not even locally simply connected. (Received November 13, 1956.)

326t. K. T. Chen: *Integration of paths—a formal method for Lie groups.*

The exponential homomorphism Θ with respect to the differentials $\pi_1, \cdots, \pi_n$ in a C^r (or complex analytic) manifold M of dimension n is given by the formal power series $\Theta(\alpha)=1+\sum_{p=1}^{\infty}\sum\int_{\alpha}\pi_{i_1}\cdots\pi_{i_p}X_{i_1}\cdots X_{i_p}$ in the noncommutative indeterminates $X_1, \cdots, X_m$ for any path $\alpha: [a, b]\rightarrow M$. Denote by $\alpha[a, t]$ the portion of α with parameter running from a to t Both $\log\Theta(\alpha)$ and $(d/dt)\log\Theta(\alpha[a, t])$ are Lie elements which converge under the relations $[X_i, X_j]=\sum c_{ijk}X_k$ for α sufficiently short, c_{ijk}'s being the structural constants of a Lie algebra. Thus one constructs the corresponding local Lie group and its Maurer-Cartan forms. The above results rely upon no differential equations. If M is real with $r\geq 2$ and $\pi_1, \cdots, \pi_m$, of rank n over M, then any irreducible piecewise regular continuous path α is uniquely determined (up to change of parameter) by $\Theta(\alpha)$ and $\alpha(a)$. Let $\bar{\theta}$ be the exponential homomorphism with respect to a base $\omega_1, \cdots, \omega_m$ of Maurer-Cartan forms of a real Lie group G, then any irreducible piecewise regular continuous path $\tilde{\alpha}$ in G is uniquely determined by $\bar{\theta}(\alpha)$ up to left translation. Consequently G, if connected, is a homomorphic image of the group generated by all $\exp\sum\mu_i X_i$, μ_i's running over real numbers. (Received October 15, 1956.)

327. W. F. Davison: *Topologies for function spaces of curves.*

Let (X, T) be a curve space, with topology obtained from a mosaic of Peano spaces (see W. F. Davison, *Convergent sequences and mosaics*, Bull. Amer. Math. Soc. vol. 62 (1956) p. 180). On the set X^I of all continuous functions from the unit interval into (X, T) let T_{CO} be the compact-open topology. Since T_{CO}-convergent sequences have unique limits the compact metric subspaces of (X^I, T_{CO}) form a mosaic, which yields a mosaic topology T_M on X^I. The Peano subspaces of (X^I, T_{CO}) yield a curve topology T_C. The resulting inclusions $T_{CO}\subset T_M\subset T_C$ are in general proper. Every T_C-convergent sequence has a subsequence in a homotopic deformation of curves in (X, T). Each topology T_{CO}, T_M, and T_C, relativized to the set K of constant functions, yields a homeomorph of (X, T). K is closed for T_M and T_C, but not necessarily for T_{CO}. The function $p: X^I\rightarrow X$ defined by $p(f)=f(0)$ is continuous and open for T_{CO}, T_M and T_C; thus (X, T) is a retract of (X^I, T_M) and (X^I, T_C). Since the set of functions X^I satisfies the compatibility condition Fréchet equivalence F is defined (see W. F. Davison, *An equivalence relation for compact Hausdorff varieties*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 567). The quotient space $(X^I/F, T_Q)$ of F and (X^I, T_C) remains a curve space, with its topology described in terms of a mosaic of Peano spaces. (Received November 13, 1956.)

328t. Robert Ellis: *Locally compact transformation groups.*

Let X be a locally compact Hausdorff space, let T be a group of homeomorphisms of X, and let $\pi: X\times T\rightarrow X$ such that $\pi(x, t)=xt$ for $x\in X$ and $t\in T$. If T is provided with a locally compact Hausdorff topology which makes π unilaterally continuous and the maps $t\rightarrow st$ and $t\rightarrow ts$ of T into T continuous for all $s\in T$, then π is continuous. Using this result, it is shown that if X is a locally compact Hausdorff space with a group structure in which multiplication is unilaterally continuous, then X is a topological group. (Received November 13, 1956.)

329. Leonard Gillman: *Real-compact spaces (Q-spaces).*

It is proposed that Hewitt's Q-spaces be termed *real-compact*. (For background, see Proc. Amer. Math. Soc. vol. 5 (1954) pp. 447–455.) Let X be a completely regular space. Hewitt proved that a maximal ideal M in $C(X)$ is real (C/M is the real field) iff its family of zero-sets has the countable intersection property (c.i.p.). Call an ultrafilter $\mathfrak{U}$ on X *real* if the family of all zero-sets in $\mathfrak{U}$ has the c.i.p. The following results, among others, prompt the proposal. I. X is compact iff every maximal ideal in C is fixed; X is real-compact iff every real maximal ideal in C is fixed. II. X is compact iff every ultrafilter converges; X is real-compact iff every real ultrafilter converges. III. Any product of compact spaces is compact (Bourbaki proof uses convergence of ultrafilters); any product of real-compact spaces is real-compact (analogous proof uses convergence of real ultrafilters). IV. In any structure, X is precompact iff every ultrafilter is a Cauchy filter; X is pre-real-compact (its completion is real-compact) iff every real ultrafilter is a Cauchy filter. (Necessity in this last was proved by Shirota; sufficiency is an unpublished result of Isbell.) (Received November 13, 1956.)

330t. A. J. Goldman: *A Cech theory of fundamental groups and covering spaces.*

Let X be a connected (but not necessarily locally connected) space. By $F(X)$ we mean the inverse system of fundamental groups of nerves of X's open coverings, together with the natural homomorphisms between these groups. I. The inverse limit $\pi'(X)$ of $F(X)$ obeys the homotopy axiom, obeys the continuity axiom on compact spaces, and is independent of the basepoint if X is pathwise connected. If X is also locally pathwise connected and semilocally-1-connected, then $\pi'(X)$ is naturally isomorphic to the ordinary fundamental group $\pi(x)$; it follows that the 1-dimensional Cech and singular homology groups of such an X coincide. II. Using $F(X)$, procedures are given for classifying those covering spaces (both connected and disconnected) over X which have "simplicial approximations." Every finite-sheeted covering space is shown to be of this type. The classification is carried out in detail for a "$\sin (1/x)$" space and for the dyadic solenoid. (Received October 24, 1956.)

331. M. E. Hamstrom (p) and Eldon Dyer: *The space of homeomorphisms on a two-manifold.*

It is known that the space $H(S^2)$ of homeomorphisms of a 2-sphere onto itself is locally arc-wise connected [M. K. Fort, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 59–62]. Using a generalization of a technique of J. H. Roberts [*Summary of Lectures and Seminars*, Summer Institute in Set Theoretic Topology, Madison, Wis. 1955, p. 100] it is shown that the space $H(C)$ of homeomorphisms of a compact 2-manifold with boundary B (possibly null) onto itself which leave B fixed has the following property: if $\epsilon>0$, there is a $\delta>0$ such that if K is a compactum, $\dim K<\infty$, and $f: K\rightarrow H(C)$ has diameter less than δ, then f can be extended to a map on the cone of K having diameter less than ϵ. Using a selection theorem of E. Michael [Ann. of Math., to appear], it is shown that if $p: X\rightarrow B$ is a 0-regular map, B is a compactum, $\dim B<\infty$, and for each $b\in B, f^{-1}(b)$ is a compact 2-manifold, then (X, p, B) is a locally trivial fibre space. It is also shown that if K is as before, P^3 is the space of rotations of the 2-sphere, and $f: K\rightarrow J$, the identity component of $H(S^2)$, then there is a map $F: K\times I\rightarrow J$ such that $F|K\times(0)=f$, $F(K\times(1))\subset P^3$, and if $f(x)\in P^3$, then $F(x, t)=f(x)$ for all $t\in I$. This shows that J is of the same weak homotopy type as P^3, the real projective 3-space. Certain of these results are similar to results of J. Cerf (unpublished). The authors are not aware of Cerf's arguments. (Part of this work was sponsored by the National Science Foundation.) (Received November 13, 1956.)

332. Melvin Henriksen (p) and J. R. Isbell: *On the stone-Čech compactification of a product of two spaces.*

As usual, βX denotes the Stone-Čech compactification of the completely regular (c.r.) space X (E. Čech, Ann. of Math. vol. 38 (1937) pp. 823–844). If X is dense in a compact space BX and there is a homeomorphism of βX upon BX keeping X pointwise fixed, we write $\beta X = BX$. A c.r. space X is *pseudo-compact* if every continuous real-valued function on X is bounded (E. Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99). Theorem 1. *If X and Y are infinite c.r. spaces such that $\beta(X\times Y) = \beta X\times\beta Y$, then $X\times Y$ is pseudo-compact.* Theorem 2. *If X is compact and satisfies the first axiom of countability, and if Y is pseudo-compact, then $\beta(X\times Y) = \beta X\times\beta Y$.* Theorem 3. *If X is compact, and if Y is pseudo-compact and has a countable dense subset, then $\beta(X\times Y) = \beta X\times\beta Y$.* (Received November 6, 1956.)

333. Melvin Henriksen and Meyer Jerison (p): *A nonnormal subspace of βN.*

I. *Theorem.* Every nonnormal space contains a closed subset which is not a zero-set ($=f^{-1}(0)$ for some continuous f). Proof. Disjoint zero-sets can always be separated by disjoint open sets (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) p. 61). II. Let $\{N_\alpha\}$ be c infinite subsets of a countable set N, such that $N_\alpha \cap N_\gamma$ is finite for $\alpha \neq \gamma$. In βN, the Stone-Čech compactification of N (discrete), let $Y_\alpha = \bar{N}_\alpha - N$, and Y consist of one point from each Y_α. Then Y has c points, is discrete in βN, and is closed in $X = N \cup Y$. X is nonnormal—the following proof also works for other examples: not every function on the closed set Y can be extended continuously to X because there are 2^c functions on Y but only c (on N, hence) on X. Every subset of X is a G_δ-set; by Theorem I, X contains a closed G_δ which is not a zero-set. III. (Known) $\beta N - N$ is not extremally disconnected. Proof. If A_1, A_2 are disjoint sets in X which cannot be separated by open sets, then $\bigcup\{Y_\alpha : Y_\alpha \cap A_i \neq \phi\}$, $i=1, 2$, are disjoint open sets in $\beta N - N$ whose closures intersect. (Received November 13, 1956.)

334. J. G. Hocking: *Extensions of a monotone mapping on a 2-sphere.*

A recent result of Floyd and Fort characterizes a monotone mapping of a 2-sphere onto itself in terms of an extension of the mapping to a homeomorphism on the bounded complement of the 2-sphere. [Proc. Amer. Math. Society vol. 4 (1953) pp. 828–830]. Similar extensions are possible for the general monotone mapping of S^2 as the following result shows: Theorem: Let f be a monotone mapping of S^2 onto the cactoid K. Then there exists an extension g of f mapping the solid sphere onto the solid cactoid such that on each open 3-cell in the solid cactoid, g^{-1} is a homeomorphism. As a corollary to the method of proof, one easily establishes the following result, too: Theorem: Let $f: S^2 > K$ be monotone. Then f may be approximated arbitrarily closely by a mapping g such that on each open 2-cell in K not containing a cut-point of K, g^{-1} is a homeomorphism. The key lemma in the argument is the factorization $f = m_n m$ of a monotone mapping f where m_n is n-monotone for each n. (Received November 13, 1956.)

335. J. G. Horne: *o-ideals in semirings and a universal compactification procedure.*

A semi-ring is a set R which is a semigroup with respect to two operations "+" and "·", and $a(b+c) = ab+ac$. For simplicity, assume both semigroups commutative. Sufficient conditions are placed on a transitive relation $\gg$ on R in order that the space of maximal $\gg$-ideals form a compact Hausdorff space in the dual Stone topology, (A $\gg$-ideal is a $\gg$-directed ideal; when $f\gg e$ means $fe=f$, they are *o*-ideals.) Consequences include: (1) a generalization of a theorem of Shirota concerning R-lattices (Osaka Math. J. vol. 4 (1952)); (2) a natural sufficient condition on a commutative ring R with identity in order that its space of maximal σ-ideals be compact (viz., if $f \gg e$ means there exists e' such that $fe' = f$ and $e'e = e'$ then $\gg$ is a dense order on R); (3) every compact extension of a completely regular space can be obtained as the maximal $\gg$-ideal space of some family S of its open sets and some relation $\gg$ on S; (4) an "internal" construction of a compact extension of any uniform space; (5) a result of Fan and Gottesman concerning a compactification of Freudenthal and Wallman (Indagationes Math. vol. 14 (1952)). (Received November 13, 1956.)

336. J. R. Isbell: *Weak uniformities induced by families of real-values functions.*

A large covering of uniform space has a uniformly continuous partition of unity subordinated to it if the covering is *star-bounded*, i.e. for some natural number n, no element of the covering meets more than n other elements of the covering. (The term and concept are due to G. D. Mostow, who will publish some Euclidean results which may help on the present problem.) This means that the covering can be realized by a mapping into a polyhedron. Now if a uniformity is the weak uniformity induced by a family of real-valued functions, then it has a basis consisting of countable star-bounded coverings. The converse is not true, because the relevant polyhedra cannot always be embedded in a Euclidean space by a uniform equivalence. (The metric Lobachewski plane is a counterexample, pointed out by J. E. Nelson.) However, the weak uniformities induced by certain kinds of families of functions, which are algebras generalizing the algebras of all continuous functions, are characterized in this paper. The key lemma is a very weak result on embeddings of polyhedra. The general embedding theorem of dimension theory is used to get a homeomorphism into $(2n+1)$-space, and the mapping is then modified in $(2n+2)$-space. (Received November 13, 1956.)

337. M. A. Kervaire: *Normal bundle to a sphere in euclidean space.*

Define two regular immersions of S_d into euclidean $(d+n)$-space as *equivalent* if they induce equivalent normal bundles over S_d. Equivalence classes of immersions of S_d into E_{d+n} form a group $J_{d,n}$ under the operation of joining the images by a tube. It is shown that $J_{d,n} = 0$ for $n \geq d+1$ and $J_{d,d} \approx Z$, Z_2 or 0 according to d is even, odd and S_d not parallelizable, or S_d parallelizable respectively. Similar groups $P_{d,n}$ may be defined using *imbeddings* of S_d into E_{d+n}. One has $P_{d,d} = 0$ for every d. It is not known, whether $P_{d,n} \neq 0$ for some pair (d, n). (Received November 13, 1956.)

338. G. R. Livesay: *Concerning real valued maps of S_n.*

The following theorem is proved: Let $0 \leq d \leq \text{diam. } S_n$, $f: S_n \rightarrow E_1$ continuous, $\epsilon > 0$. Then in a sufficiently fine triangulation in which the antipodal map, T, is simplicial, there is a representative, γ_{n-1}, of the non zero class of $H_{n-1}(S_n, T: Z_2)$ such that for $x \in |\gamma_{n-1}|$, there exists $y \in S_n$ with $\rho(x, y) = d$, $|f(x) - f(y)| < \epsilon$. (ρ is the metric is S_n.) The only notion used which may be new is a definition for the width of a curve: Let $\alpha: S_1 \rightarrow X$ (metric) be continuous, H the space of maps of degree 1 of $S_1 \rightarrow S_1$. Then define $w(\alpha) = \lim \inf_{h \in H} \max_{0 \leq \theta \leq \pi} \delta(\alpha h(\theta), \alpha h(-\theta))$, where δ is the metric in X. This width agrees with the usual definition for convex curves in the plane. (A part of this work was done under contract no. 7174 with the Office of Naval Research.) (Received November 13, 1956.)

339. D. M. Mesner: *Linear graphs with a condition on 2-chains.*

In a nonoriented linear graph with n points and L lines let s denote the largest integer such that there is a complete subgraph with s points. Graphs are considered which satisfy (1): two points not joined by a line are joined by at most D chains of two lines. There are known inequalities in n, L and s which say that if L is large, then s must be fairly large, loosely speaking. It is shown here that if (1) holds, s must be considerably larger, still speaking loosely. More specifically, and defining $Z = n(n-1) - 2L$, $Z \geqq \max [2(n-s)(s-D), 2^{-1}(n+s-2D), 2^{-1}(n-D)(n-s), n^2/s-n]$. The proof makes use of incidence matrices, in which Z is the number of off-diagonal 0's and (1) becomes a condition on the inner products of certain pairs of rows taken as vectors. Conditions on 2-chains arise naturally in graphs obtained by identifying points with treatments in a PBIB experimental design and joining pairs of points which correspond to first associates. (Received November 14, 1956.)

340. C. J. Neugebauer: *B-sets and fine cyclic elements.*

An *A-set* of a Peano space P can be defined as a nondegenerate closed subset A of P such that either $A = P$ or else every component of $P - A$ has a single frontier point. A *proper cyclic element* of P can be defined as a cyclic A-set of P. An immediate generalization of the above concepts is the following. A *B-set* B of a Peano space P is a nondegenerate continuum of P such that either $B = P$ or else every component of $P - B$ decomposes into a finite number of points. A *fine cyclic element* (introduced first in a slightly different context by L. Cesari, Bull. Amer. Math. Soc. Abstract 61-6-698) of P is then defined to be a B-set which remains connected after removing any finite set of points. In order to obtain a theory of B-sets and fine cyclic elements as complete as possible as the theory of A-sets and proper cyclic elements one is forced to restrict P to be of finite degree of multicoherence. The paper shows that the properties of B-sets and fine cyclic elements of such Peano spaces are suitable extensions of the corresponding properties of A-sets and proper cyclic elements. (Received November 13, 1956).

341. C. D. Papakyriakopoulos: *A proof of Dehn's lemma.*

The following theorem known as *Dehn's lemma* [Math. Ann. vol. 69 (1910) p. 147] is proved: *Let M be a 3-manifold, compact or not, with boundary which may be empty, and in M let D be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve C and such that, there exists a closed neighborhood of C in D which is an annulus (i.e. no point of C is singular). Then there exists a 2-cell D_0 with boundary C, semi-linearly imbedded in M.* A consequence of Dehn's lemma is that, a knot K in the 3-sphere S^3 is *unknotted if and only if $\pi_1(S^3-K)$ is free cyclic* [Math. Ann. vol. 69 (1910) p. 158, Satz 2]. The theorem is proved in the case where M is orientable. From this and a theorem of I. Johansson [Math. Ann. vol. 115 (1938) p. 695] follows that, the theorem holds also for any nonorientable M. (Received October 31, 1956.)

342*t*. C. D. Papakyriakopoulos: *The sphere theorem.*

The following theorem is proved: *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$, and which can be topologically imbedded in a 3-manifold N, having the following property: The first homology group of any nontrivial (but not necessarily proper) subgroup of $\pi_1(N)$, has an element of infinite order (n.b. in particular this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere S semi-linearly imbedded in M, such that S is not homotopic to 0 in M.* The proof of this theorem is analogous to the author's proof of Dehn's lemma [Bull. Amer. Math. Soc. Abstract 63-2-341]. (Received October 31, 1956.)

343t. C. D. Papakyriakopoulos: *A proof of the asphericity of knots.*

The following theorem is proved: *Let U be a nonempty proper open connected subset of the 3-sphere S^3. Then U is aspherical if and only if S^3-U belongs to one and the same component of S^3-S^2, for any 2-sphere S^2 semi-linearly imbedded in U.* This solves a problem of J. H. C. Whitehead [Fund. Math. vol. 32 (1939) p. 161], and its proof is based on the theorem contained in the author's abstract. *The sphere theorem* [Bull. Amer. Math. Soc. Abstract 63-2-342]. A corollary of the above theorem is the following: *If F is a nonempty proper closed connected subset of S^3, then each component of S^3-F is aspherical.* From this follows immediately: *If F is a connected graph or knot, then S^3-F is aspherical.* (Received October 31, 1956.)

344t. C. D. Papakyriakopoulos: *A proof of Hopf's conjecture.*

The following theorem known as *Hopf's conjecture* is proved: *If U is an open connected subset of the 3-sphere, then $\pi_1(U)$ has no element of finite order.* The proof is based on the theorem contained in the author's abstract. *The sphere theorem* [Bull. Amer. Math. Soc. Abstract 63-2-342], and the following simple consequence of a theorem due to P. A. Smith: The fundamental group of an aspherical polyhedron (of finite dimension) has no element of finite order [Hurewicz, Proc. Acad. Amsterdam vol. 39 (1936) p. 216, l. 31]. (Received October 31, 1956.)

345. J. E. L. Peck (p) and A. L. Dulmage: *Games on a compact set.*

The fundamental theorem of the theory of games has been extended by Kneser (C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2418–2420) and Nikaidô (J. Math. Soc. Japan vol. 5 (1953) pp. 86–94). We give an extension which goes beyond both of these. The theorem is as follows: $\tilde{X}$ is a convex subset of a real linear space, Y is any subset of a real linear space and $\tilde{Y}$ is the convex set generated by Y. A real function f is defined on $\tilde{X}\times\tilde{Y}$ and $f(x, y)$ is a concave function of $x\in\tilde{X}$ and a convex function of $y\in\tilde{Y}$. If $\tilde{X}$ is compact in a topology which is such that for every $y\in Y$, $f(x, y)$ is an upper semi-continuous function of $x\in\tilde{X}$, then $\sup_{x\in\tilde{X}}\inf_{y\in Y} f(x, y) \geq \inf_{y\in\tilde{Y}}\sup_{x\in\tilde{X}} f(x, y)$. As a corollary one obtains that if in a game, one of the pure strategy spaces X is a compact and Hausdorff, while the pay-off function is such that if y is any pure strategy for the other player, $f(x, y)$ is a continuous function of x, then the game has a value. Some pertinent examples are given. (Received November 7, 1956.)

346. R. W. Rector: *A system of linear equations for the seven ring.*

Birkhoff and Lewis (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 355–451) conclude their attack on the Four Color Problem with a partial analysis of the n-ring with particular attention to the 6-ring and the 7-ring. The present author has previously indicated in a preliminary report (Bull. Amer. Math. Soc. vol. 62 (1956) p. 399) an extension of the list of linear relations for the 7-ring and shown the existence of 126 linearly independent relations for the 7-ring. Utilizing these results a system of 36 linear equations can be obtained; the solution of this system would yield formulas expressing the constrained polynomials of the 7-ring in terms of free chromatic polynomials of maps containing fewer regions than the original map. (Received November 13, 1956.)

347. P. E. Thomas: *The Pontrjagin cohomology operations and rings with divided powers.* Preliminary Report.

Following H. Cartan (see Séminaire H. Cartan, E.N.S. 1954–1955, Chapter 7) we define a *ring with divided powers* to be a commutative, graded ring $A=\sum A_k$ (degree $A_k = 2k$), together with a set of functions γ_r ($r = 0, 1 \cdots$) such that: (1) γ_r: $A_k \to A_{kr}$, (2) $\gamma_0(x) =$ unit of A, $\gamma_1(x) = x$, (3) $\gamma_r(x)\gamma_s(x) = (r, s)\gamma_{r+s}(x)$, (4) $\gamma_t(x+y) = \sum \gamma_r(x)\gamma_s(y)$, (5) $\gamma_r(xy) = r!\gamma_r(x)\gamma_r(y)$ for $r \geqq 2$, (6) $\gamma_r(\gamma_s(x)) = c_{r,s}\gamma_{rs}(x)$. Here $(r, s) = (r+s)!/(r!s!)$ and $c_{r,s} = (s, s-1)(2s, s-1) \cdots ((r-1)s, s-1)$. The concept also clearly extends to bi-graded rings. A ring with divided powers will be called *p-cyclic* if each summand A_k is a cyclic group of infinite order or order a power of a prime (after Eilenberg-MacLane). Let K be a complex, and let A be a ring with divided powers. Define the cohomology ring of K with coefficients in A, $H^*(K; A)$, as the bi-graded ring $\sum_{i,j} H^i(K; A_j)$. Restricting the ring A to be p-cyclic, we have *THEOREM*: *$H^*(K; A)$ is a ring with divided powers. The functions γ_r ($r = 0, 1 \cdots$) are given by the Pontrajagin cohomology operations* $\mathfrak{P}_r$: $H^{2n}(K; A_k) \to H^{2rn}(K; A_{rk})$. The Pontrjagin operations are defined in Proc. Nat. Acad. Sci. vol. 42 (1956) pp. 266–269; however, the definition must be changed slightly in order to use the ring A as coefficients. (Received November 14, 1956.)

348. R. F. Williams: *Local contractions of continuous curves.*

The purpose of this paper is to find an additional, purely topological, hypothesis for compact metric spaces X, such that the following theorem holds: if f is a local contraction of X onto itself, then f is a local isometry. For definitions, counter-examples, and reference, see an earlier paper, (Proc. Amer. Math. Soc. vol. 5 (1954) p. 652). That X be a dendrite (acyclic continuous curve) with only countably many end-points is shown to suffice. It is shown to be insufficient to assume any of the following: X is a simple closed curve; X is a dendrite, even in the plane; X is a (topological) sphere of any dimension; or X is a (topological) n-cell, for $n \geqq 2$. (Received November 13, 1956.)

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BOOK REVIEWS

The geometry of geodesics. By Herbert Busemann. New York, Academic Press, 1955. 10+422 pp. $9.00.

This book represents a most interesting and successful effort to treat in a unified fashion the geometry of a class of metric spaces general enough to include Riemannian and Finsler spaces. Only the barest minimum is postulated; in particular, the spaces are metric; bounded infinite sets have limit points; any two points can be connected by a (geodesic) segment, i.e. a curve whose length equals the distance of the points; every point has a neighborhood in which a geodesic segment can be prolonged in a unique way. It is a consequence that a given segment can be prolonged indefinitely in both directions to yield a geodesic, i.e. a curve which is locally a segment. These are more or less paraphrases, the five axioms themselves being stated in terms of the metric, and in very simple form. The spaces satisfying them are called G-spaces and are the object of study throughout the book. They include the complete Riemannian and Finsler manifolds. It should be pointed out however that it is not assumed that the spaces are differentiable, or even topological manifolds. In fact, it is an important unsolved question of the theory as to whether the axioms imply this. It is shown that the dimension, which may, so far as is known, be infinite, is the same at each point, and that should it be two, then the space is a manifold. It is a remarkable fact that in many instances it is shown that the addition of a single property or axiom is enough to characterize well-known geometries, of various types. For example if $B(x, y)$ denotes the bisector (set of points equidistant from) x and y, then the assumption that bisectors are flat, i.e. contain with each pair of points a segment joining them, implies that the space is euclidean, hyperbolic or spherical and of dimension greater than one. This implies a second characterization of these same spaces as solutions to the Helmholtz-Lie problem, to wit: if for any two isometric point triples of a G-space a motion exists which carries the first into the second, the same conclusion as above follows.

In all there are six chapters; the first deals with general concepts and contains as a highlight a solution in the large of the inverse problem of the calculus of variations for a system of curves in the plane: if each curve of the system goes to infinity at each end and any two points lie on exactly one curve, then the plane may be metrized as a G-space with these curves as geodesics. Chapter II is devoted to G-

spaces which can be imbedded topologically (but not isometrically) and with preservation of geodesics in projective spaces. These spaces the author calls Desarguesian. He shows that they can be characterized by two conditions: (i) the geodesic through two distinct points is unique and (ii) either the space is two dimensional and Desargues' theorem and its converse hold or it is a higher dimensional space and any three points lie on a plane. Special Desarguesian spaces are discussed, in particular Minkowski (or finite dimensional Banach) spaces which provide the local geometry in Finsler spaces in the same way as does euclidean space for Riemannian spaces. Chapter III deals with the theory of perpendiculars and parallels in a G-space. Definitions, too technical to repeat here, are given for these two notions and in terms of them a form of the axiom of parallels is stated, all of course, reducing to the usual notions in euclidean space. Elliptic, Minkowskian and euclidean geometries of dimension greater than two are characterized in this context.

Chapter IV is devoted to covering spaces of G-spaces, and using the theory developed there it is shown that in a G-space satisfying (i) above either all geodesics are straight lines or all are great circles of the same length. Theorems on the existence of closed geodesics, geodesics on tori, and on geodesic transitivity in G-spaces conclude the chapter. Chapter V, entitled *The influence of the sign of the curvature on geodesics* is of special interest in connection with the author's contention that most of the more geometric theorems on Riemannian spaces can be proved, by his methods, without differentiability assumptions and indeed without the assumption that the metric is Riemannian, so that they still hold, when suitably formulated, in the general class of spaces here considered. Sectional curvature is not defined as such, but simple geometric definitions of negative, positive, non-negative and nonpositive curvature are given for G-spaces, and it is shown that many classical theorems such as those of Hadamard for spaces of nonpositive curvature and the famous theorems of Cohn-Vossen for open surfaces hold equally well in G-spaces. To this end a generalized Gauss-Bonnet theorem is developed. The sixth and last chapter deals with homogeneous spaces and touches on questions related to the work of H. C. Wang and J. Tits on two point homogeneity (closely related to the Helmholtz-Lie problem mentioned above). Several examples are discussed in detail.

This is a book with a very original method of approach to problems of what might be called "general differential geometry" (with a slant toward the foundations of geometry). It assumes very little and proves a great many theorems, many of which were not known, certainly not in so general a form, before the work of the author. For this reason it is not a book that can be skimmed lightly, but rather it must be studied to be followed. The organization, however, is good and the proofs clearly written, their length in many instances being due to the extremely weak assumptions made. It must certainly be counted an important addition to the literature and will deserve the careful consideration of mathematicians interested in the geometry of Riemannian and, particularly, Finsler spaces.

WILLIAM M. BOOTHBY

Structure of a group and the structure of its lattice of subgroups. By Michio Suzuki. (Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge. Heft 10.) Berlin, Springer, 1956. 6+96 pp. DM 16.50.

The lattice of subgroups of a given group has been studied for a long time, even before lattice theory was recognized or named. The converse problem—what can be told about a group from knowledge of its lattice of subgroups—was first studied by Ada Rottländer in 1928, followed in the next few years by R. Baer and O. Ore. In the past fifteen years much more has been learned, through the efforts of a number of mathematicians including the author. The present monograph is the first collected presentation of this work. The author thoroughly surveys the known facts, rounding them out with additional results not previously published. Substantial familiarity with groups and lattices is assumed; thereafter the presentation is self-contained except for the omission of proofs or details (for which references are given) in the advanced stages of some developments.

Chapter I discusses groups which have special kinds of lattices of subgroups, such as distributive, modular, etc. In this respect, knowledge seems relatively weak in the case of complemented lattices. Chapter II considers the case of two groups G and H whose lattices of subgroups are isomorphic; then the author calls H a projectivity of G (more commonly G and H have been called lattice-isomorphic or structurally-isomorphic). This situation has been essentially completely characterized for finite groups, and largely so in the infinite case. One might add to the bibliography the recent paper of B. Jónsson [Mathematica Scandinavica, vol. 1 (1953) pp. 193–206]. Chapter III presents analogous results for lattice-homomorphic groups; for most of these results it is required that the homomorphism be complete (i.e., hold even for infinite joins and meets). Chapter IV considers cases where the lattices of subgroups are dually-isomorphic.

It is impressive to see how much is known about these matters as a whole, although the author points out substantial unsolved questions. This comprehensive view is aided by the very readable exposition. The book is recommended to all readers interested in the subject.

PHILIP M. WHITMAN

The world of mathematics. A small library of the literature of mathematics from A'h-mosé the Scribe to Albert Einstein, presented with commentaries and notes by James R. Newman. New York, Simon and Schuster, 1956. 18+2537 pp., 4 volumes. $20.00.

While it is not customary to review popular books on mathematics in this Bulletin, this one so far exceeds the norm both in range and in sales that it demands notice. (It is undoubtedly the all-time best-seller among mathematics books other than textbooks.) A nonmathematician with an amateur's interest in the subject might well wonder at first why he should buy these volumes rather than one of the more compact (and less expensive) popular books, of which there are many excellent ones that have enjoyed a far smaller sale. However, most short popular books on mathematics cover only a limited selection of topics that are not too technical to discuss superficially and are conceded to possess universal appeal. Most of these topics are included here too, but so is much more, and the reader can make his own choice. The subtitle is in a sense misleading, since the contents are much more literature *about* mathematics than mathematics as such. This is of course inevitable in any popular book. A nonmathematician will not learn much mathematics from these volumes, although he is told a great deal about mathematics and about cognate subjects, such as mathematicians, physics, logic, and foreign politics; whether this will help him understand what mathematics is about and what mathematicians do is not for a professional mathematician to say. However, there is also a great deal here of value for the professional mathematician, collected from sources that are not on everyone's bookshelf. Some at least of this material will be helpful to teachers, and it would be hard to find any mathematician who will not be entertained by some of it, or who will not find something that is new to him.

The contents are highly varied. Some of the selections are actually from the mathematical literature in the strict sense, some are written specifically for the layman, and some are mathematics only by the editor's fiat. Some are extremely interesting, some are exasperating, and some are downright dull. It would be neither practical nor illuminating to list the contents in detail: the following remarks are indicative rather than exhaustive. There are numerous assorted discussions of the nature of mathematics and mathematical thinking, some old-fashioned and some up-to-date. They illustrate the principle that there are at least as many ways of thinking about mathematics as there are mathematicians. There are many biographical and historical selections. Since the chief interest of mathematicians, outside of mathematics, seems to be the personalities of other mathematicians, there is much here to interest the professionals. There are remarks on the general subject of "numbers," ranging from an enquiry into the question of whether birds can count to Dedekind's own account of irrational numbers. There is fascinating material on applied mathematics (my term, not the editor's), much of which seems to be more applications than mathematics: the discovery of Neptune, the problem of determining longitude, the periodic table of the elements, Haldane's famous essay *On being the right size*, Eddington on the constants of nature, Malthus on population. There is an assortment of essays on probability and statistics (it is a pity that room could not have been found for Feller's deflation of the St. Petersburg paradox, when so much of the traditional well-meant nonsense about it is included). There is a lucid exposition of Gödel's theorem by Nagel and Newman, and there is an essay that Lewis Carroll would have enjoyed (written especially for this anthology by Nagel) on *Symbolic notation, Haddocks' Eyes, and the dog-walking ordinance*. There are particularly interesting essays on computing machines by von Neumann, Turing and Shannon. Apparently just to show how far one can attempt to go, there are selections from G. D. Birkhoff's writings on ethics and aesthetics. A real novelty is the inclusion of five selections from mathematics in fiction, ranging from *Gulliver's Travels* to *The New Yorker*.

All in all, this is an anthology with the faults of its genre and more virtues than most specimens of its kind, especially in the set of mathematical anthologies of which it is almost the only example. It has as legitimate a place in any mathematician's library as the *Oxford Book of English Verse* has in that of a specialist in English literature.

R. P. Boas, Jr.

Logic, semantics, metamathematics. Papers from 1923 to 1938. By Alfred Tarski; translated by J. H. Woodger. Oxford, Clarendon Press, 1956. 14+471 pp. 60 shillings.

Since most of the papers that make up this volume have become an essential part of the modern logician's equipment by now, it would be out of place to offer a review of their content at this time; what follows is a brief descriptive report.

Although Tarski has written extensively on several parts of mathematics (e.g., set theory, geometry, Boolean algebra, measure theory), his principal contributions are along the lines of logic. The present volume contains his early logical work. (It does *not*, for instance, contain the Banach-Tarski paradox.) The seventeen papers that appear here in English translation, occasionally with slight changes, corrections, and additions, were first published in German, or French, or Polish. Since some of them were published in more than one part, or in more than one version, a complete list of the original sources would contain not seventeen but something more like thirty titles.

The longest paper is the famous *Wahrheitsbegriff;* it covers 127 pages. Since the first version of this work is in Polish, and the second (in German) appeared in a journal not readily available to mathematicians (Studia Philosophica), the reviewer, for one, would have rushed out to buy the book even if it contained nothing else. It does, however, contain a lot more. Here are a couple of tempting samples: the 1930 paper on the sentential calculus (with Łukasiewicz), and the 1933 paper on ω-consistency and ω-completeness.

The translation seems smooth. A noteworthy feature is the fact that, frequently, the notation was "translated" as well as the text; the letters (or digrams) used in this volume remind the reader, quite properly of course, of the English words "truth," "consequence," etc., rather than, say, the corresponding German words. The usability of the volume is further enhanced by a unified bibliography, a subject index, an index of names, and an index of symbols.

According to a letter from Tarski to the reviewer, Tarski's post-1938 papers on the subject of this volume are all in English. Armed with this volume, therefore, and with the reviews in the *Journal of Symbolic Logic*, the English-speaking mathematical logician can be confident of having Tarski's complete logical output easily accessible to him—a circumstance for which many logicians should and will be grateful to the translator and publisher.

PAUL R. HALMOS

La théorie des fonctions de Bessel, exposée en vue de ses applications à la physique mathématique. By Gérard Petiau. Centre National de la Recherche Scientifique, Paris, 1955. 477 pp. 2500 francs.

This book is the first comprehensive treatise on Bessel functions since Watson's standard work on this subject. A comparison between the two is somewhat out of place since their purposes are different. The book under review, as its subtitle and its application part (more than 100 pages) indicate, is aimed to appeal mainly to the applied mathematician and physicist. Consequently those results and properties which are likely to be of more importance with regard to applications than others are treated in more detail. Among the contributions since the appearance of Watson's book, mainly those with connection to applications are included. This of course is quite consistent with the purpose of this book. However, it seems regrettable to the reviewer that Chapter 5 (*Asymptotic expansions*) does not contain anything of the work done by R. E. Langer, T. M. Cherry, F. W. Olver and others. Likewise regrettable is the absence of a more detailed and organized bibliography. But these points may be of minor concern for the majority of the users of this book.

F. OBERHETTINGER

Symposium on Monte Carlo methods. Held at the University of Florida, March 16–17, 1954. Edited by H. A. Meyer. New York, Wiley, 1956. 16+382 pp. $7.50.

The term Monte Carlo methods was coined during the second World War to describe the use of random sampling for the numerical solution of mathematical problems. The questions that gave rise to these techniques had to do with the penetration of radiation into matter, in particular with the shielding properties important in the technology of atomic energy. Instead of solving the complicated integro-differential equations describing these processes by analytical or orthodox numerical methods it was proposed to simulate the physical phenomenon on computing machines by constructing artificially a sufficiently large number of particle biographies. Thus, in dealing with a process in which an elementary particle is subject to collisions resulting in absorption or reflection, possibly with a partial loss of energy, a moving particle with given initial energy and direction would be assumed and its subsequent history decided by playing a sequence of games of chance. The outcome of these games would decide from step to step the free path lengths, the nature of the collisions, the angles of reflection, the loss of energy in the collision, etc. The probability distributions underlying each game must, of course, be taken from physical theory. If a sufficient number of such fictitious biographies have been calculated, physically relevant questions such as the amount of radiation penetrating a given slab can be answered with reasonable accuracy.

The idea of solving mathematical problems by statistical experimentation is not new. Buffon's needle experiment for the determination of π is well known, and statisticians have occasionally used such procedures to obtain preliminary information on the shape of distribution functions under mathematical investigation. However, these new applications were more interesting, since they were farther removed from questions in statistics proper and at the same time numerically useful. Also, the picturesque name given to the method helped to arouse the interest of numerous mathematicians. The advent of high speed computing machines made it likely that this procedure would have a bright future, since it requires that a fairly simple algorithm be repeated a large number of times, something for which fast machines are excellently suited.

From the purely mathematical point of view it is a challenging question to give a probabilistic interpretation to problems which did not originate from a physical situation involving probability considerations. The classical example for this is the finite difference approximation to Dirichlet's problem based on the equation

$$U(x+h, y) + U(x-h, y) + U(x, y+h) + U(x, y-h) - 4U(x, y) = 0,$$

which has a well known simple interpretation in terms of random walks in a grid in the presence of an absorbing boundary. Many other differential and integral equations, as well as quadratures and problems in linear algebra can also be connected with problems in probability.

It soon turned out that even with the fastest machines available simple straightforward sampling was frequently inadequate. This is a consequence of the fact that the accuracy of the results increases only as $n^{1/2}$ with the sample size n. Therefore a considerable literature has sprung up devoted to the improvement of the method by means of more refined sampling procedures such as quota, stratified, importance sampling etc., some of which had long been used by statisticians.

Another problem that had to be confronted was the generation of random sequences of numbers. In order to make full use of the speed of modern computing machines it is desirable that these sequences be generated in the machines as the computation proceeds. Also, it is important for the purpose of checking the calculations, that such a sequence be reproducible at will. It is therefore agreed now that the best way to supply such random sequences is to use some number theoretical process which generates sequences of numbers that satisfy the common criteria of randomness. There are some philosophical questions inherent in the simulation of a chance process by a strictly deterministic procedure. These misgivings are brought out well in D. H. Lehmer's definition of such pseudo-random sequences as "A vague notion embodying the idea of a sequence in which each term is unpredictable to the uninitiated and whose digits pass a certain number of tests, traditional with statisticians and depending somewhat on the uses to which the sequence is to be put." But these doubts do not seem to affect the practical usefulness of such methods.

If the numbers of the random sequence employed have a uniform probability distribution while the problem at hand requires some other distribution a problem of conversion arises which turns out to be less trivial than one might first think. This is another line of research on which some effort has been spent.

The volume reviewed here gives an excellent survey of the present stand of the Monte Carlo method. It may not be unfair to apply a statistical analysis to a collection of papers on this subject. The table below lists the number of articles devoted to the various aspects of the theory.

Generating and testing of random numbers	4
Sampling methods	4
Diffusion, reflection and scattering problems	4
Statistical problems	2
Linear algebra	2
Numerical quadratures	2
Partial differential equations	2
Integral equations	2
Game theory	1

(Some articles fall into more than one category. Therefore the total is 23, while there are only 20 papers in the book.)

This list shows that the "how" is rather heavily emphasized by comparison with the "what." This impression is strengthened if it is noted that only four papers contain genuine numerical applications other than illustrative examples. This reviewer cannot be sure whether this bias reflects accurately the ratio of theory and practice in current work on or with Monte Carlo methods, or if there exists a large body of useful routine applications which do not warrant publication as a research paper in such a volume, but he suspects that the first interpretation is correct.

The book contains an extremely useful bibliography with abstracts of about 125 articles on Monte Carlo proper and a list, mostly without abstracts, of about 250 papers and books on related topics.

A breakdown of this material gives the impression that by far the most important field of application of Monte Carlo techniques is still the original one of particle diffusion. Problems not intrinsically connected with probability have given rise to a considerable theoretical literature, but since no worth while computational application has been forthcoming, the interest has flagged in recent years, perhaps only temporarily.

There is no space here to report extensively on the contents of the individual articles. This reader found the paper by H. Kahn a very readable introduction to Monte Carlo sampling techniques. The paper by O. Taussky and J. Todd contains a particularly useful exposition of the generation of random sequences. H. F. Trotter and J. W. Tukey have contributed an article on "Conditional Monte Carlo for normal samples" which contains some very clever ideas by means of which the sample size necessary for satisfactory accuracy is reduced by the factor 5000 in a statistical application reported by H. J. Arnold, B. D. Bucher, H. F. Trotter and J. W. Tukey. The first of these two papers is rather difficult to read, partly because its authors adopt the extremely colloquial style frequently met in work on Monte Carlo. The results of J. H. Curtiss concerning the relative efficiency of Monte Carlo procedures and ordinary numerical methods for the solution of systems of linear algebraic equations raises the hope that there is still a future for Monte Carlo in this particular field.

This volume is almost indispensable to mathematicians doing research on or with Monte Carlo methods, and it can be highly recommended to readers who wish to find out what the Monte Carlo method really is.

WOLFGANG WASOW

Automata studies. Edited by C. E. Shannon and J. McCarthy, Princeton, New Jersey, Princeton University Press, 1956, viii +285 pp. $4.00.

This collection of essays is divided into three sections: "Finite Automata," "Turing Machines," and "Synthesis of Automata." We will discuss the first two sections together and take up the third later.

The essays in these two sections treat mainly the mathematical and logical theory of quantized or discrete automata, as contrasted with analog machines. The automata of the first section have a fixed number of elements and states, while those of the second section have a changing, but always finite, number of states. An equally basic division is into *deterministic* and *probabilistic* automata, according to whether the state of an automaton is a deterministic or probabilistic function of the preceding state (including the inputs); this classification crosses the one used by the editors (e.g., there are two kinds of deterministic automata, fixed automata and Turing machines). Since realizability by a deterministic automaton is equivalent to (partial) recursiveness, the essays of these two sections are all contributions to the logic of recursive functions. Some are concerned with recursiveness in general, others with the special kind of recursiveness involved in fixed automata, and the remainder with extending the theory of recursiveness to include probabilistic considerations in fixed automata and Turing machines.

A discrete, fixed, deterministic nerve net is a useful model for studying the human brain. S. C. Kleene's *Representation of events in nerve nets and finite automata* is one of the most important papers in the volume. In it Kleene investigates the logic of such nets, particularly the classes of past histories that a nerve net can detect in the sense that there is a net junction which is active at a given time if and only if the history of the inputs up to this time belongs to the given class. He defines a concept of "regular" and shows that any regular class of histories is detectable and conversely. M. L. Minsky's *Some universal elements for finite automata* and J. Culbertson's *Some uneconomical robots* are also concerned with nerve nets. Minsky characterizes some sets of elements that are sufficient to construct all neural nets; we will comment on Culbertson's paper in connection with the third section.

E. F. Moore's paper, *Gedanken-experiments on sequential machines*, is concerned with finite, deterministic automata in general. He specifies ways of deducing the detailed internal structure of an automaton from some general information about its structure and detailed information about its response to various inputs.

John von Neumann's *Probabilistic logics and synthesis of reliable organisms from unreliable components* is another very important paper. It is a study of methods of synthesizing fixed, probabilistic automata whose overall reliability is greater than the reliability of their parts. A novel method of accomplishing this—called multiplexing—is introduced, and quantitative results concerning its effectiveness are established. This paper contains many suggestive ideas. K. de Leeuw, E. F. Moore, C. E. Shannon, and N. Shapiro in *Computability by probabilistic machines* also treat probabilistic automata; they investigate the extent to which adding a random element to a Turing machine increases its powers.

Three other papers on (deterministic) Turing machines complete the first two sections of the volume: C. E. Shannon, *A universal Turing machine with two internal states*, M. D. Davis, *A note on universal Turing machines*, and John McCarthy, *The inversion of functions defined by Turing machines*.

The third section, *Synthesis of automata*, contains four papers on the general problem of designing automata to perform human functions: W. R. Ashby, *Design for an intelligence-amplifier*, D. M. MacKay, *The epistemological problem for automata*, and two papers by Albert M. Uttley, *Conditional probability machines and conditioned reflexes* and *Temporal and spatial patterns in a conditional probability machine*. Culbertson's paper in the first section also makes many remarks pertinent to this area.

We can think of the various possible stimuli presented by an environment as input states to an automaton, and the human reactions to these stimuli as output states of an automaton. For example, in language translation the input is a Russian text, the output is an English translation of it. A basic theoretical problem is to describe the stimuli and human responses to them with sufficient precision and detail that the relation between inputs and outputs may be expressed as recursive or statistical functions. Not any recursive function will do; it must be one simple enough to be exemplified in an existing machine or a machine that can be built. Thus, complexity is one factor, but it is not the only one. For example, there does not now exist a precise description (of any complexity) of the relation between Russian and English suitable for automatic translation. The point can be put this way: there exist machines which are potentially capable of performing many human functions which they cannot now perform because there do not exist programs (sequences of instructions) which tell the machines how to perform these functions.

These problems, which clearly belong to applied rather than pure mathematics, are very difficult and complicated. What is needed are specific formulations of ways in which automata could perform the more complicated human functions. MacKay recognizes this need, and gives valuable specific suggestions on how to make an automaton learn new concepts. Culbertson seems to be unaware of this need. Thus he says, "we will describe a general method for designing robots with any specified behavior properties whatsoever. They can be designed to do *any* desired physically possible thing under any given circumstances and past experience, and certainly any naturally given 'robot,' such as Smith or Jones, can do no more" (p. 110). His "general method" consists merely of showing how to construct a special type of net without cycles!

Ashby tackles the problem of constructing an automaton which can do original thinking. He suggests using a random source for producing a tremendous number of possible solutions to a problem and then using an equilibrium method for selecting a correct one. That leaves untouched the difficult problem of formulating mechanical criteria for selecting a correct answer from so many wrong ones in a given field. In view of the difficulty of this problem, it is not clear that his is the best way to approach the overall problem of designing an automaton with originality. Perhaps it is better to produce fewer more promising possibilities than to produce so many wild ones.

ARTHUR W. BURKS

BRIEF MENTION

Theory of approximation. By N. I. Achieser. Trans. from the Russian by Charles J. Hyman. New York, Frederick Ungar, 1956. 10+307 pp. $8.50.

For a review of the German translation of this work, cf. this Bulletin, vol. 61, pp. 369–371.

Foundations of the theory of probability. By A. N. Kolmogorov. 2d English edition. Trans. from the German by Nathan Morrison, with an added bibliography by A. T. Bharucha-Reid. New York, Chelsea, 1956. 8+84 pp. $2.50.

A translation of the classic Ergebnisse monograph of 1933.

The theory of groups. Vol. II. By A. G. Kurosh. Trans. from the 2d Russian edition by K. A. Hirsch, with supplementary material by the translator. New York, Chelsea, 1956. 308 pp. $4.95.

For Vol. I, cf. this Bulletin, vol. 62, p. 277. This second and last volume includes Part III: *Group-theoretical constructions*, and Part IV: *Solvable and nilpotent groups.*

Contributions to the theory of nonlinear oscillations. Vol. III. Ed. by S. Lefschetz. Annals of Math. Studies no. 36, Princeton University Press, 1956. 7+285 pp. $4.00.

This collection contains papers by G. Seifert, L. Markus, E.Pinney, V. B. Haas, R. E. Gomory, S. Barocio, F. Haas, G. Hufford, W. T. Kyner, S. P. Diliberto, M. D. Marcus, and P. Koosis.

Proceedings of the conference on differential equations, held at the University of Maryland, March 17–19, 1955. Ed. by J. B. Diaz and L. E. Payne. College Park, University of Maryland Book Store, 1956. 12+294 pp.

This collection contains papers by W. M. Whyburn, S. Bochner, E. Hopf, M. Riesz, J. B. Diaz and G. S. S. Ludford, M. H. Protter, E. T. Copson, Y. W. Chen, G. Szegö, A. Huber, Z. Nehari, L. Amerio,

F. G. Dressel and J. J. Gergen, A. Pleijel, B. Friedman, D. V. Widder, R. V. Churchill, K. Millsaps and K. Pohlhausen, and W. Feller.

Opere. Vol. 4. Part 1. By L. Bianchi. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1956. 481 pp. 4500 lire.

For earlier volumes, cf. this Bulletin vol. 59, pp. 415–416, vol. 60, p. 288 and p. 414, vol. 62, p. 74.

Opere. Vol. I: *Note e memorie.* By G. Ricci-Curbastro. Ed. by the Unione Matematica Italiana with the assistance of the Consiglio Nazionale delle Ricerche. Rome, Cremonese, 1956. 441 pp. 4000 lire.

RESEARCH PROBLEMS

1. Richard Bellman: *Uniform approximation of roots.*

Let $f(x)$ be a monotone increasing function of x with positive continuous derivative for $x \geq 0$, with $f(0) = 0$, $f(\infty) = \infty$. Consider the equation

$$f(x) = y, \tag{1}$$

possessing the unique solution $x = f^{-1}(y)$ for $y \geq 0$. Let

$$x_{n+1} = x_n + \frac{y - f(x_n)}{f'(x_n)}, \qquad x_0 = s, \tag{2}$$

be the sequence of successive approximations to $f^{-1}(y)$ furnished by Newton's method.

Determine $s = s(a, b, n)$ so that

$$\operatorname*{Max}_{a \leq y \leq b} \left| x_n - f^{-1}(y) \right| \tag{3}$$

is a minimum, where $0 < a < b < \infty$, and determine the asymptotic behavior of $s(a, b, n)$ as $n \to \infty$.

For $f(x) = x^2$, it is known that $s(a, b, n) \to (ab)^{1/4}$ as $n \to \infty$. (Received November 26, 1956.)

2. Richard Bellman: *Maximisation of linear functions.*

At the present time, there is no systematic technique for solving the problem of maximizing the linear form $L(x) = \sum_{i=1}^{N} a_i x_i$ subject to the constraints $\sum_{j=1}^{N} b_{ij} x_j \leq c_i$, $i = 1, 2, \cdots, M$, where the a_i and b_{ij} are positive integers, or zero, and the x_j are constrained to be positive integers or zero. On the other hand, if this constraint on integral solutions is removed, the solution is readily obtained for small M, and there exist effective algorithms for large M.

For the case $M = 1$, let $f_N(c_1)$ denote the maximum of $L(x)$ under integral constraints and $g_N(c_1)$ denote the solution under the constraint $x_i \geq 0$. Define the function

$$\phi(N) = \operatorname*{Sup}_{a_i, b_{1j} \geq 1} \left[\operatorname*{Sup}_{c \geq \operatorname{Min}_j b_{1j}} \frac{g_N(c)}{f_N(c)} \right].$$

What is the order of magnitude of $\phi(N)$ as $N \to \infty$, and in particular, is it bounded?

Consider the corresponding problem for general M where

$$\phi_M(N) = \operatorname*{Sup}_{a_i, b_{ij} \geq 1} \left[\operatorname*{Sup}_{c_i \operatorname{Min}_j b_{ij}} \frac{g_N(c_1, c_2, \cdots, c_M)}{f_N(c_1, c_2, \cdots, c_M)} \right]$$

(Received October 3, 1956.)

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SOME HYDRODYNAMICAL METHODS[1]

L. M. MILNE-THOMSON

The famous lines

> Geography is about maps,
> Biography is about chaps

leave little to be desired in conciseness. About what is mathematics? To use a hydrodynamical metaphor I shall take the plunge and assert that mathematics is about the logical consequences of assumed propositions, nowadays called axioms. Thus all mathematics is one. The fancied distinction between "pure" and "applied" is a modern and false dichotomy unknown to Euler and Cauchy. To see this clearly, reflect on the investigations to which the objects of nature have given rise. Insofar as they are mathematical, these investigations have been concerned, not with the objects of nature themselves, but with models, that is to say with certain ideal objects which are defined solely by a set of precise statements concerning the properties to be attributed to these objects. With such a set of statements the mathematician is on firm and familiar ground, and he can follow the path wherever it leads; perhaps to paradox or perhaps to such results as may induce the opinion, it can be no more, that his model furnishes an explanation of the physical counterpart which inspired the original postulates. From this point of view the ocean wave of the mathematician smacks as little of salt as does the Riemann Hypothesis. Indeed the flavor of fluid is a property from which no mathematical inference can be made, and so flavor, perhaps regrettably, must be consigned to the consideration of the chemist or the connoisseur.

1. **Tensors.** It is becoming increasingly realized that the most insight giving statement of the equations of motion of continuous media in general and of fluids in particular is by means of tensors.

There are at least two ways of regarding tensors, namely as quantities attached to a coordinate system, or as intrinsic entities, the latter way presenting many advantages not the least of which is the absence of an irrelevant coordinate system. Our intrinsic definition of a tensor of rank n is recursive.

An address delivered before the Lexington meeting of the Society, November 30, 1956, by invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings; received by the editors November 29, 1956.

[1] The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research, Contract Nonr-567 [07].

DEFINITION. A tensor or rank n is a linear vector operator which, operating on an arbitrary vector x by scalar multiplication, gives rise to a tensor of rank $n-1$.

This definition together with the statement that a tensor of rank zero is a scalar, completely characterizes tensors of all positive integral rank.

Thus, for example, a vector a is a tensor of rank 1 since $a\cdot x$ is a scalar, or tensor of rank zero.

Similarly the dyadic product $\rho q;q$ combines with x to give $\rho q(q\cdot x)$ a tensor of rank 1 and so $\rho q;q$ is a tensor of rank 2, a 2-tensor, indeed the important momentum transfer tensor. The equation of steady motion of a fluid under no body forces can be written [1]

$$\nabla\cdot[\Phi - \rho q;q] = 0,$$

where

$$\Phi = -pI - \frac{2}{3}\mu(\nabla\cdot q)I + \mu(\nabla;q + q;\nabla)$$

is the stress tensor, I being the idemfactor or unit 2-tensor.

From this form of the equation of motion, by integration over a sphere of large radius an expression for the force on a moving solid, for example an aerofoil, is readily obtained. The application of Oseen's approximation at a distance permits analysis of this force into a lift and drag. The drag is of particular interest since its expression [2] is VF where V is the velocity of the body and F is an influx of liquid into the sphere, predominately an influx into the wake behind the body. Moreover the result is of an asymptotic character improving in accuracy as the radius of the sphere is increased. The two-dimensional form of this theorem was obtained by Filon [3] thirty years ago.

Tensor expression also pinpoints the Lagrangian form of the equation of motion [4]

$$\frac{\partial;r}{\partial r_0}\cdot\left(\frac{\partial^2 r}{\partial t^2} - F\right) + \frac{1}{\rho}\cdot\frac{\partial p}{\partial r_0} = 0,$$

where r is the position vector at time t of the particle originally at r_0. The equation of continuity is

$$\rho\left(\frac{\partial;r}{\partial r_0}\right)_{\mathrm{III}} = \rho_0$$

where the notation indicates the third scalar invariant of the tensor derivative.

Integration from 0 to t leads directly to Weber's transformation

$$\frac{\partial ; \boldsymbol{r}}{\partial \boldsymbol{r}_0} \cdot \boldsymbol{q} - \boldsymbol{q}_0 = -\frac{\partial \chi}{\partial \boldsymbol{r}_0},$$

where

$$\chi = \int_0^t \left\{ \int \frac{dp}{\rho} + \Omega - \frac{1}{2} q^2 \right\} dt, \qquad \boldsymbol{F} = -\nabla \Omega.$$

The Lagrangian form of the equation of motion has been applied to the one-dimensional motion of a gas, to two-dimensional steady motion and to free surface problems which I shall mention later.

The point that I want to make here is that the Lagrangian form is not quite so repulsive as the three equations which result from its expression in coordinates would seem to indicate. In the case of incompressibility we have $\rho = \rho_0$ and a consequent simplification. The equation should repay further study.

2. **The theorems of Gauss and Stokes.** The two great weapons of general fluid mechanics are the theorems of Gauss and Stokes [5] and their vector forms are suggestive;

$$\int_\tau \nabla \circ X d\tau = \int_S dS \circ X, \qquad \int_S (dS \wedge \nabla) \circ X = \int_C dC \circ X,$$

where, in the first the closed surface S encloses the region τ, and in the second the diaphragm S spans the closed curve C. Here the small circle indicates scalar, vector, or dyadic multiplication and X is a general function of position, scalar, vector, or tensor.

For the rate of change of circulation in a circuit which always consists of the same fluid particles we have

$$\frac{d}{dt} \operatorname{circ} C = -\int_C dC \cdot \frac{1}{\rho} \nabla p = \int_S dS \cdot P,$$

where $P = \nabla p \wedge \nabla(1/\rho)$, so that P is a vector along the intersection of surfaces of constant pressure and constant density. Also $\nabla P = 0$, and so by Gauss's theorem P defines tubes of constant intensity. Thus we have the famous meteorological theorem of Bjerknes [6] that the rate of change of circulation in C is measured by the number of unit P tubes which C embraces. When p is a function of ρ alone, $P = 0$ and we obtain Kelvin's theorem on the constancy of circulation.

3. **Complex variable.** For plane flow the vector notation leads directly to the complex variable [7]. The use of the complex variable in

two-dimensional problems has a long history, but it is only in recent years that full advantage has begun to be taken of the methods of function theory as opposed to resolution into equations in x and y. What is beginning now to be more fully realized is that the variables most generally useful are not x, y but the conjugate pair z, $\bar{z}$. For example if $\phi(x, y)$ is a plane harmonic function, it is the real part of a holomorphic function $f(z)$. The identity $2\phi(x, y) = f(z) + \bar{f}(\bar{z})$ leads, on putting $\bar{z} = 0$, to

$$f(z) = 2\phi\left(\frac{1}{2}z, -\frac{1}{2}iz\right) - \phi(0, 0) + i\gamma,$$

where γ is an arbitrary real constant.

Again the circle theorem [8] states that the motion of an unbounded liquid whose complex potential is $f(z)$ when disturbed by the circle $|z| = a$ is governed by the complex potential

$$w(z) = f(z) + \bar{f}(a^2/z),$$

for on the boundary $\bar{z} = a^2/z$ so that the boundary is a streamline. That no new singularities are introduced is clear from the fact that of the points z and a^2/z only one lies inside the circular boundary and therefore if $f(z)$ has no singularities inside $|z| = a$, $\bar{f}(a^2/z)$ will have none outside.

The circle theorem enables us to deal with the perturbation produced by a cylinder of any form of cross-section, which can be mapped conformally on the circle.

From another point of view if the cross-section is the curve $\xi = \alpha$ in the net $z = n(\zeta)$, $\zeta = \xi + i\eta$, we have $\zeta + \bar{\zeta} = 2\alpha$ on the boundary of the cross-section. Since $f(z) = f[n(\zeta)] = F(\zeta)$, the complex potential

$$F(\zeta) + \bar{F}(2\alpha - \zeta)$$

makes $\psi = 0$ on the boundary, but the second term may introduce new singularities. For example if the stream Uz is disturbed by the elliptic cylinder $\xi = \alpha$ in the net $z = c \cosh \zeta$, we shall have

$$F(\zeta) + \bar{F}(2\alpha - \zeta) = \frac{1}{2} Uc\{e^{\zeta} + e^{-\zeta} + e^{2\alpha-\zeta} + e^{-2\alpha+\zeta}\}.$$

The last term in the brackets introduces a singularity at infinity but the second term and the last are conjugate complex on the boundary and can therefore be omitted. Thus we get

$$w = \frac{1}{2} Uc(e^{\zeta} + e^{2\alpha-\zeta}) = Uce^{\alpha} \cosh (\zeta - \alpha).$$

Similarly if the parabola $\xi = \alpha$ in the net $z = \zeta^2$ disturbs the stream Uz, we have

$$F(\zeta) + \bar{F}(2\alpha - \zeta) = U\zeta^2 + U(2\alpha - \zeta)^2$$

which gives at infinity the velocity $2U$ instead of U. Halving gives

$$w = \frac{1}{2} U\zeta^2 + \frac{1}{2} U(2\alpha - \zeta)^2.$$

In the same line of thought we know that a stream function, say $\psi(z, \bar{z})$, can be defined for any two-dimensional flow whether irrotational or rotational, and the complex velocity is given by

$$u - iv = -2i \frac{\partial \psi}{\partial z} \cdot$$

In the case of steady streaming past a fixed cylinder, ψ is constant on the boundary, and so

$$\frac{\partial \psi}{\partial z} dz + \frac{\partial \psi}{\partial \bar{z}} d\bar{z} = 0.$$

The Blasius theorem for the force (X, Y) then gives [9]

$$X - iY = \frac{1}{2} i\rho \int_C (u - iv)(u + iv) d\bar{z} = -2i\rho \int_C \left(\frac{\partial \psi}{\partial z}\right)^2 dz,$$

where the integral is taken round the contour C of the cross-section. But the equation of C is of the form $h(z, \bar{z}) = 0$. Thus on the boundary $\bar{z}$ can be eliminated and Cauchy's residue theorem can be used even in the case of rotational motion.

For example a circular cylinder of radius a exposed to a stream U on which is superposed uniform shear flow of vorticity ω undergoes the lift $\pi\rho a^2 \omega U$.

4. **Complex Stokes's theorem.** Just as the theorem of Stokes is fundamental in the theory of "solid flow" if I may use that term to contrast with plane flow, so is the form which the theorem assumes in two dimensions in terms of z and $\bar{z}$ a powerful tool. This plane form I have named the complex Stokes's theorem [10]. The complex Stokes's theorem refers to a plane area S bounded by a closed curve C and states that

$$\int_C f(z, \bar{z}) dz = 2i \int_S \frac{\partial f}{\partial \bar{z}} dS.$$

Cauchy's theorem is a particular case, namely when $\partial f / \partial \bar{z} = 0$, which is the single equation implying the Cauchy-Riemann equations.

One immediate and important application is to the calculation of the kinetic energy [11] of liquid in irrotational motion, with its bearing on virtual mass. Thus

$$\text{Kinetic energy} = \frac{1}{2}\rho\int_S q^2 dS = \frac{1}{2}\rho\int_S \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\,dS$$

$$= \frac{1}{2}\rho\int_S \frac{\partial}{\partial\bar{z}}\left(\bar{w}\frac{dw}{dz}\right)dS = -\frac{1}{4}i\rho\int_C \bar{w}\,dw.$$

If the region S is multiply-connected, suitable barriers must be introduced as part of the boundary C. Note that on the boundary $\bar{z}$ is a function of z so that the integral can be evaluated by the residue theorem.

The complex Stokes's theorem effects a simple proof of a theorem on conformal mapping due to Bieberbach [12].

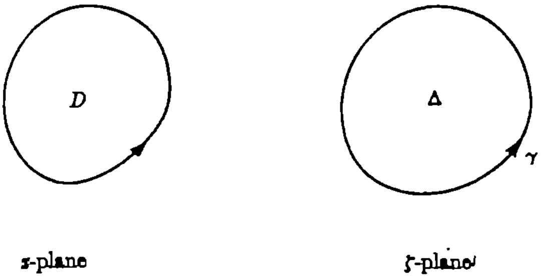

FIG. 1

Let us map the area D bounded by the contour C in the z-plane on the area Δ bounded by the contour γ in the ζ-plane by the relation

$$z = \zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots = f(\zeta).$$

We then have

$$2iD = 2i\int_D dS = \int_C \bar{z}\,dz = \int_\gamma \bar{f}(\bar{\zeta})f'(\zeta)\,d\zeta = 2i\int_\Delta \bar{f}'(\bar{\zeta})f'(\zeta)\,dS.$$

Suppose now that Δ is the circle $|\zeta| \leqq r$. Then on γ, $\bar{\zeta} = r^2/\zeta$ and therefore by Cauchy's theorem applied to the above integral round γ

$$D = \pi r^2\{1 + 2a_2\bar{a}_2 r^2 + 3a_3\bar{a}_3 r^4 + \cdots\} > \pi r^2.$$

Thus of all areas on which D is mapped by choice of $a_2, a_3, \cdots$, the circle is the least. Alternatively if we regard Δ as the given area the problem of mapping on a circle D is that of choosing $a_2, a_3, \cdots$, to make $\int_\Delta f'(\zeta)\bar{f}'(\bar{\zeta})dS$ a minimum. These are Bieberbach's results.

5. **Theorem of Plemelj.** Another theorem important for the study of plane flow is due to Plemelj [13]. Let C be a closed contour, $\phi(t)$ a function defined at every point t of C, and consider the holomorphic function

$$(*) \qquad \Phi(z) = \frac{1}{2\pi i}\int_C \frac{\phi(t)}{t-z}\,dt$$

it being assumed that the integral exists as a Cauchy principal value. Let

$$\lim_{z\to t_0} \Phi(z) = \Phi^L(t_0) \text{ or } \Phi^R(t_0)$$

according as z approaches t_0 from within C or from without i.e. from the left or right. Then Plemelj has proved that (under certain restrictions on $\phi(t)$)

$$\phi(t_0) = \Phi^L(t_0) - \Phi^R(t_0).$$

It easily follows that the function $\Phi(z)$ which satisfies this condition, which is holomorphic in the whole plane (except perhaps on C), and which vanishes at infinity is given uniquely by $(*)$.

As a simple illustration in potential theory we seek the function $f(z)$ holomorphic within the unit circle γ, $|z| \leq 1$ and zero at infinity, whose real part at the point $t_0 = e^{i\theta}$ on the circumference is $\sin 2\theta$. Then

$$2 \sin 2\theta = f(e^{i\theta}) + \bar{f}(e^{-i\theta}).$$

If we continue $f(z)$ outside the circle by the definition

$$f(z) = -\bar{f}(1/z), \qquad |z| > 1,$$

we have

$$f^L(t_0) = \lim_{z\to t_0} f(z), \ |z| < 1 \quad \text{and} \quad f^R(t_0) = \lim_{z\to t_0} -\bar{f}(1/z), \quad |z| > 1.$$

Thus on the circle

$$f^L(t_0) - f^R(t_0) = -i(t_0^2 - 1/t_0^2)$$

and so

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{-i(t^2 - 1/t^2)dt}{t-z} = -iz^2,$$

the continuation of which outside the circle is $-i/z^2$ which tends to zero as $|z|$ tends to infinity.

Another method for such problems is to apply the formula of Schwarz [14] which is substantially equivalent to the foregoing.

6. **Quaternion variable.** The generalization of the complex variable to three dimensions leads to Hamilton's quaternions. Alan Rose [15] defines a stream function $(x, y, z, \xi, \eta, \zeta)$ as the flux across the triangle formed by the origin, the point (x, y, z), and the point $(x+\xi, y+\eta, z+\zeta)$. If we introduce the vector

$$\Psi = (\psi_1, \psi_2, \psi_3) = \left(\frac{\partial\psi}{\partial\xi}, \frac{\partial\psi}{\partial\eta}, \frac{\partial\psi}{\partial\zeta}\right)_{\xi=\eta=\zeta=0},$$

the velocity is $q = -\nabla\wedge\Psi$.

When the motion is axisymmetrical and irrotational with velocity potential ϕ, the function

$$F = \phi + i\psi_1 + j\psi_2 + k\psi_3$$

satisfies the condition for it to be a right-regular quaternion [16] function of the quaternion variable

$$w + ix + jy + kz,$$

where w is an imagined coordinate whose axis is perpendicular to the axes of x, y, z. This condition of right regularity is

$$\Box F = 0, \quad \text{where} \quad \Box = \frac{\partial}{\partial w} + \nabla.$$

This condition, which has four scalar components, may be regarded as the analogue of the Cauchy-Riemann equations for a function $f(z)$ which are in fact the two scalar components of $\nabla f = 0$.

Thus the theory of analytic quaternion functions is in principle available.

Rose has shown that it is possible to deduce the perturbation of a uniform stream by a sphere (in terms of the quaternion variable), by starting with a needle-shaped body along the axis of the stream and which therefore does not disturb the uniform flow. This method is entirely analogous to that for deducing the perturbation of a uniform stream by a circle in terms of the complex variable, by starting with a needle-shaped body.

7. **Virtual mass.** To take the simplest case when a body of mass M moves, with uniform speed V in a straight line, in inviscid liquid the total kinetic energy of the system is of the form $(M+H)V^2/2$ and

the body moves as if the liquid was absent and the mass of the body were increased from its actual mass M to its virtual mass $M+H$. Here H is the added or hydrodynamic mass for this particular motion and is the coefficient of $V^2/2$ in the expression for the kinetic energy of the liquid.

It is only quite recently that a physical interpretation of hydrodynamic mass as an actual mass of the liquid has been given by Sir Charles Darwin [17]. To understand Darwin's interpretation consider the particular case of a circular cylinder which moves (two-dimensional motion) along the x-axis from minus to plus infinity. Suppose that when the cylinder is at $x=-\infty$ blue dye is used to color the particles of the fluid in a plane perpendicular to the direction of motion. Since the cylinder in its motion displaces a certain volume of liquid the gap left behind must be filled up, and it might appear reasonable to suppose that when the cylinder has attained the position $x=+\infty$, the blue wall of liquid will have retreated a certain distance to the rear of its initial position. Now the paths of the particles are elasticas as indicated in Fig. 2. So that a particle which is at

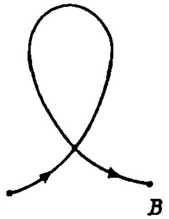

FIG. 2

A when the cylinder is at $x=-\infty$ will have drifted to B when the cylinder is at $x=+\infty$. If then we consider those particles which at a given instant lie in an axial plane of the cylinder perpendicular to the direction of motion, the positions of these particles when the cylinder is at $x=-\infty$ and at $x=+\infty$ define surfaces which, in the plane of the motion are typified by curves A, A, and B, B in Fig. 3.

Thus the particles on the curves A, A move forwards not backwards to the positions B, B. The intuitive idea of reflux of a wall of dyed particles is false. Darwin's discovery is this, that the mass of liquid enclosed (per unit thickness of fluid) between the surfaces A, A and B, B is in fact the hydrodynamic mass of the cylinder (per unit thickness) for this particular motion.

The fact can be established by direct integration since the coordinates of the points on the elastica are expressible rationally in

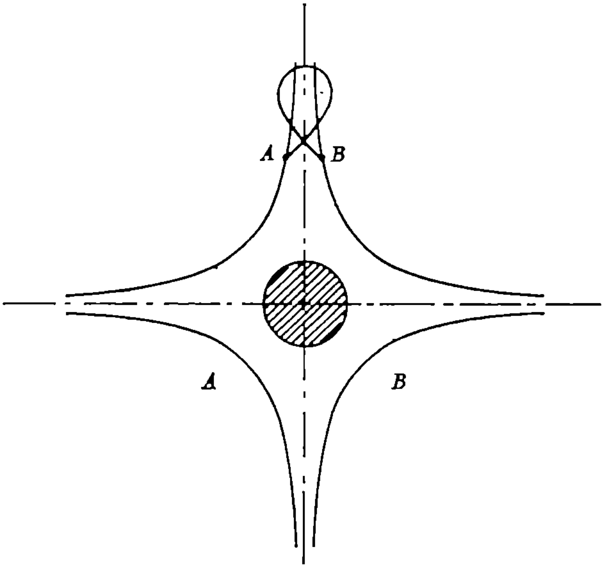

Fig. 3

terms of Jacobian elliptic functions [18]. The argument is however capable of general formulation independent of the particular shape of the cross-section of the cylinder. Moreover as Darwin has shown, the proof can be extended to three-dimensional motion. Thus we have here, I believe for the first time, a genuine physical interpretation of hydrodynamic mass as a mass of fluid entrained by the body.

The corresponding problem of interpretation with a free surface still awaits investigation, although some computations of the hydrodynamic mass itself have been made by Bloh [19] for spheres and ellipsoids half immersed and totally immersed. The problem of rotation also offers opportunities for investigation. If we consider a plate rotating in two-dimensional motion, it is found that two regions A and B (Fig. 4) exist in which the liquid is trapped and moves round with the plate, not as a rigid body but consistently with irrotational flow. This points to a method of interpreting hydrodynamic mass due to rotation.

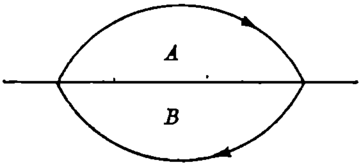

Fig. 4

8. **Free streamlines.** Problems concerning free streamlines have been intensively studied in recent years in relation to the cavities formed behind bodies moving at high speed and the water entry of missiles.

The plane problem has been greatly simplified by Max Shiffman's method [20] of reflection across free streamlines whereby an image in the free streamline is obtained of the actual flow; in effect a method of analytical continuation of the flow across the free streamline. Apart from the insight-giving character of this method, one of the integrations is already performed by geometrical considerations, thus greatly simplifying the technique. The method also affords a direct geometrical interpretation of the drag coefficient.

To give the simplest possible illustration, it is a provable theorem that the image of flow in an angle is flow outside an equal angle. Thus for a jet running along a wall *ABC* (Fig. 5) with a free streamline (shown dotted) we reduce the problem to flow in a channel formed by the original wall and a parallel wall got by translation along the bisector of the angle *ABC*.

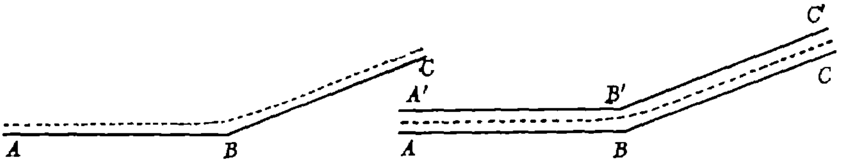

FIG. 5

The method also takes care of re-entrant jet problems such as the impact of a stream on a wedge (Fig. 6).

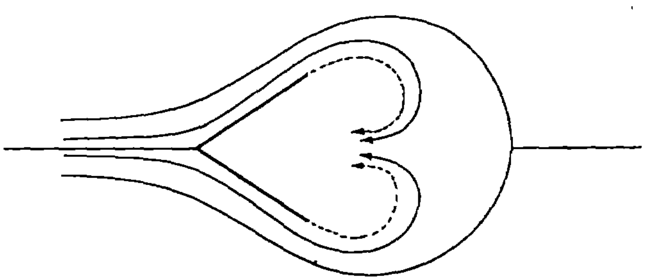

FIG. 6

Applied to the impact of a stream on a lamina the image flow indicates a layer of sources on the down stream face, and thus offers at

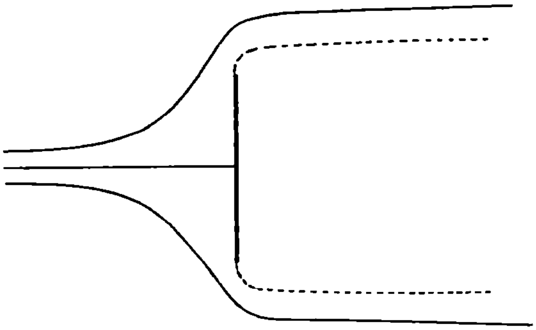

FIG. 7

least a suggestion for tackling the corresponding problem of a stream impinging on a circular disc.

Riabouchinsky [21] 30 years ago discovered a method of finding an approximation to the drag on a lamina exposed to a stream.

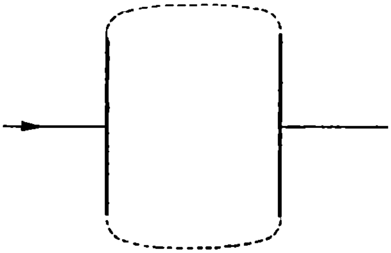

FIG. 8

The method consists in placing an image lamina downstream, the two being joined by free streamlines. This problem is capable of exact solution [22] in terms of Jacobian elliptic functions, where Prandtl's cavitation number $\sigma = (p_\infty - p_c)/2^{-1}\rho V^2$, where p_∞ is the pressure at infinity and p_c is the pressure in the cavity between the free streamlines, appears in the parameter of the elliptic functions. The case of an unlimited cavity can be approximated to any degree of accuracy by increasing the distance between the plate and its image.

Quite recently Garabedian [23] has undertaken the numerical study of the axisymmetrical problem, in particular that of the circular disc with an equal image disc behind it on the Riabouchinsky model. The two discs are joined by a free stream surface to enclose a region containing water vapor. A convergent iterative process is set up, taking an initial form of the free streamline based on the curve afforded by the plane flow solution of Riabouchinsky.

9. **Gravity flow with a free surface.** Let us now turn to the problem of free surfaces when the liquid moves under gravity. By a free surface we shall mean a surface which always consists of the same fluid particles and on which the pressure is constant. The grand illustration in nature is the surface of the ocean.

But few simple complete solutions of this problem are known. Two nontrivial cases are Gerstner's trochoidal wave and Rankine's combined vortex. In the Gerstner wave [24] the free surface is a trochoid and the motion is rotational. In Rankine's combined vortex [25] the motion is rotational within a vertical cylindrical core and is irrotational outside the core, reminiscent of Poe's "Maelstrom."

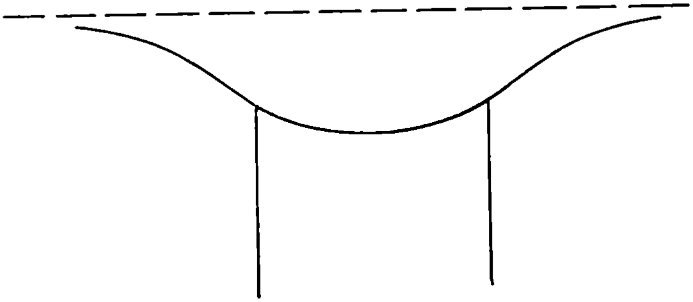

FIG. 9

There are three main problems whose solution still eludes us, namely the problem of periodic surface waves on deep water, surface waves on water of finite depth, and the solitary wave.

Let us examine what is involved in the problem of the two-dimensional progressive wave of permanent type on water of infinite depth. As long ago as 1925 Levi-Cività [26] stated the boundary value problem in the form

$$\frac{\partial\theta}{\partial\psi} = -\frac{g}{c^3}e^{-3\tau}\sin\theta, \qquad \psi = 0, \qquad q = ce^{\tau}.$$

Here θ is the inclination of the velocity vector to the horizontal. By mapping the strip defined by a period on the unit circle and then obtaining a Taylor series expansion Levi-Cività established the existence of this type of wave.

One might enquire why such an existence theorem should be necessary, when waves are to be seen any day. But it is only fair to remember that the wave here considered is perfectly regular and is propagated in an inviscid fluid, conditions to which observed waves only approximate.

The intrinsic difficulty of the problem here envisaged is the nonlinearity of the boundary condition, quite apart from the fact that the form of the boundary is part of the solution.

10. **Method of T. V. Davies.** Write $\omega = \theta + i\tau$. The linearized approximation is obtained by assuming $|\omega| = (\theta^2+\tau^2)^{1/2}$ to be small of the first order. From this assumption follows the usual theory of waves of small amplitude and slope.

There is, however, a serious limitation to the use of the linearized approximation. A wave will break at the crest when the fluid velocity there exceeds the velocity of advance of the wave. The critical case is when the fluid velocity at the crest is equal to the velocity of the wave that is to say when

$$q = ce^{\tau} = 0$$

so that $\tau = -\infty$. It follows that no approximation based upon the assumption that τ is small can throw any light on the case of breaking.

A way to avoid this difficulty has been proposed by T. V. Davies [27], namely in Levi-Cività's boundary condition to replace $\sin\theta$ by $(\sin 3\theta)/3$. This substitution replaces one nonlinear boundary condition by another. It still preserves the essential feature but allows τ to be large. Mathematically the nonharmonic function $e^{-3\tau}\sin\theta$ is replaced by the harmonic function $e^{-3\tau}\sin 3\theta$. The boundary condition then becomes

$$\frac{\partial\theta}{\partial\psi} = -\frac{g}{3c^3}e^{-3\tau}\sin 3\theta, \qquad \psi = 0,$$

which leads to

$$e^{-3i\omega} = 1 - 3A\exp(2\pi i w/c\lambda),$$

where $w = \phi + i\psi$ is the complex potential, $c^2 = g\lambda/2\pi$, and λ is the wave length.

The critical condition for breaking at the crest is $u - iv = 0$ when $w = 0$, which, since $u - iv = ce^{-i\omega}$, leads to $3A = 1$. In the neighborhood of the crest the wave then forms a wedge of angle $2\pi/3$.

Again if we write $A = 2\pi a/\lambda$, where a is small we recover the ordinary linearized theory.

Thus the method discovered by Davies yields an approximation which applies over the whole range from waves of small amplitude to those on the point of breaking at the crest.

The method has also been applied by my colleague B. A. Packham [28] of the Royal Naval College to obtain a unique solution in closed form of the problem of the solitary wave on the surface of water of depth h, namely

$$e^{-2i\omega} = 1 - \sin^2 kch \operatorname{sech}^2 \frac{1}{2} k(w - ich),$$

$$c^2/gh = (\tanh kch)/kch, \qquad 0 \leqq kch \leqq \pi/3.$$

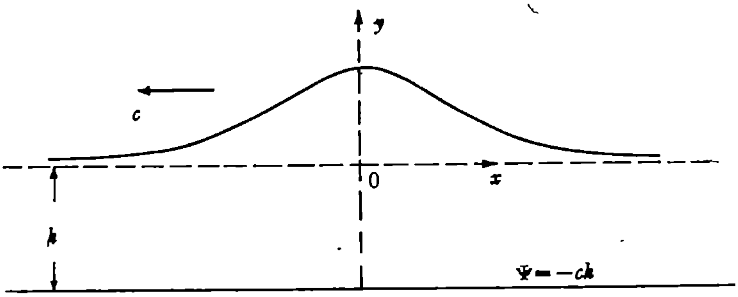

FIG. 10

In this result $k=0$ corresponds to rest and $kch=\pi/3$ corresponds to breaking at the crest.

11. **Exact methods for free surfaces.** We can now turn to exact methods of treating the free surface problem. The equation of motion of an inviscid liquid under gravity can be written in the form [29]

$$\boldsymbol{a} - \boldsymbol{g} = -\frac{1}{\rho} \nabla p,$$

where $\boldsymbol{a}$ is the acceleration of a fluid particle and $\boldsymbol{g}$ is the gravitational force per unit mass. Since ∇p is a vector normal to the surfaces of constant pressure, of which the free surface is one, it follows that for particles in the free surface, the vector $\boldsymbol{a}-\boldsymbol{g}$ is normal to the free surface. This remark has been applied by Fritz John [30] to reduce the attack on the two-dimensional case to the solution of a linear partial differential equation of the second order, of parabolic type as follows. If the free surface has the equation

$$z = f(\alpha, t), \qquad z = x + iy,$$

where α is a real Lagrangian parameter, we have

$$\frac{\partial^2 f}{\partial t^2} + ig = ir(\alpha, t) \frac{\partial f}{\partial \alpha},$$

where $r(\alpha, t)$ is an arbitrary real-valued function. Every two-dimensional continuous free surface must satisfy this equation, whether the motion is rotational or irrotational, steady or time-dependent.

When the motion is irrotational it is not difficult to deduce the equation satisfied by the complex potential. John has also shown that the form $y = k(x, t)$ of the free surface can be prescribed. For a particle, x and y are functions of t and the constant pressure condition then implies that x, as a function t, satisfies a nonlinear ordinary differential equation of the second order.

In the case of irrotational steady motion the problem can be simplified still further to depend on the solution of an ordinary second order linear equation of the form

$$f''(\beta) + ig = iS(\beta)f'(\beta),$$

where $S(\beta)$ is an arbitrary function.

It follows from these considerations that two-dimensional free surface problems can be reduced to the study of a limited class of differential equations. Nevertheless progress will necessarily depend on divining the proper form of the arbitrary function involved.

On the other hand we have here a means of generating an unlimited number of free surfaces by assigning the arbitrary function. In particular, in the steady irrotational case, putting $S(\beta)$ equal to a constant leads to a trochoidal free surface from which we can proceed to a trochoidal progressive wave. Unfortunately this wave has to be associated with a moving ocean bed since the singularities of the progressive wave are no longer fixed.

A different approach to steady irrotational flow is due to H. Lewy [31]. By proper choice of axes and units the surface condition can be expressed in the form

$$\frac{dz}{dw}\left(\frac{dz}{dw} - 2i\,\frac{dy}{dw}\right) + \frac{1}{2y} = 0, \qquad \psi = 0, \qquad w = \phi + i\psi.$$

Let us regard this as an equation in the complex domain involving a complex function $y = -\eta(w)$, real when $\psi = 0$. The equation has the solution

$$z = -i\eta(w) + \int \left[\frac{1}{2\eta} - \left(\frac{d\eta}{dw}\right)^2\right]^{1/2} dw.$$

Provided the integral is real on some segment of $\psi = 0$, we have

$$y = -\eta(w) \text{ on } \psi = 0$$

and the point z describes a free streamline.

Lewy has also proved that the flow is analytic on the free surface in the steady case and has thus established that the formula gives the most general steady irrotational motion with a free surface.

M. J. Vitousek [32] has studied in detail flows obtained by attributing certain forms to $\eta(w)$ and in particular waves of trochoidal and cycloidal profile.

12. **Analogies.** It sometimes happens that problems in apparently diverse branches of science lead to equations and boundary conditions which have the same mathematical form and differ only in the physical interpretation attributed to the symbols. We then say that an analogy exists. Examples are Kirchhoff's analogy [33] between the equations of equilibrium of a thin rod, straight and prismatic when unstressed, held bent and twisted by forces and couples applied at the ends, and the equations of motion of a heavy rigid body turning about a fixed point: the analogy between the torsion of a prismatic beam [34] and the two-dimensional hydrodynamic problem of a cylinder of the same shape as the surface of the beam, hollow, filled with inviscid liquid, and rotating about an axis parallel to the generators: the analogy between electric circuits [35], electric current, magnetic force, positive magnetic pole, and vortex filaments, strength of vorticity, fluid velocity, source respectively. I quote here as further illustrations two analogies between flow of a viscous and an inviscid liquid.

The first concerns flow in a straight horizontal pipe [36] in which viscous liquid flows steadily under a pressure gradient P. The velocity of a particle at distance r from the axis is $\psi - Pr^2/(4\mu)$, where μ is the coefficient of viscosity and ψ is the stream function for inviscid liquid filling the same pipe and rotating about the axis with angular velocity $P/(2)$.

The second analogy concerns the slow motion of a solid of revolution which rotates with angular velocity ω about its axis of revolution in unbounded viscous liquid [37]. The fluid particles describe circles about the axis of revolution with velocity ψ/r, where r is the distance of the particle from the axis and ψ is the stream function when the solid moves in inviscid liquid in the direction of its axis with velocity -2ω.

13. **Motion of a body through a fluid.** The classical case, governed by Kirchhoff's equations, when a vessel deeply submerged (e.g. a submarine) moves with velocity $u = u(t)$ and angular velocity ω in liquid otherwise at rest gives [38] for the force $\boldsymbol{R}_0$ and the moment $\boldsymbol{M}_0$, exerted by the liquid on the vessel,

$$\boldsymbol{R}_0 = -\frac{\partial \boldsymbol{K}}{\partial t} - \boldsymbol{\omega}\wedge\boldsymbol{K},$$

$$\boldsymbol{M}_0 = -\frac{\partial \boldsymbol{L}}{\partial t} - \boldsymbol{\omega}\wedge\boldsymbol{L} - \boldsymbol{u}\wedge\boldsymbol{K},$$

where

$$\boldsymbol{K} = \rho\int \phi d\boldsymbol{S}, \qquad \boldsymbol{L} = \rho\int \phi \boldsymbol{r}\wedge d\boldsymbol{S}$$

the integrals being taken over the surface of the body, and $d\boldsymbol{S}$ denoting the vector element of area directed into the fluid. When a surface ship moves with speed less than $(gh)^{1/2}$ where h is the depth of the water, it is accompanied by a wave train whose energy is maintained at the expense of the fuel supply of the ship and consequently the ship experiences a wave making resistance. A deeply submerged submarine experiences little resistance from this cause, and that is one reason why the submarine is coming into prominence as a war vessel. M. D. Haskind [39] has this year extended the above formulae to include the case where the liquid has velocity $\boldsymbol{v}=\boldsymbol{v}(t)$ at infinity and consequently acceleration $\boldsymbol{f}=d\boldsymbol{v}/dt$. If Φ is the velocity potential, he defines $\phi=\Phi-\boldsymbol{v}\cdot\boldsymbol{r}$ and the corresponding force and moment are

$$\boldsymbol{R} = \boldsymbol{R}_0 + M'\boldsymbol{f},$$

$$\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{v}\wedge\boldsymbol{K} + M'\boldsymbol{r}\wedge\boldsymbol{f},$$

where $\boldsymbol{M}_0$, $\boldsymbol{R}_0$ are calculated from the Kirchhoff formulae, M' is the mass of liquid displaced and $\boldsymbol{r}$ is the position vector of the centre of buoyancy.

The motion of a submarine when gravity is taken into account has been discussed also this year by P. V. Harlamov [40]. The problem he studies is that of a body totally immersed with buoyancy equal to the weight of the body so that, in general, the weight and buoyancy form a couple. The results of Harlamov's thorough-going discussion are not suited to reproduction here but I mention one as a specimen. If $\boldsymbol{r}$ is the position vector of the centre of buoyancy relative to the centre of gravity of the body, permanent translation is impossible if the body is immersed so that the plane containing the vector $\boldsymbol{r}$ and one of the principal axes of the virtual mass ellipsoid is vertical but the vector $\boldsymbol{r}$ is not vertical.

14. A general solution of the hydrodynamical equations. Let me conclude this necessarily brief review of some of the methods used in hydrodynamics today, (I have purposely kept exclusively to the mathematical aspect), by stating a general result [41].

The equation of motion and the equation of continuity of a fluid, viscous or inviscid, compressible or incompressible, are satisfied identically by

$$\rho = \nabla^2\chi, \qquad \rho q = -\nabla(\partial\chi/\partial t),$$
$$\Phi = \rho(q; q) + (V - \partial^2\chi/\partial t^2)I + \nabla\wedge\Psi\wedge\nabla.$$

Here Φ is the stress tensor, χ is an arbitrary function, Ψ is an arbitrary 2-tensor, and $V=\epsilon\rho\Omega$, where Ω is the potential of a conservative field of external force and $\epsilon=1$ if the fluid is incompressible, and $\epsilon=0$ otherwise.

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THE FEBRUARY MEETING IN NEW HAVEN

The five hundred thirty-second meeting of the American Mathematical Society was held at Yale University in New Haven, Connecticut, on Saturday, February 23, 1957. The meeting was attended by about 130 persons including 110 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings Professor J. T. Tate of Harvard University delivered an address entitled *Class formations* at a general session presided over by Professor A. A. Albert. Sessions for contributed papers were held in the morning and afternoon, presided over by Professor Nathan Jacobson and Dr. E. C. Schlesinger.

Abstracts of the papers presented follow. Those having the letter "*t*" after their numbers were read by title. Where a paper has more than one author, the paper was presented by that author whose name is followed by "(p)". Mr. Bomberault was introduced by Mr. Michael Held.

ALGEBRA AND THEORY OF NUMBERS

349*t*. J. W. Andrushkiw: *A note on elimination.*

Let $f(x)=a_0x^n+\cdots+a_n$, $g(x)=b_0x^m+\cdots+b_m$, $a_0\neq 0$, $b_0\neq 0$, $n>m$, be polynomials with complex coefficients, and denote by $r_1, r_2, \cdots, r_n$, $r_i\neq r_j$, $i\neq j$, the roots of $f(x)$. If s_k are the coefficients in expansion $g(x)/f(x)=\sum_{k=0}^{\infty} s_kx^{-k-1}$, it follows that (1) $\sum s_ia_j=0$, $i+j=k$, $k=0, 1, \cdots, n-m-2; n, n+1, \cdots$, and $\sum s_ia_j=b_h$, $i+j=h$, $h=n-m-1+k$, $k=0, 1, \cdots, m$; (2) for $i, j=1, 2, \cdots, n-k$; $k=0, 1, \cdots, m$ there is $\|s_{i+j-2}\|=\sum_{\lambda>\mu}^{1,n-k}\prod(r_{i_\lambda}-r_{i_\mu})^2g(r_{i_1})g(r_{i_2})\cdots g(r_{i_{n-k}})/f'(r_{i_1})f'(r_{i_2})\cdots f'(r_{i_{n-k}})$ $=(-1)^{\sigma(n-m,k)}a_0^{k-n-m}A_{n+m-2k}$ where A_{n+m-2k} are the principal diagonal minors of $n+m-2k$ order of A_{n+m}. The elements of A_{n+m} are the coefficients of $f(x)$ and $g(x)$, and the summation is taken over all combinations of $n-k$ roots. If $r_1=r_2=\cdots r_s=r$, the factors $f'(r_1), \cdots, f'(r_s)$ have to be replaced by $f^{(s)}(r)$, and in the product occur only differences of distinct roots. If r also a root of $g(x)$, the factors $g(r_1), \cdots, g(r_s)$ have to be replaced by $g^{(s)}(r)$ where $g^{(s)}(x)$ is a proper derivative depending upon the multiplicity of r. The relation is still true if $n=m$; (3) $f(x)$ and $g(x)$ have exactly k common roots if and only if $A_{n+m}=A_{n+m-2}=\cdots=A_{n+m-2k+2}=0$, $A_{n+m-2k}\neq 0$. The determinant A_{n+m} is related to the resultant $R(f, g)$ by the equation A_{n+m} $=(-1)^{\sigma(n,0)-\sigma(n-m,0)}R(f, g)$. If $g(x)=f'(x)$, $A_{n+m}/a_0=$ represents the discriminant $D(f)$. (Received December 31, 1956.)

350*t*. Maurice Auslander and Alex Rosenberg: *Dimension of prime ideals in polynomial rings.*

Let R be a commutative ring and let p be a prime ideal in R such that R_p is a regular local ring. Let P be a prime ideal in $S=R[x_1, \cdots, x_n]$ with $R\cap P=p$ and denote the transcendence degree of $S-P$ over $R-p$ by d_P. Using homological methods we then show that rank $P+d_P=$ rank $p+n$. If R is a Dedekind ring, P and Q two prime ideals in S, and I the ideal in $S \otimes_R S$ generated by $P \otimes 1$ and $1 \otimes Q$, we use this equality to show that $d_I = d_P + d_Q$. (Received January 7, 1957.)

351t. R. W. Bass: *The real parts of eigenvalues.*

Let A be any nonsingular matrix with eigenvalues λ_i $(i=1, \cdots, n)$ and let $\Lambda = \operatorname{diag}[\mathcal{R}e\lambda_1, \cdots, \mathcal{R}e\lambda_n]$. Theorem: *There is a nonsingular matrix R with the property that the symmetric part of $(R'R)A$ has the same signature as Λ.* (Here $'$ denotes transpose.) When the eigenvalues are all distinct a simple construction for R is given; the theorem is shown in this case to be equivalent to a theorem of Lewis (Amer. J. Math. vol. 73 (1951) pp. 48–58). See also Abstract No. 362. (Received January 10, 1957.)

352. Chandler Davis: *All invariant convex functions of hermitian matrices.*

Any function of $n \times n$ hermitian matrices with values in a partially ordered vector space, if it is convex and unitary-invariant, gives rise in an obvious way to a convex symmetric function of n real variables (values in the same partially ordered vector space). The main purpose of the present paper is to point out the converse. The proof is simple. Several well-known results are special cases. (Received January 9, 1957.)

353. Emil Grosswald: *Some theorems concerning partitions.*

Let q be an odd prime, let $\{a\} = \{a_1, \cdots, a_m\}$ stand for a set of m least positive residues mod q. $\{a\}$ is called symmetric, if with a also $q-a$ belongs to $\{a\}$. Let $p_n(q)$ be the number of partitions of n into summands congruent (mod q) to elements of $\{a\}$; define $p_n(q, l)$ similarly, if no summand may appear more than l times; $p_n^+(q)$, $p_n^-(q)$, $p_n^+(q, l)$, $p_n^-(q, l)$ are, respectively, the number of partitions of n into quadratic residues, into quadratic nonresidues (mod q) without, or with, restriction to at most l repetitions of a summand. For these, and more general, partition numbers Petersson (Abh. D. Akad. Wiss. (2) 1954) has obtained asymptotic formulae in the case of symmetric sets $\{a\}$. In the present paper the Hardy-Rademacher method is used to obtain asymptotic formulae for these partitions, corresponding to any set $\{a\}$. Typical results are: For $q = 1$ (mod 4), $p_n^+(q)/p_n^-(q) = \epsilon^h(1 - A\alpha_3(q)n^{-1/2} + O(n^{-1}))$, $p_n^+(q, l)/p_n^-(q, l) = 1 + B\alpha_3(q)n^{-1/2} + O(n^-)$, while for $q = 3$ (mod 4), $p_n^+(q)/p_n^-(q) = Cn^{h/2}(1 - Dn^{-1/2} + O(n^{-1}))$, $p_n^+(q, l)/p_n^-(q, l) = (l+1)^h(1 + O(n^{-1}))$. Here $A, B, \cdots$ are trivial constants, $\alpha_1(q)$ is the number of representations of q as a sum of five squares, $\epsilon > 1$ is the fundamental unit of $R(q^{1/2})$ and h is the class number of $R(\pm q)^{1/2}$ for $q > 3$, $h = 1/3$ for $q = 3$. (Received January 10, 1957).

354. E. C. Posner: *Derivations in prime rings.*

A prime ring is a ring in which the zero ideal is a prime ideal. *Theorem* I. In a prime ring of characteristic not two, if d_1 and d_2 are derivations such that $d_1(d_2)$ is also a derivation, then d_1 or d_2 is zero. *Theorem* II. If d is a derivation in a prime ring such that, for all elements a in the ring, a and $d(a)$ commute, then either the ring is commutative or d is zero. The proofs proceed by manipulation of identities. (Received January 10, 1957.)

355. M. O. Rabin: *The generalized word problem for free groups with an application to Post languages.*

Let $\Pi = (x : r(x))$ be a (finite) presentation. The generalized word problem of Π is the problem of finding a method for deciding, for any given set of words $w_1, \cdots, w_n$,

u (n is arbitrary), whether u is in the subgroup U generated by $w_1, \cdots, w_n$. When $n=0$ this reduces to the word problem for groups. *Theorem: There exists a general and effective solution for the generalized word problem for free groups (i.e. for those presentations having no defining relations). In fact, let* $l(u)$ *be the length of* u *as a product of the (free) generators of* Π *and let* $L(u)$ *be the minimal length of* u *as a product of the* w_i *and their inverses; there exists an effectively computable function* $c(k)$ *such that if* $u \in U$ *then* $L(u) \leqq (l(u)+1)c(l(w_1)+ \cdots +l(w_n))$. As a corollary the following result is obtained. *Theorem: Let* L *be normal Post language on the letters* $a_1, \cdots, a_n$*, having the axiom* H *and productions* $A_iP \rightarrow PB_i$, $i=1, \cdots, m$. *If* L *is undecidable then the words* H, A_i, B_i, $i=1, \cdots, m$, *considered as elements of the free group on the* a_i, *satisfy a nontrivial algebraic relation.* (Received January 11, 1957.)

356*t*. Irving Reiner: *A new type of automorphism of the general linear group over a ring.*

Let $GL_2(R)$ be the general linear group over the ring $R=K(x)$, where K is a field. Let $(a\ b)X_m=(a\ ax^m+b)$, $m=1, 2, \cdots$. The elements X_m and those in $GL_2(K)$ generate $GL_2(R)$. Let Y_m be obtained from X_m by replacing x^m by y_m, where $1, y_1, y_2, \cdots$ form a K-basis of R. The map ϕ_y which leaves $GL_2(K)$ elementwise fixed, and maps X_m onto Y_m, induces an automorphism of $GL_2(R)$. This automorphism is not expressible in terms of previously known types of automorphisms. Automorphisms of this new type do not arise for the group $GL_2(Z)$, $Z=$ring of rational integers, nor for $GL_n(R)$, $n \geqq 3$. The proofs make use of properties of continued fractions. A set of generators for the group of automorphisms of $GL_2(R)$ is obtained. (Received January 4, 1957.)

357. E. V. Schenkman: *A characterization of some metacyclic groups.*

Szász (*On groups every cyclic subgroup of which is a power of the group*, Acta Mathematica Acad. Scient. Hungarian vol. 6 (1955) pp. 475–477), has recently shown that a group is cyclic if and only if it satisfies condition (A) as follows. (A) Every cyclic subgroup of the group is for some positive integer k the subgroup generated by the kth powers of the elements of the group. This idea will be extended to show that a metacyclic group whose commutator subgroup has order relatively prime to its index is characterized as a solvable group satisfying condition (B) as follows. (B) Every member of a composition series (i.e., every subinvariant subgroup) is for some positive integer k the subgroup generated by the kth powers of the elements of the group. (If G denotes the group, the subgroup will be denoted by $G(k)$.) (Received January 9, 1957.)

ANALYSIS

358. R. W. Bass: *The generalization to n dimensions of Bendixson's nonexistence criterion.*

This refers to the well known criterion for the nonexistence of limit cycles of an autonomous two dimensional system of ordinary differential equations. Let $f(x)$ be an n-vector function of class C^1 on E^n. Consider the dynamical system S defined on E^n by $\dot{x}=f(x)$ ($\dot{}=d/dt$). A trajectory of S will be called a *normal almost-periodic orbit* if it is (i) uniformly (Bohr) almost-periodic, and (ii) if its closure M has dimension $\geqq (n-1)$ at some point. THEOREM. *Let* G *be any open set in which* div $f(x)$ *does not change its sign and is not identically zero. Then* G *does not contain any normal almost-*

periodic orbit of S. This theorem cannot be improved, as can be shown by constructing compressible flows in E^3 which contain a (1 dimensional) periodic orbit. The proof of the theorem is an immediate consequence of the following nontrivial lemma. *The closure* M *of any normal almost-periodic orbit of* S *is an* $(n-1)$ *dimensional torus.* (Received January 10, 1957.)

359*t*. R. W. Bass: *Irrotationality, invariance, and instability* (I).

Let S be a dynamical system defined in E^n by $\dot{x}=f(x)$ $(\dot{}=d/dt)$, where $f(x)$ is a class C^1 n-vector function. Let $F(x, t)$ denote the flow defined by S so that e.g. $F(x, 0)=x$. A set R is *invariant* [positively semi-invariant] if $F(R, I)=R$ $[F(R, I^+)\subset R]$. A *rest point* is an invariant point. Define a compact set to be *rotational* if its Euler-Poincaré characteristic $E(R)=0$ and define a flow F to be *P-irrotational in* R if there exists a continuous matrix $P(x)$ such that the tensor curl $P(x)f(x)=0$ and $f(x)\cdot P(x)f(x)>0$ for all x in R, where $\cdot$ denotes the scalar product in E^n. *In all that follows we assume* G *to be an open set whose closure* $\bar{G}$ *is* (i) *compact, and* (ii) *contains no rest point.* Lemma. There is a positive number which is smaller than the period of any periodic orbit in $\bar{G}$. Theorem 1. *If* $\bar{G}$ *is positively semi-invariant and is a Lefschetz space* (*e.g. a compact ANR*) *then* $\bar{G}$ *is a rotational set.* (Received January 10, 1957.)

360*t*. R. W. Bass: *Irrotationality, invariance, and instability* (II).

Further define a compact set R to be *Poisson unstable relative to itself* if to every x in R there correspond finite numbers $T_-(x)<0<T_+(x)$ with the property that $F(x, T_\pm(x))$ is not in R. A closed subset K of R is a *transversal* if each orbit in R intersects K exactly once (hence K is a local cross section). Theorem 2. *The following statements are entirely equivalent*: (a) $\bar{G}$ *is Poisson unstable relative to itself*; (b) $\bar{G}$ *admits a portion of an orientable* $(n-1)$*-manifold as a transversal*; (c) *there is a lower semi-continuous function* $U(x)$ *defined on* $\bar{G}$ *whose* (*positive*) *directional derivative along any orbit exists and is positive.* Theorem 3. *If* $\bar{G}$ *is connected, triangulable, and has a vanishing first Betti number, and if the boundary of* $\bar{G}$ *is a differentiable* $(n-1)$*-manifold nowhere tangent to the flow, then statements* (a)–(c) *of Theorem 2 are equivalent to*: (d) *the flow is P-irrotational in* $\bar{G}$. Theorem 4. *Whether or not* $\bar{G}$ *is "simply connected" as in Theorem* 3, *statement* (d) *implies statements* (a)–(c) *of Theorem* 2. Corollary. *If* $\bar{G}$ *is a semi-invariant* (*hence rotational*) *set, then the flow in* $\bar{G}$ *cannot be P-irrotational for any* P. (Received January 10, 1957.)

361*t*. R. W. Bass: *Global asymptotic stability of equilibrium* (I).

Notations as in Abstracts No. 359 and No. 360. A rest point x_0 of S is *asymptotically stable in the large* if x_0 is stable and if for every x_* in E^n, $F(x_*, t)\to x_0$ as $t\to+\infty$. *Theorem.* Let x_0 be the only rest point of S. Then the following statements are entirely equivalent: (i) x_0 is asymptotically stable in the large; (ii) E^n-x_0 admits a compact connected orientable $(n-1)$ dimensional spherelike manifold K as a transversal; x_0 lies in the conditionally compact component of the complement of K, and (at least) for $n\leqq 3$ K is a sphere; (iii) the flow in E^n-x_0 is P-irrotational, x_0 is asymptotically stable in some (small) neighborhood of x_0, and every positive semi-orbit of S is bounded; (iv) the flow is P-irrotational in E^n-x_0 and the class C^1 function $V(x)=\int_0^1 u\cdot P(\theta u)f(\theta u)d\theta$, where $u=x/\|x\|$, satisfies (a) $0<V(x)$ for $x\neq 0$, and $V(0)=0$, and (b) $V(x)\to+\infty$ as $\|x\|\to\infty$; (v) there exists a C^1 function having properties (a) and (b) of (iv) and also: (c) $f(x)\cdot\text{grad } V(x)<0$ for $x\neq 0$; (vi) the differential equation $\dot{x}=f(x)$ can be re-written as $\dot{x}=-Q(x)\text{ grad } V(x)$, where $V(x)$ satisfies (a), (b), as well as: (d) grad $V(x) \neq 0$ for $x \neq 0$, and (e): grad $V(x) \cdot Q(x)$ grad $V(x) > 0$ for $x \neq 0$. (Received January 10, 1957.)

362t. R. W. Bass: *Alternatives to the Routh-Hurwitz criterion.*

A matrix is called stable if all of the real parts of its characteristic roots are negative. Theorem. *A is stable if and only if* $\det(\mathfrak{A}) \neq 0$ *and* $\det(B_i) > 0$, $(i = 1, \cdots, n)$ *where* $\mathfrak{A}$ *and* B_i *are certain symmetric* $(\mathfrak{A}' = \mathfrak{A}, B_i' = B_i)$ *matrices composed of the elements of* A. ($\mathfrak{A}$ *and* B *can be found readily by inspecting* A.) This solves Bellman's Research Problem No. 24, this Bulletin vol. 60 (1954) p. 501. Theorem. *If* A *is stable and* $x(t)$ *satisfies* $\dot{x} = Ax$ $(\dot{} = d/dt)$, *then* $\|x(t)\| \leq \gamma \|x(0)\| \exp(-\lambda t)$, *where* $\gamma = (\sigma_n/\sigma_1)^{1/2}$ *and* $\lambda = \mu_1/2\sigma_n$; σ_1, μ_1 *and* σ_n, μ_n *are respectively the least and the largest, in absolute value, of the (necessarily real, hence easily computable) characteristic roots of* B_1 *and of* $(B_1A + A'B_1)$. See also Abstract No. 169. (Received January 10, 1957.)

363. Barry Bernstein: *On analytic functionals derivable from differential equations.*

The system of equations $dz_i/d\tau = G_i[\tau, z, w(\tau)]$, where z is a complex n-vector and w is a complex m-vector assigns to each function $w(\tau)$, $0 \leq \tau \leq t$, a function $z(\tau)$ such that $z_i(0) = c_i$, a fixed initial value. Thus z is a functional of w. If the quantities $G_1, \cdots, G_n$ are analytic in the components of z and w as well as continuous in all their variables, then this functional is expressible as a series of Riemann Integrals. (Received January 9, 1957.)

364. A. M. Bomberault: *A generalized minimum principle and applications.*

An extension of the minimum principle in the calculus of variations is proposed to study interacting physical systems. Necessary conditions on the functions which satisfy the principle are obtained. These correspond to the Euler-Lagrange equations, boundary conditions and Legendre conditions of the ordinary minimum principle. Using the principle and fixed point theorems, general existence theorems for systems of nonlinear partial differential equations are proved. One theorem concerns a system of equations such that each equation is quasi-linear elliptic in one variable, with its coefficients continuous functions of the remaining variables; and each variable appears as the variable in one and only one equation. For such a system one can define a Rayleigh-Ritz method which converges in the mean to the solution. The proposed principle can be enunciated as finding the solution, using Nash's definition of best play, of the following n-person game. Each player's allowable play is a function from a given function space and his payoff is an integral functional of the plays of all the players. (January 14, 1957.)

365. Frederic Cunningham, Jr.: *Uniform norm structure in Banach spaces.* Preliminary report.

Given for each t in a compact space S a Banach space $X(t)$; a *uniform direct integral* in $X(t)$ over S shall mean a linear subspace X of $\prod_{t \in S} X(t)$ invariant under the ring R of multiplications by continuous functions, with $N_x(t) = \|x(t)\|$ upper semi-continuous on S for each $x = x(t)$ in it, and closed under the norm $\|x\| = \|N_x\|_\infty$. Then R satisfies the Kakutani relation: $\|(f \vee g)x\| = \|fx\| \vee \|gx\|$ for all $f \geq 0$, $g \geq 0$ in R and x in X. A function ring of operators with this property will be called an *M-ring*. *Theorem: Any Banach space X has an essentially unique representation as a uniform direct* *integral over an appropriate compact Hausdorff space S, such that every M-ring on X is a closed subring of* $R=C(S)$. This refines a similar canonical decomposition of X into L^∞-subspaces, a notion dual to the L^1-decompositions reported on earlier (Bull. Amer. Math. Soc. Abstract 60-2-217). (Received January 10, 1957.)

366t. I. S. Gál: *On subbases for uniform structures.*

The object of this note is to give a satisfactory definition of a subbase $\mathcal{S}$ for a uniform structure $\mathfrak{U}$ and to give an example where such subbases can be used to show that a given topology is a uniform topology. The notion of a subbase occurs in the present literature (See: J. Kelley, *General topology*, p. 178) but the definition is not satisfactory because not all subbases satisfy the axioms given for a subbase. Definition: *A family $\mathcal{S}$ of nonvoid sets $S\subseteq X\times X$ is a subbase for a uniform structure if it satisfies the following axioms*: (i) *If $S\in\mathcal{S}$ then $I\subseteq S$.* (ii) *If $S\in\mathcal{S}$ then there exist sets $T_1, \cdots, T_m$ such that $(T_1\cap\cdots\cap T_m)\circ(T_1\cap\cdots\cap T_m)\subseteq S\cap S^{-1}$. The uniform structure generated by $\mathcal{S}$ is determined by the base $\mathcal{B}$ consisting of all finite intersections $V=S_1\cap\cdots\cap S_n$ where $S_1, \cdots, S_n\in\mathcal{S}$.* It is easy to see that there are no other subbases for uniform structures than the ones which satisfy (i) and (ii). The proof of the sufficiency is straightforward. Using the present definition it can be proved directly without any reference to complete regularity and the Haar-König theorem that the interval topology of a linearly ordered set is a uniform topology. (Received December 27, 1956.)

367. Fritz John: *Nonadmissible data for partial differential equations with constant coefficients.*

Let $P(\xi_1, \cdots, \xi_n, \tau)$ denote a polynomial of degree m with constant coefficients. The characteristic form $Q(\xi_1, \cdots, \xi_n, \tau)$ consists of the terms of P of degree m. We assume that $Q(0, \cdots, 0, 1)\neq 0$, and that P is "strictly nonhyperbolic" in the sense that the equation $Q(\xi_1, \cdots, \xi_n, \tau)=0$ has a nonreal root τ for some real vector $(\xi_1, \cdots, \xi_n)$. The following theorem is proved: If the polynomial P is irreducible, every solution $u(x_1, \cdots, x_n, t)$ of $P(\partial/\partial x_1, \cdots, \partial/\partial x_n, \partial/\partial t)u=0$, whose Cauchy data on $t=0$ have compact support, vanishes identically. (Received December 17, 1956.)

368t. Dorothy Maharam: *On a theorem of von Neumann.*

Let S be any measure space with a σ-finite complete measure; S need not be separable. Then, from each class x of measurable sets modulo null sets of S, a member X of x may be chosen so that the correspondence $x\leftrightarrow X$ is a finite Boolean isomorphism (i.e., $x\vee y\leftrightarrow X\cup Y$, $-x\leftrightarrow S-X$, and therefore $0\leftrightarrow\emptyset$). This theorem is due to von Neumann (for the separable case see J. für Math. vol. 165 (1931) pp. 109–115); his proof for the nonseparable case has not been published, but the present proof is believed to be different. As a corollary to the method of proof, every open subset of a product of (not necessarily countably many) unit intervals, with the usual topology and measure, is measurable. (Received January 10, 1957.)

369. J. C. Mairhuber, I. J. Schoenberg (p) and R. Williamson: *On variation diminishing transformations for the circle.*

Important special examples of transformations as described in the title were given by G. Polya in his paper *Qualitatives über Warmeausgleich*, Z. für Angew. Math. und

Mech. vol. 13 (1933) pp. 125–128. Applications of some of these were made by G. Polya and N. Wiener, *On the oscillation of the derivative of a periodic function*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 249–256. In the present paper Polya's ideas are shown to lead to a wide class of well described periodic kernels furnishing convolution transformations having the variation diminishing property. In analogy with the corresponding problem for the real axis (see I. J. Schoenberg, *On Polya frequency functions.* II: *Variation diminishing integral operators of the convolution type*, Acta Szeged. vol. 12 (1950) pp. 97–106), one would expect the class of kernels thus derived to contain all variation diminishing kernels. Using essentially new ideas the authors are able to construct infinitely many new kernels which are shown not to be included in Polya's class. (Received January 14, 1957.)

370*t*. R. R. Phelps: *Convex sets and nearest points*. II.

If S is a subset of a normed linear space E and $s \in S$ we define $S_s = \{x: \|x-s\| = \inf_{y \in S} \|x-y\|\}$, the (closed) set of all points having s as a nearest point in S. Using a result of Motzkin (Atti Acad. Naz. Lincei, Rend 6 vol. 21 (1935) pp. 773–779) it is easily shown that E is an inner product space if and only if S_s is convex for any $S \subset E$ and $s \in S$. A subset T of E is called a *nearest-point set* if there exists $S \subset E$ and $s \in S$ such that $T = S_s$. It is shown that if E is a complete inner product space then every closed convex subset of E is a nearest-point set. The converse is true if the dimension of E is at least three. If $S \subset E$ a *nearest-point map for* S is a function f which assigns to each point of E a nearest point in S. It is shown that the property "a nearest-point map shrinks distances whenever it exists for a convex set" characterizes inner product spaces of three or more dimensions. The convexity of a closed set in E^n is characterized by the fact that its nearest-point map f shrinks distances. (Received December 10, 1956.)

371*t*. M. O. Reade: *On quasi-conformal maps in three space*. Preliminary report.

Let $y = y(x)$ be a differentiable quasi-conformal map defined for $\|x\| < 1$ such that $\iiint_{\|x\|<1} J(y/x)dx < \infty$. Then the following results are obtained. 1. Almost all radii, in the sense of two dimensional measure on $\|x\| = 1$, have rectifiable images. 2. There is a set E of capacity zero, on $\|x\| = 1$, such that if $\|x_0\| = 1$, $x_0 \notin E$, then almost all of the radii of sphere $\|x - x_0\| = 1/2$ that lie in $\|x\| < 1$ have rectifiable images. A method due to Tsuji [Tohoku Mathematical Journal vol. 2 (1950) pp. 113–125; Journal of the Japanese Mathematical Society vol. 5 (1953) pp. 307–320] is used. (Received December 17, 1956.)

372*t*. M. O. Reade: *A radius of univalence for* $\int_0^z e^{-\zeta^2} d\zeta$. Preliminary report.

It is shown that $\int_0^z e^{-\zeta^2} d\zeta$ is univalent for $|z| < R$, where R is the largest positive root of the equation $(4R^4-1)^{1/2} - \mathrm{ArcTan}\,(4R^4-1)^{1/2} = \pi$. $R = 2.1$, approximately, which improves upon a result due to Rogozin, $R = \pi/2$ [Rostov. Gos. Univ. Uc. Zap. Fiz-Mat. Fak. volume 32 (1955) pp. 135–137], and Nehari's $R = (((\pi^2+1)^{1/2}-1)/2)^{1/2}$. [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 545–551]. A result due to Umezawa is used [Tohoku Math. J. vol. 7 (1955) pp. 212–228] to obtain our radius R. (Received December 17, 1956.)

373t. M. O. Reade: *On Umezawa's criteria for univalence.*

Recent results due to Umezawa [Tohoku Math. J. vol. 7 (1955) pp. 212–228] are examined. Some of Umezawa's proofs are simplified, and some of his results are generalized. A typical result is the following one, first announced by the author at the International Congress of Mathematicians, Amsterdam, 1954. If $f(z)=1/z+a_0+a_1z+\cdots$ is analytic, with nonvanishing derivative, for $0<|z|<1$, and if $\int_C d\arg df(z)<\pi$ holds for all arcs C on $|z|=r$, for all $0<r<1$, then $f(z)$ is univalent for $0<|z|<1$; moreover, $f(z)$ maps each circle $|z|=r$ onto a close-to-convex curve. The methods used to prove the results contained in this note are extensions of one due to Kaplan [Michigan Mathematical Journal vol. 1 (1952) pp. 169–185]. (Received December 24, 1956.)

374t. Martin Schechter: *On estimating partial differential operators.*

Let G be a bounded n-dimensional domain with closure $\bar{G}$ and boundary B of class C^∞. Denote by $C_{mr}(G)$ the set of complex valued functions of class C^m in $\bar{G}$ having derivatives of order $<r$ vanish on B $(0\leq r<m)$. Let A be a linear mth order partial differential operator with complex valued coefficients continuous in $\bar{G}$. Consider the statement (I). There is a constant K such that $\|D^m u\|^2\leq K(\|Au\|^2+\|u\|^2)$ for all $u\in C_{mr}(G)$, where $D^m u$ denotes the generic mth order derivative of u and $\|u\|$ its $L_2(G)$ norm. For $n>2$, (I) holds if and only if A is elliptic in $\bar{G}$ and $m\leq 2r$. For $n=2$, one must consider the vector roots of the characteristic polynomial of A. Properly defined, there are, at B, m roots with imaginary parts perpendicular to B. If p is the number of these roots with imaginary parts directed inward and $q=\max(p, m-p)$, then (I) holds if and only if A is elliptic in $\bar{G}$ and $q\leq r$. (For $n>2$, this is equivalent to the first statement.) The proofs employ a method due to Gårding and Aronszajn. (Received January 7, 1957.)

375t. I. J. Schoenberg and George Polya: *The de la Vallée Poussin summation method is variation diminishing.*

The de la Vallée Poussin means $V_n(x)$ of a function $f(x)$, of period 2π, are obtained by convoluting $f(x)$ with the periodic kernel $(2\cos x/2)^{2n}/\binom{2n}{n}$. It is now shown that, if $f(x)$ is real, the number of real zeros, in a period, of the nth order trigonometric polynomial $V_n(x)$ never exceeds the number of variations of signs of $f(x)$ in a period. As an application of this new property it is shown that the de la Vallée Poussin means furnish elegant necessary and sufficient conditions for a power series $z+a_2z^2+\cdots$ to map the unit circle onto a schlicht convex domain; also criteria for such a power series to map the unit circle onto a star-shaped domain. Another consequence is as follows: If $f(x)$ is the function of support of a plane convex domain D, then also $V_n(x)$ is the function of support of a convex domain D_n. The D_n are all isoperimetric, of nonincreasing areas as n increases and, of course, converge to D as n tends to infinity. (Received January 14, 1957.)

376t. V. L. Shapiro: *A best possible result in the uniqueness of Laplace series.*

Let S designate the surface of the unit sphere in three dimensional Euclidean space and let x denote a point on S. Furthermore for every surface spherical harmonic of order n, $Y_n(x)$, defined on S let the value $(\int_S|Y_n(x)|^2dS(x))^{1/2}$ be designated by $\|Y_n\|$ where $dS(x)$ is the two-dimensional area element on S. Then in this paper the following theorem is proved: Given a series $\sum_{n=0}^{\infty} Y_n(x)$ of surface spherical harmonics and a fixed point x_1 on S. Suppose that (i) $\|Y_n\| = o(n^{1/2})$; (ii) $\lim_{r\to 1} \sum_{n=0}^{\infty} Y_n(x)r^n = 0$ for x in $S - x_1$. Then the given series is identically zero (i.e. $Y_n(x) = 0$ for all n). This result is in a certain sense best possible for condition (i) cannot be changed to read $\|Y_n\| = O(n^{1/2})$. For letting (x, x_1) designate the scalar product it is well-known that the series $\sum_{n=0}^{\infty} (2n+1)P_n[(x, x_1)]$ is Abel summable to zero for $x \neq x_1$, and a computation shows that $\|(2n+1)P_n[(x, x_1)]\| = O(n^{1/2})$ but not $o(n^{1/2})$. (Received January 4, 1957.)

377. Norbert Wiener and E. J. Akutowicz (p): *The definition and ergodic properties of the stochastic adjoint of a unitary transformation.*

The concepts and results of the present paper depend upon a combination of elementary Hilbert space notions with the theory of the Brownian motion. To begin with, integrals of the form $\int_{-\infty}^{\infty} f(t)dX(t, \alpha)$, where $f \in L^2(-\infty, \infty)$ and $X(t, \alpha)$ is the complex Brownian motion process, are well-defined complex Gaussian random variables, the covariance of two such quantities being $\int_{-\infty}^{\infty} f_1(t)\bar{f}_2(t)dt$. With any unitary transformation U in L^2 there is associated a measure-preserving point transformation $T = T(U)$ defined almost everywhere on the space of the α's, and such that $\int_{-\infty}^{\infty} Uf(t)dX(t, \alpha) = \int_{-\infty}^{\infty} f(t)dX(t, T\alpha)$ almost everywhere. The definition of T is such that from a one-parameter group U^{λ}, $-\infty < \lambda < \infty$, of unitary transformations there results a well-defined one-parameter group of measure-preserving point transformations T^{λ}. The ergodic properties of T^{λ} are determined as follows. *T^{λ} is weakly mixing if and only if the point spectrum of U^{λ} is absent. If the spectrum of U^{λ} is absolutely continuous, then T^{λ} is strongly mixing.* Using results of Wiener and Wintner, Amer. J. Math. vol. 60 (1938), it is shown that the last statement cannot be reversed. If the point spectrum is present, then ergodicity necessarily fails. These results have application in quantum mechanics. (Received January 9, 1957.)

378t. F. M. Wright: *On weighted mean Stieltjes sigma integrals of order p.*

Let $[a, b]$ be a finite closed interval of the real x axis, and let f, g be real-valued functions on $[a, b]$. Let p be any integer ≥ 2, and let $r = (r_1, r_2, \cdots, r_p)$ be any p real numbers satisfying $r_1 + r_2 + \cdots + r_p = 1$. If $P = (a = x_0 < x_1 < \cdots < x_n = b)$ is any partition of $[a, b]$, we consider sums $S(P) = \sum_{i=1}^{n} [\sum_{j=1}^{p} r_j \cdot f(x_{i,j})] \cdot [g(x_i) - g(x_{i-1})]$, where $x_{i-1} = x_{i,1} < x_{i,2} < \cdots < x_{i,p} = x_i$ for $i = 1, 2, \cdots, n$. If $S(P)$ has a finite limit in the sense of successive subdivisions, it is said that f has a mean Stieltjes sigma integral of order p and weight r with respect to g on $[a, b]$. It is shown that this integral exists in case f has only discontinuities of the first kind and g is monotone nondecreasing, and a formula is obtained for the integral in this case involving the Lebesgue-Stieltjes integral $(LS)\int_a^b f(x)dg(x)$ and the weights r_1, r_p. For $p = 3$ and such an f, g, the integral either assumes the value $(LS)\int_a^b f(x)dg(x)$ for all (r_1, r_2, r_3) or else assumes an arbitrary value for a suitable (r_1, r_2, r_3). The usual theorem on integration by parts for the Lebesgue-Stieltjes integral follows. (Received January 11, 1957.)

GEOMETRY

379t. M. O. Reade: *On certain conformal maps in three space.*

In this note the following theorem is proved. If $y = y(x)$ is a differentiable homeomorphism of $\|x\| < 1$ onto $\|y\| < 1$, with nonvanishing Jacobian, and if $y(0) = 0$, then

$y(x)$ is a rotation. The method is an elementary one depending upon properties of subharmonic functions. (Received December 17, 1956.)

380*t*. Stephen Smale: *A classification of immersions of the 2-sphere.*

An immersion of one C^1 manifold in another is a regular map (a C^1 map whose Jacobian is of maximum rank at every point) of the first into the second. A homotopy of an immersion is called regular if at each stage it is regular and if the induced homotopy of the tangent bundle is continuous. If M is a C^1 manifold, $F_2(M)$ denotes the bundle of ordered 2-frames of M. Let N be a C^2 manifold of dimension greater than two and let $x_0 \in F_2(S^2)$ and $y_0 \in F_2(N)$. It will be assumed hereafter that all immersions of S^2 in N are C^2 in some neighborhood of the base point of x_0 and their induced maps send x_0 into y_0. If $f, g\colon S^2 \to N$ are given immersions, an invariant $\Omega(f, g) \in \pi_2(F_2(N); y_0)$ is defined. The following theorems are proved. Two immersions $f, g\colon S^2 \to N$ are regularly homotopic if and only if $\Omega(f, g) = 0$. If $\Omega_0 \in \pi_2(F_2(N), y_0)$ and an immersion $f\colon S^2 \to N$ are given there is an immersion $g\colon S^2 \to N$ such that $\Omega(f, g) = \Omega_0$. For example any two C^2 immersions of S^2 in E^3 are regularly homotopic. Given even $\gamma \in H_2(S^2)$, then there is an immersion of S^2 in E^4 such that the characteristic class of the normal bundle is γ. Any two such C^2 immersions are regularly homotopic. (Received January 2, 1957.)

LOGIC AND FOUNDATIONS

381*t*. J. W. Addison and S. C. Kleene: *A note on function quantification.*

The question raised in Bull. Amer. Math. Soc. vol. 61 (1955) p. 211, whether for $k>0$ each predicate expressible in both the $k+1$-function-quantifier forms is hyperarithmetical in predicates expressible in the k-function-quantifier forms, is answered in the negative. The predicates of the finite $\mathfrak{E}$-hierarchy $\mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4, \cdots$ (after $\mathfrak{E}_0, \mathfrak{E}_1$) and its extension into the transfinite up through the constructive third number class lie properly between $\mathfrak{R}_1$ and $\mathfrak{R}_3$ in hyperdegree. (Received January 2, 1957.)

382*t*. Angelo Margaris: *A problem of Rosser and Turquette in many-valued logic.*

An affirmative solution is given for the following problem posed by Rosser and Turquette [*Many-valued logics*, Amsterdam, 1952, p. 110]. "Given an S and T such that $1 \leqq S < T < M$, can one define a many-valued logic so that if P is a statement formula with the truth-value function p, then the following conditions are satisfied? (1) If p is such that we always have $p \leqq S$, then $\vdash P$. (2) If p is ever such that $p > T$, then not $\vdash P$. (3) If neither condition 1 nor condition 2 is satisfied, then we have $\vdash P$ in some cases and not $\vdash P$ in other cases." A solution is given for the case $S=1$, $T=2$, $M=3$, and is then extended to the general case. A further extension solves the problem on the level of the restricted predicate calculus. The key device is the use of two valuations, one of which is the classical one. (Received December 11, 1956.)

STATISTICS AND PROBABILITY

383*t*. K. S. Miller and R. I. Bernstein: *Generalized Rayleigh processes.*

Let $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_N$ be N independent stationary Gaussian processes with means

a_s respectively and with correlation function $\psi(\xi)$. A generalized Rayleigh process $\Re$ is defined as $\Re^2=\sum_{s=1}^{N}\mathfrak{X}_s^2$. Properties of $\Re$ including second and third order distributions are investigated. Typical results are: (i) The joint probability density of $\Re$: $p(R_1, R_2)=(R_1R_2)^{N/2}[\psi_0^{(N+2)/2}(1-\lambda^2)\Gamma(N/2)(2\lambda)^{(N-2)/2}]^{-1}I_{N/2-1}(\lambda R_1R_2\psi_0^{-1}(1-\lambda^2)^{-1})$ $\cdot\exp\left[-(R_1^2+R_2^2)/2\psi_0(1-\lambda^2)\right]$ where $I_{N/2-1}$ is a Bessel function, $\lambda=\psi(\xi)/\psi_0$ is the normalized autocorrelation function and $a_1=a_2=\cdots=a_N=0$. (ii) The correlation function of $\Re$: $C(x)=[2\psi_0(1-\lambda^2)^{\alpha+1}][\Gamma(\alpha)]^{-1}D^{\alpha-1}[E(\lambda)/(1-\lambda^2)^2-K(\lambda)/2(1-\lambda^2)]$ where $N=2\alpha$, $D^{\alpha-1}$ is the $(\alpha-1)$st derivative with respect to λ^2 and E and K are complete elliptic integrals. Applications to radar and other problems are considered. A brief survey is made of similar formulas which occur in the literature. (Received January 2, 1957.)

384*t*. Valdemars Punga: *On solution of Dirichlet problem by Monte Carlo method in regions covered by a nonuniform network.*

Consider the two-dimensional $\nabla^2\psi=0$, ψ prescribed at the boundary of the region. The stochastic model for the above stated Dirichlet problem is usually formulated in the following way: subdividing the region of integration into square network of mesh size h, a random walk along the mesh lines starts at an interior nodal point (x, y) and ends upon reaching any boundary point. It is assumed that it is equally likely to proceed from (x, y) to any one of the four neighboring points, i.e. the probability of each of the four directions is 1/4. If $p(x, y)$ is the probability starting from (x, y) to reach a boundary point B, then (1) $p(x, y)=p(x+h, y)/4+p(x, y+h)/4+p(x-h, y)/4+p(x, y-h)/4$, $p(B)=1$, $p=0$ at all other points of the boundary. Now, if the distances from the point (x, y) to the four neighboring points are not all equal (what usually happens near the curved boundary), (1) is not adequate for definition of $p(x, y)$. It is shown in this paper that, in case the distances from (x, y) to $(x+h, y)$, $(x, y+h)$, $(x-h, y)$ and $(x, y-h)$ are h_1, h_2, h_3, and h_4, respectively, (1) can be replaced by (2) $p(x, y)=A_1p(x+h, y)+A_2p(x, y+h)+A_3p(x-h, y)+A_4p(x, y-h)$, where $A_1=h_2h_3h_4/(h_1h_3+h_2h_4)(h_1+h_3)$, $A_2=h_1h_3h_4/(h_1h_3+h_2h_4)(h_2+h_4)$, $A_3=h_1h_2h_4/(h_1h_3+h_2h_4)(h_1+h_3)$, $A_4=h_1h_2h_3/(h_1h_3+h_2h_4)(h_4+h_2)$. Since $p(x, y)$ is a Green function of Dirichlet problem, the solution of $\nabla^2\psi=0$ follows in usual pattern. Note that $\sum_{i=1}^{4}A_i=1$, so that $A-s$ can be interpreted as the probabilities of proceedings to the neighboring points. (Received January 11, 1957.)

TOPOLOGY

385*t*. R. W. Bass: *Generalizations to n dimensions of the combined theorems of Poincaré-Bendixson, Poincaré-Denjoy, and Haas* (I).

Notations as in Abstract No. 359, except that E^n is replaced by an n-manifold D^n. Other definitions as in Nemickii, Amer. Math. Soc. Translations, no. 103. Define a dynamical system S to be ***ultimately dissipative*** if its center is compact. define S to be ***smooth*** if every Poisson stable orbit is uniformly Lyapunov stable relative to (at least) itself, and to be ***extremely smooth*** if in addition every quasi-minimal set of dimension $\leq(n-2)$ is locally connected. S is defined to be ***very regular*** if every compact quasi-minimal set of S is actually minimal. S is defined to be ***nice*** if every compact quasi-minimal set containing no rest point is a manifold. Theorem 1. *The positive limit set of any bounded asymptotic semi-orbit is nowhere dense.* Theorem 2. *If S is very regular and ultimately dissipative then the closure of any nonasymptotic orbit is either* (i) *nowhere dense, or* (ii) *equal to D'', in which case S has no rest points and D^n*

is a rotational manifold. Corollary 1. *If S is very regular and ultimately dissipative then either the positive limit set of any given orbit is nowhere dense, or the orbit is dense on D^n and D^n is a rotational manifold.* (Received January 10, 1957.)

386t. R. W. Bass: *Generalisations to n dimensions of the combined theorems of Poincaré-Bendixson, Poincaré-Denjoy, and Haas* (II).

Theorem 3. *If S is ultimately dissipative and nice, and if the presence of a rest point in the positive limit set of an orbit implies that there are no other points in the limit set, then there are only two kinds of orbits in S:* (a) *asymptotic orbits, and* (b) *Poisson stable orbits dense on rotational manifolds. Moreover, each bounded positive semi-orbit of type* (a) *contains at least one of type* (b) *in its positive limit set.* The dimensional restrictions in the following theorem are nontrivial. Theorem 4. *Let $D^n = E^n$. If S is ultimately dissipative and smooth, then there are only three kinds of orbits of S:* (a) *asymptotic orbits, and* (b) *almost-periodic orbits ergodic on* (b_1) *k-dimensional tori, $k \in \{0, 1, \cdots, n-1\}$; or* ($b_2$) *$k$-dimensional solenoids, $k \in \{0, 1, \cdots, n-2\}$. Moreover, each bounded positive semi-orbit of type* (a) *contains at least one orbit of type* (b) *in its positive limit set.* Theorem 5. *If, in addition to the hypotheses of Theorem* 4, *S is extremely smooth, then the orbits of type* (b_2) *cannot occur.* One can show by examples that Corollary 1 and Theorems 4 and 5 cannot be improved. If S is defined by a C^1 vector field in the plane, or by a C^3 vector field on a compact orientable 2-manifold, then S is very regular, nice, and extremely smooth. Since the only rotational orientable 2-manifold is the 2-torus, Corollary 1 and Theorems 3 and 5 are each generalizations of the (combined) theorems of Poincaré-Bendixson, Poincaré-Denjoy, and Haas. (Received January 10, 1957.)

387t. R. H. Bing: *Pseudo-arc is only nondegenerate homogeneous chainable compact continuum.*

It is shown that if M is a homogeneous snake-like continuum, then M contains an end point p in the sense that for each positive number ϵ there is an ϵ-chain covering M one of whose end links contains p. This insures that M is hereditarily indecomposable. It is known [R. H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. of Math. vol. 1 (1951) pp. 43–51] that any two nondegenerate hereditarily indecomposable chainable compact continua are topologically equivalent and that each is a pseudo-arc. (Received January 7, 1957.)

388t. M. R. Demers and Herbert Federer: *The norm of a cohomology class, with applications to the theory of area.*

With each integral Čech cohomology class $\lambda \in H^k(X, A)$, the authors associate the norm $|\lambda|$, a nonnegative integer or ∞. In case X is a k dimensional finitely triangulable space, $|\lambda|$ measures the minimal multiplicity with which a continuous map, representing λ, of (X, A) into a relative k-cell (S, T) must cover S. Using this norm, the authors define the multiplicity $M(f, y)$ with which a map f of X into Euclidean k-space assumes the value y. It is proved that the *Lebesgue area* of f equals $\int M(f, y)dy$, an integral with respect to *Lebesgue measure* over k-space. Furthermore, the Lebesgue area of a map f of X into n-space, where $n > k$, does not exceed the sum of the Lebesgue areas of the $\binom{n}{k}$ principal projections of f into k-space, provided the image of f has $k+1$ dimensional Hausdorff measure zero. Previously such results had been established only for the special case in which X can be embedded in a k dimensional manifold. (Received December 21, 1956.)

the group or the space is not compact, the result may fail. Upon adding a point at infinity to euclidean space, it is seen that Montgomery's conjecture holds for a compact Lie group operating in euclidean space E. Consequently (E, G) admits an equivariant embedding in euclidean space. (Received January 10, 1957.)

R. D. SCHAFER,
Associate Secretary

BOOK REVIEWS

Linear inequalities and related systems. Ed. by H. W. Kuhn and A. W. Tucker. Annals of Mathematics Studies, no. 38, Princeton, Princeton University Press, 1956. 22+322 pp. $5.00.

This is the first collection of papers devoted to a subject standing midway between the study of convex sets in linear spaces which it uses and that of certain optimization problems, including minimax approximation, linear programming and matrix games, to which it is applied, and which in turn bear on decision processes in statistical and economic theory. Delineated by Fourier in connection with mechanical equilibrium as well as minimax error problems, given (at the occasion of determining a fundamental region for the set of all positive definite quadratic forms under a discrete transformation group) its substance and essential results by Minkowski, developed by Farkas, Dines, Schoenberg, Weyl and others and systematically established in the reviewer's inaugural dissertation, the theory of linear inequalities with its ramifications has, in the last decade, attracted increasing attention in this country, and also in the U.S.S.R., while modern high-speed computing has made its practical application feasible.

The present study contains nine papers dealing with general linear systems and their geometric and extremizational counterparts, followed by nine papers on special cases. Of the latter, those by Dantzig, Fulkerson, Heller, Hoffman, Kruskal, Kuhn, and Tompkins consider systems whose basic solutions belong to the ring (not merely field) of coefficients, in particular array problems; the paper by Gale shows that, in $2n$-space, convex polyhedra with an arbitrary number of vertices exist every n of which form a face (cf. Motzkin, *Comonotone curves and polyhedra*, Bull. Amer. Math. Soc., vol. 63 (1957), p. 35); Kuhn and Gale re-prove the theorems of Wald and von Neumann on economic equilibrium; a related economic problem is brought into game-theoretic form by Thompson and shown to have only a finite number of solutions.

Of the papers on general systems, Fan's is the first systematic study of linear inequalities which includes infinite systems, systems in infinite dimensional linear spaces and also systems in complex linear spaces. Duffin, *Infinite programs*, likewise deals with linear operators. A new algorithm for linear programs is given by Dantzig, Ford and Fulkerson. Wolfe considers matrix games with polyhedral, instead of simplicial, strategy sets. Mills establishes an elegant theorem on the variation of the value of a game as function of its matrix.

The first four papers, by Goldman and Tucker, culminate in a *Theory of linear programming*, preceded (also logically) by an examination of the structure of polyhedral convex cones under set inclusion, by variants of the reviewer's theorems on the resolution of a polyhedral convex set into the sum of a polyhedron and a polyhedral cone and on systems related by matrix transposition, and by Tucker's opening paper on *Dual systems of homogeneous linear relations*, which consolidates previous transposition theorems and introduces the concept of slackness, stressing the importance of equated constraints (casually appearing in Stiemke, Math. Ann. vol. 76, p. 342; cf. also Miller, Amer. Math. Monthly vol. 17, p. 137; Černikov, Mat. Sb. N.S. vol. 38 (80), pp. 506–507).

As might be expected from the editors of Annals Studies 24 and 28, the value of this representative cross-section of recent research in the field is enhanced by careful revision, pleasant appearance, a clear and instructive résumé of the individual papers in the preface to the study, and an extensive and up-to-date bibliography.

THEODORE S. MOTZKIN

Einführung in die Geometrie der Waben. By W. Blaschke. (Elemente der Mathematik vom höheren Standpunkt aus, Band IV) Birkhäuser Verlag, Basel and Stuttgart, 1955. 108 pp. SFr 15.25.

This monograph contains four chapters. Chapter I deals with 3-honeycombs $\mathfrak{W}$ of curves in a Euclidean plane E_2, called 3-webs of curves in *Geometrie der Gewebe* (Springer, Berlin, 1938) by the author and G. Bol. For a $\mathfrak{W}$ three Pfaffian forms are introduced, and the connection γ and the curvature k are derived by exterior differentiation. It is proved that a necessary and sufficient condition for $\mathfrak{W}$ to be a hexagonal honeycomb is that k vanishes identically. Furthermore k is expressed in terms of a honeycomb function W. There are also obtained a complete system of invariants and a canonical expansion of the function W, from which G. Thomsen's geometric interpretation of the curvature k is deduced. Rotations and conformal mappings of honeycombs are studied by means of complex Pfaffian forms.

Chapter II is devoted to 4-honeycombs $\mathfrak{W}$ of surfaces in a three-dimensional Euclidean space E_3. As in Chapter I for a $\mathfrak{W}$ the author introduces four Pfaffian forms and obtains the connection γ, the curvatures a_1, a_2, a_3, a complete system of invariants and a canonical expansion of a honeycomb function W. It is proved that a necessary and sufficient condition for a $\mathfrak{W}$ to be an octahedral honeycomb is that a_1, a_2 and a_3 vanish identically. Hexagonal honeycombs of surfaces and octahedral and hexagonal honeycombs of planes are also discussed.

In the last two chapters n-honeycombs of curves in a plane E_2 and in a space E_3 are studied for different values of n. At the end of Chapter II and Chapter III there are listed a few problems, some of which are referred to current papers.

C. C. HSIUNG

Darstellungen von Gruppen, mit Berücksichtigung der Bedürfnisse der modernen Physik. By H. Boerner (Die Grundlehren der mathematischen Wissenschaften, vol. 74) Berlin-Göttingen-Heidelberg, Springer-Verlag, 1955. 11+287 pp. DM 36.60.

This book is concerned with the representation theory of finite groups and of the classical Lie groups, with the exception of the symplectic group. The literature on the classical groups has not been noted for its lucidity up to now. This book does a great deal to clarify matters and to render the theory more accessible. This is due largely to the tangibility and explicitness with which the author treats things. For example, the necessary parts of the Lie theory are dealt with in a very neat and concrete way so that it would be possible for a person unfamiliar with it to obtain a good insight into how it works. On the other hand, some more classical matters such as the Wedderburn theorems and Maschke's theorem are treated in a very conservative manner—lots of idempotents and matrices—which possibly is not in accord with the highest aesthetic ideals. The first chapter is devoted to elementary linear algebra and the next two develop the general representation theory of finite groups and linear Lie groups. Lie algebras and group integration are introduced and the character relations for compact Lie groups are obtained. Throughout the whole book the ground field is assumed to be of characteristic zero and, for the most part, algebraically closed. There is an extensive and detailed account of the representation theory of the symmetric group which is certainly one of the most complete available. The rest of the book deals with the classical groups. Generally speaking, the treatment is as purely algebraic as possible; comparatively little use is made of group integration. The representations of the full linear group are obtained as in Weyl's book (*The Classical Groups*, Princeton, 1936). In this connection there is one elementary point which the author does not make sufficiently clear. It is shown that the semigroup of all square matrices of degree n induces the full algebra of bisymmetric transformations on the space of tensors of given rank. However it is necessary to show that the group of nonsingular matrices will do it. This is not exactly trivial and requires some such device as Weyl's principle of irrelevance of algebraic inequalities. A rather unusual and commendable feature of the book is the treatment of the rotation group by the method of Stiefel. The spin representations are obtained in several ways, including the method of Brauer and Weyl. The author's handling of the topological matters relevant to the rotation group is perforce rather sketchy, as he readily admits. However it could be made somewhat less so with very little extra effort. For instance, it would be no trouble to give at least a precise analytic definition of homotopy before constructing covering groups. The final chapter is devoted to the finite-dimensional representations of the Lorentz group.

The subject of which this book treats is a difficult one and it is far from easy to achieve clarity of exposition in dealing with it. In the reviewer's opinion, Professor Boerner has succeeded in this respect better than anybody so far. Not the least of the consequences is that it renders more accessible the rest of the literature on the subject, especially Weyl's book which even after all the years since its publication still contains such a wealth of stimulating ideas. There is only one major regret, namely that the symplectic group has been ignored, presumably because of the absence of applications to physics. It is sincerely to be hoped that it will come in for its proper share of attention in a future edition.

W. E. JENNER

The convolution transform by I. I. Hirschman and D. V. Widder. Princeton, Princeton University Press, 1955. 10+268 pp. $5.50.

At a time when so much mathematics writing has had a tendency to become ponderous and unreadable it is a pleasure to have this excellent book appear. The style of the authors has become well known through their extensive papers and the precision and care that has marked their writing in the past is scrupulously maintained throughout the book. Furthermore the subject is itself of considerable interest being classical in every sense of the word, and appearing again and again in the most varied contemporary contexts. The problems treated are solved on their own ground and the theorems are not based on changing the problem. The ratio of theorems to definitions in this book is high.

The main part of the book is devoted to a study of the convolution equation whose kernel has the property that the reciprocal of its Laplace transform is entire and of the form $\exp(-cs^2+bs)\prod(1-s/a_k)\cdot\exp s/a_k$ where a_k is real, $c\geqq 0$ and $\sum a_k^{-2}<\infty$. This class of kernels has been studied extensively in recent years by Schoenberg and some of his work is reproduced in Chapter IV. Except for this the first seven chapters are devoted to the authors' own investigations of the case $c=0$. Chapters VI and VII have detailed discussions of the inversion and representation theories. Chapter VIII contains a discussion of the Weierstrass transform. This is the only case in which $c\neq 0$ that is discussed in the book, although it is actually the whole class described above that appears in several connections as the "natural" one. Chapter IX is concerned with the complex inversion of convolutions whose kernel has a Laplace transform whose reciprocal is of the form $\prod(1-s^2/a_k^2)$ with real a_k and $\lim k/a_k^2=\Omega$, $0<\Omega<\infty$. The final chapter has a discussion of Bernstein polynomials, the asymptotic behavior of the convolution transform, the analytic behavior of the transforms of the kernels and a theorem on quasi-analyticity.

In some respects the title of the book is misleading. While agreeing that it would be a mistake to try to include all of the work in convolution transforms that has been done one cannot help but feel that what is here presented is actually so restricted in scope as to present a distorted view to the reader studying the material for the first time. Moreover the existence of other problems in the field is not even hinted at. Even if it had to be at the expense of some of the finer details in, say the representation or inversion theory, a chapter devoted to a discussion of the state of other questions such as spectral synthesis, equations of the Wiener-Hopf type or local and non-local inversions would have been valuable, particularly from these authors.

In this connection the bibliography is also somewhat inadequate. Even papers of basic importance to the matters discussed are not mentioned. For example, although there is a chapter devoted to the Weierstrass transform the only references to work other than their own is one to Hille's book on semi-groups and one to Tychonoff's paper on his theorem of uniqueness for solutions of the heat equation. Also Pollard's basic paper on the material of Chapter IX (Ann. of Math. vol. 49 (1948) pp. 956–965) is not mentioned although other papers of his of less relevancy are. As a final example Post's work on differential operators (Trans. Amer. Math. Soc. vol. 32 (1930) pp. 723–781) is not listed in spite of the fact that it is not only important for the work already done in the field of convolutions but also shows promise of being important in some of the unsolved problems.

The above comments should not be construed as meaning that the book is not an excellent one. It is just not possible to satisfy everyone on all points.

J. BLACKMAN

Handbuch der Laplace-Transformation, Vol. III. *Anwendungen der Laplace-Transformation*, Part 2. By Gustav Doetsch. Basel and Stuttgart, Birkhäuser, 1956. 300 pp. 40 Swiss francs.

This final volume of Doetsch's treatise contains applications to partial differential equations, difference equations, integral equations, and entire functions of exponential type. (Vols. 1 and 2 were reviewed in this Bulletin, vol. 58 (1952), pp. 670–673, and vol. 62 (1956), p. 628.) An appendix gives a number of remarks supplementing various points in vol. 1. An extensive bibliography for vols. 2 and 3 appears at the end of vol. 3.

For differential equations, the author continues to give a presentation that is mathematically satisfying and at the same time maintains contact with the physical background. At the end of the section on differential equations there is a chapter on "Huygens's and Euler's principles": the former is regarded as a source of relations among special functions which appear as Green's functions; the latter is the author's term for the fact that one can sometimes solve the same equation in two ways and thus obtain an identity connecting the solutions. There are three chapters on difference equations. The first deals with the equation $c_1Y(t)+\cdots+c_nY(t+n)=G(t)$, where the "initial conditions" are that $Y(t)$ is given in $0\leqq t<n$; this is an unfamiliar analogue of the linear differential equations that are usually treated by transforming the whole equation, and is given an analogous treatment here. The second chapter deals with difference equations in analytic functions, which are treated essentially by applying the inverse Laplace transform. The third chapter gives some examples of how partial difference equations can be solved. The chapters on integral equations take up various types which can be attacked by various methods, including that of Wiener and Hopf; the equation of renewal theory, Abel's equation, and derivatives and integrals of fractional order illustrate the general methods. Special chapters are devoted to functional relations for special functions, obtained from algebraic relations for their transforms. Convolution integral equations (where the convolution is an integral either over a vertical line or a closed path) are studied in the complex domain, again with applications to derivatives and integrals of fractional order. It is to be hoped, but hardly expected, that would-be writers on the subject of differentiability of fractional order will consult the

author's discussions and heed his warning that there cannot be any universal definition.

The last two chapters deal with finite Laplace transforms and entire functions of exponential type. Particularly noteworthy is the most elementary proof yet given of the theorem about the indicator diagram of a finite Laplace transform (the author seems to be unaware that the equivalent result for Fourier transforms was already known).

R. P. Boas, Jr.

RESEARCH PROBLEMS

3. D. R. Hughes. *A problem in group theory.*

Let G be a group and p a prime. Define $H_p(G)$ to be the subgroup of G generated by all the elements of G which do not have order p. Is the following conjecture true: *either* $H_p(G)=1$, $H_p(G)=G$, *or* $[G:H_p(G)]=p$? For $p=2$, the conjecture is easily established (Lemma 4, *Partial difference sets*, Amer. J. Math. vol. 78 (1956) pp. 650–674). It is possible that the conjecture has a different status according as G is finite or not. (Received December 26, 1956.)

4. V. L. Klee, Jr. *Banach algebras.*

Suppose A is a real Banach algebra and P is the set of all sums of (finitely many) squares in A. What can be said about the Borel type of P? In particular, must P be a G_δ set? (Received December 26, 1956.)

NOTES

The following are among those who received doctorates in the mathematical sciences and related subjects from universities in the United States and Canada during 1956. In each case, the university, the month in which the degree was conferred, minor subjects (other than mathematics), and title of the dissertation are given.

Smbat Abian, Cincinnati, June, *Invariants and covariants of systems of linear differential and integro-differential equations.*

J. H. Ahlberg, Yale, June, *Algebraic properties of topological significance.*

F. E. Alzofon, California at Berkeley, September, *Multiple valued functions with two circles as branch curves and the Sommerfeld method.*

Douglas Anderson, California Institute of Technology, June, minor in aeronautics, *Invariant measures on groups.*

L. W. Anderson, Tulane, May, *Topological lattices.*

R. R. Archer, Massachusetts Institute of Technology, June, minor in electrical engineering, *Post buckling behavior of thin spherical shells.*

Steve Armentrout, Texas, June, Minor in physics, *On spirals in the plane.*

D. G. Aronson, Massachusetts Institute of Technology, February, minor in electrical engineering, *A boundary layer problem for a linear parabolic differential equation.*

R. L. Ashenhurst, Harvard, June, *The structure of multiple-coincidence selection systems.*

R. P. Authement, Louisiana State, August, minor in physics, *Sums of irreducible polynomials with coefficients in* $GF(c)$.

George Bachman, New York, February, *Geometry in groups.*

J. L. Bagg, Michigan State, *A probability model for theory of organization of groups with multi-valued relations between persons.*

R. S. Ballance, Illinois, June, *Cauchy type representations for functions of a complex variable.*

L. C. Barrett, Utah, August, *Analytical solutions for the optimization of rocket trajectories.*

W. E. Baxter, Pennsylvania, June, *Lie simplicity of a special class of associative rings.*

Anatole Beck, Yale, June, *On the random ergodic theorem.*

H. F. Becksfort, Syracuse, September, *The solution of a functional equation arising from a fluid flow problem.*

Barry Bernstein, Indiana, June, minor in physics, *The differential equations of the stream lines for compressible flow of an ideal gas.*

A. A. Betser, Illinois Institute of Technology, June, *Studies in dynamic photoelasticity fringe values and beams under impact.*

R. E. Block, Chicago, August, *New classes of simple Lie algebras of characteristic p.*

R. M. Blumenthal, Cornell, September, *An extended Markov property.*

A. M. Bomberault, Carnegie Institute of Technology, September, *The application of variational methods to the flow of viscous fluids through elastic channels.*

G. W. Booth, New York, May, *A uniqueness theorem for the reduced plate equation.*

F. G. Brauer, Massachusetts Institute of Technology, June, minor in physics, *Singular self-adjoint boundary value problems for the differential equation* $Lx=\lambda Mx$.

C. F. Briggs, Michigan, February, *Semi-topological linear spaces.*

K. A. Brons, Illinois, June, minor in physics, *Groups, all of whose partials endomorphisms are extendable.*

F. P. Brooks, Jr., Harvard, June, *The analytic design of automatic data processing systems.*

Leon Brown, Minnesota, December, minor in physics, *Nonlinear equations in a Banach space.*

W. F. Brown, Indiana, June, minor in physics, *Certain problems related to pressure determination on two-dimensional supersonic airfoils.*

S. J. Bryant, Missouri, June, *Ordered rings.*

R. G. Buschman, Colorado, June, *Substitution theorems for integral transforms.*

G. C. Caldwell, North Carolina, August, *Application of method of successive approximations to the solution of functional equations.*

J. F. Carpenter, Jr., Oregon State, June, minor in physics, *On the Doppler frequency spectrum presented to a moving radar system by reflection from a rough plane earth.*

R. V. S. Chacon, Syracuse, June, *Some theorems on continuous parameter Markov chains.*

John Cocke, Duke, June, minor in physics, *The regular point.*

A. E. Danese, Rochester, June, *On Turan functions associated with the classical orthogonal polynomials, Bessel functions, and their derivatives.*

W. F. Davison, Virginia, June, *Mosaics.*

E. C. DeLand, California at Los Angeles, June, *On the separation of variables for Laplace's equation.*

A. P. Dempster, Princeton, June, *The two sample multivariate problem in the degenerate case.*

R. F. Dennemeyer, California at Los Angeles, June, *Quadratic forms in Hilbert space and second order elliptic differential equations.*

B. J. Derwort, St. Louis, June, *An extension of the theory of cumulative frequency functions.*

F. C. DeSua, Pittsburgh, June, *An extension of well-chainedness in metric spaces to Hausdorff spaces.*

Gus DiAntonio, Pittsburgh, June, *Summability of double series by abelian methods.*

Roberto Diaz-Fernandez, California at Berkeley, January, *Alternative rings and their regular ideals.*

Sylvain Ehrenfeld, Columbia, May, *Complete class theorems in design of experiments.* Part I. *Complete class theorems in experimental design.* Part II. *On the efficiency of experimental design.*

H. A. Eliopoulos, Toronto, November, *Methods of generalized metric geometry with applications to mathematical physics.*

T. A. Elkins, Carnegie Institute of Technology, June, *Conjugate harmonic functions in space.*

D. I. Epstein, New York, February, *Diffraction problems for the parabolic cylinder.*

I. J. Epstein, New York, February, *A study of integrals and integral equations arising in diffraction theory.*

C. C. Faith, Purdue, January, *Normal bases and Galois theory.*

E. H. Farr, Carnegie Institute of Technology, June, *On pseudo-analytic functions and elliptic equations.*

A. V. Fend, Illinois, June, *Unbiased estimation and admissibility and the treatment of ties in the sign test.*

T. S. Ferguson, California at Berkeley, June, Part I: *On the existence of linear regression in linear structural relations*; Part II: *A method of generating best asymptotically normal estimates with application to the estimation of bacterial densities.*

Barbara A. Foos, Notre Dame, July, *On the values of certain sets of modules.*

Marguerite J. S. Frank, Radcliffe, June, *New simple Lie algebras.*

Aubyn Freed, Illinois, June, minor in philosophy, *On the ergodic theorem in dynamical systems with variant measure.*

D. A. Gardiner, North Carolina State College, May, *Some third order rotatable designs.*

R. C. Gast, Carnegie Institute of Technology, June, I. *Half-plane diffraction with line source excitation.* II. *Diffraction by two parallel half-planes with line source excitation.*

D. P. Gaver, Jr., Princeton, June, *Some results in the theory of queues.*

Jesus Gil de Lamadrid, Michigan, February, *Topology of mappings in locally convex topological vector spaces, their differentiation and integration and application of gradient mappings.*

R. P. Goblirsch, Wisconsin, August, *Approximating the area of a surface with the area of a nearby polyhedral one.*

R. R. Goldberg, Harvard, June, *Generalized Lambert transforms.*

Lawrence Goldman, Columbia, May, *Specializations and Picard-Vessiot theory.*

Basil Gordon, California Institute of Technology, June, minor in physics, *Some Tauberian theorems connected with the prime number theorem.*

D. G. Gosslee, North Carolina State College, August, minor in climatology, *Level of significance and power of the unweighted means' test.*

E. E. Grace, North Carolina, August, *Certain properties of continua related to non-aposyndesis.*

Donald Greenspan, Maryland, June, *On vertices of space curves with respect to families of surfaces.*

H. P. Greenspan, Harvard, June, *Generation of edge waves by moving pressure distributions.*

R. M. Gundersen, Brown, June, *The flow of a compressible fluid with weak entropy changes.*

S. S. Gupta, North Carolina, August, *On a decision rule for a problem in ranking means.*

H. M. Gurk, Pennsylvania, February, *Extreme games, simple games, and finite solutions.*

Herman Hanish, New York, May, *On a certain class of Tauberian theorems.*

M. A. Hanhauser, St. Louis, June, minor in physics, *The generalized Bäcklund transformation.*

J. R. Hanna, Colorado, June, *The Dirichlet series transformation.*

Bruno Harris, Yale, June, *Galois theory of Jordan algebras.*

J. J. Harton, Jr., California at Berkeley, June, *Extremal problems for real star mappings.*

A. S. Hausner, Yale, June, *Group algebras of vector-valued functions.*

L. O. Heflinger, California at Berkeley, September, *Applications of the multiplication-interpolation method.*

L. L. Helms, Purdue, June, *Convergence properties of martingales indexed by directed sets.*

G. R. Herd, Iowa State, December, minor in economics, *Estimation of the parameters of a population from a multi-censored sample.*

W. A. Hijab, University of Florida, August, minor in civil engineering, *Application of Papkovitch functions to three-dimensional problems of elasticity.*

T. W. Hildebrandt, Michigan, June, I, *Iterative methods for the approximate solutions of linear algebraic systems;* II. *Self-adjointness in one-group multi-region diffusion problems.*

Harry Hochstadt, New York, February, *Addition theorem for and applications of functions of the paraboloid of revolution.*

K. M. Hoffman, California at Los Angeles, June, *Boundary behaviour of generalized analytic functions.*

Walter Hoffman, Michigan, June, *Group logics and restricted implication.*

V. E. Hoggatt, Jr., Oregon State, June, minor in physics, *The inverse Weierstrass p-function: numerical solution, related properties and applications.*

J. G. Horne, Jr., Tulane, August, *Concerning o-ideals.*

W. A. Howard, Chicago, December, *k-fold recursion and well-ordering.*

W. G. Howe, North Carolina, June, *Some contributions to factor analysis.*

Bernard Jacobson, Michigan State, August, *Sums of distinct divisors of rational integers and sums of distinct divisors of quadratic integers.*

Earl Janssen, California at Los Angeles, January, minor in engineering, I. *An analog method for solving the hydro-dynamic equations for two dimensional viscous flow.* II. *Application of the method to the case of flow past a flat plate.*

K. A. Johannes, Pittsburgh, February, *Plane anticollineations with distinct fixed or interchanging points and their representation in four space.*

M. V. Johns, Jr., Columbia, June, *Contributions to the theory of empirical Bayes procedures in statistics.*

G. P. Johnson, Minnesota, December, *Spaces of functions with values in a Banach algebra.*

R. D. Johnson, Jr., Virginia, June, *Homology regular convergence and local connectedness.*

R. W. Jollensten, Virginia, June, *Topological applications to functions of several complex variables.*

Meyer Jordan, New York, February, *A mixed boundary potential problem arising from a problem in ship motion.*

M. A. Kastenbaum, North Carolina State College, January, *Analysis of data in multiway contingency tables.*

C. L. Keller, Jr., Illinois, minor in philosophy, *Approximations of functions of several complex variables.*

R. R. D. Kemp, Massachusetts Institute of Technology, June, minor in physics, *The eigenvalue problem for a non-selfadjoint differential operator on the interval* $(-\infty, \infty)$.

O. M. Kloee, University of Washington, June, *Topics in distribution-free statistics.*

Shochichi Kobayashi, University of Washington, June, *Theory of connections.*

E. E. Kohlbecker, Illinois, June, *Asymptotic properties of partitions.*

J. J. Kohn, Princeton, June, *A non-selfadjoint boundary value problem on pseudo-Kähler manifolds.*

Anthony Kooharian, Brown, June, *On a class of singular perturbations.*

W. E. Koss, Illinois, June, minor in philosophy, *On four parameter families of quadratic surfaces.*

Frank Kozin, Illinois Institute of Technology, June, *Probability measure and integration of functionals.*

A. H. Kruse, Chicago, June, *Introduction to the theory of block assemblages of and development in the theory of retraction.*

T. E. Kurtz, Princeton, June, *An extension of a multiple comparisons procedure.*

Anneli Lax, New York, June, *Cauchy's problem for a partial differential equation with real multiple characteristics.*

Walter Littman, New York, February, *On the existence of periodic waves near critical speed.*

T. P. G. Liverman, Pennsylvania, June, *Zeros of neighboring infinitely differentiable functions.*

Leo Lopidus, Michigan State, August, *Lattice metrized spaces.*

D. B. Lowdenslager, Virginia, June, *Duality in partially ordered vector spaces.*

R. A. Luebbe, Cincinnati, June, *Cesàro summability factors for L.P. spaces.*

E. J. Lytle, Jr., University of Florida, June, minor in economics, *The determination of some distributions for which the mid-range is an efficient estimator of the mean.*

W. S. McCulley, Texas, June, minor in physics, *Integration formulae and boundary conditions for the hyperbolic differential equation with three independent variables and regions interior to the cone.*

Edward McDowell, Illinois Institute of Technology, June, *On the steady-state thermoelastic problem.*

F. S. McFeely, Virginia Polytechnic Institute, June, *Decision procedures for the comparison of exponential and geometric populations.*

H. C. McKenzie, Colorado, August, *On the zeros of successive derivatives of an analytic function.*

Michel McKiernan, Illinois Institute of Technology, June, *The functional differential equation, $Df^{-1}=f$.*

R. C. MacCamy, California at Berkeley, January, *Linear boundary problems arising in the diffraction of waterwaves by surface obstacles.*

J. H. MacKay, North Carolina, June, *On the efficiency of certain tests for 2×2 tables.*

T. A. Magness, California at Los Angeles, *The use of cumulants in the theory and applications of stochastic processes.*

Angelo Margaris, Cornell, September, *A problem of Rosser and Turquette in many-valued logic.*

C. E. Marshall, Iowa State, December, minor in economics, *Cost control of sample surveys by two-step design.*

Ceslovas Massaitis, Tennessee, June, *The complement of a dendrite in compactified 3-space.*

J. S. Maybee, Minnesota, December, minor in philosophy, *On some problems suggested by the mathematical theory of compressible flow.*

Z. A. Melzak, Massachusetts Institute of Technology, June, minor in physics, *Existence of the solutions of a scalar transport equation.*

Pinchas Mendelson, Princeton, June, *On isolated singularities of differential equations with one zero characteristic root.*

T. J. Mentel, Brown, June, *Some problems in the large plastic deformation of beams under dynamic loading.*

D. M. Mesner, Michigan State, June, *An investigation of certain combinatorial properties of partially balanced incomplete block experimental designs and associated schemes, with study of latin square and related types.*

Marshall Middleton, Jr., Pittsburgh, August, *A method for finding the characteristic roots and vectors of an arbitrary real symmetric matrix.*

Irwin Miller, Virginia Polytechnic Institute, June, *Tests of hypotheses involving desirability relations and some distribution theory connected with Gaussian processes.*

S. K. Mitra, North Carolina, June, *Contributions to the statistical analysis of categorical data.*

Barbara C. Morrison, Brown, June, *Stress-deformation relations for isotropic time-independent materials.*

J. A. Morrison, Brown, June, *Small, forced oscillations in an ideal, rotating liquid and the slow motion of a disc along the axis of a viscuous rotating liquid.*

D. E. Moser, Pittsburgh, June, *Inclusion relations involving the circle method of summability.*

J. R. Munkres, Michigan, February, *Some applications of triangulation theorems.*

Isaac Namioka, California at Berkeley, June, *On partially ordered linear topological spaces.*

J. D. Neff, University of Florida, August, minor in electrical engineering, *A study of Heun's differential equation.*

Mary M. Neff, University of Florida, August, minor in philosophy, *On lattice-ordered rings.*

G. L. Neidhardt, Illinois Institute of Technology, January, *On the transmission of a concentrated load into the interior of an elastic body.*

A. R. S. Nerode, Chicago, June, *Composita, equations and recursive definitions.*

T. D. Oxley, Jr., Purdue, June, *A study of a generalised factorial series.*

R. S. Palais, Harvard, June, *A global formulation of the Lie theory of transformation groups.*

F. J. Palas, Oklahoma, June, Minor in physics, *The polynomials generated by* $f(t)$ exp $(p(x)u(\tau))$.

G. I. Paul, North Carolina State College, January, minor in genetics, *A method of estimating epistatic variance in random mating populations.*

W. H. Peirce, Wisconsin, August, *Numerical integration over planar regions.*

W. E. Perrault, St. Louis, June, *Contributions to distribution-free population comparisons.*

J. W. Pratt, Stanford, January, *Some results in the decision theory of one-parameter multivariate Pólya type distributions.*

R. W. Preisendorfer, California at Los Angeles, June, *A mathematical foundation for radiative transfer theory.*

J. J. Price, Pennsylvania, June, I. *Some duality theorems.* II. *On the characters of certain compact abelian groups.*

P. A. Puhach, McGill, May, *On the contributions of Mesen exchange currents to the radiative moments of nuclei.*

Ronald Pyke, University of Washington, August, *On one-sided distribution-free statistics.*

Roy Radner, Chicago, *Team decision problems.*

Mushfequr Rahman, McGill, October, *A statistical problem in the geometry of numbers (star-shaped domains of quadratic and hexagonal symmetry).*

L. B. Rall, Oregon State, June, minor in physics, *The quadratic formula in Banach space.*

Anthony Ralston, Massachusetts Institute of Technology, June, minor in electrical engineering, *Two problems in the buckling of thin elastic shells.*

Elvira S. Rapaport, New York, February, *On the reduction of words in the free group on three generators.*

R. W. Rector, Maryland, June, *Coloring seven-rings.*

B. L. Reinhart, Princeton, June, *Harmonic integrals on almost product manifolds.*

P. D. Ritger, New York, February, *The evaporation of a drop of liquid in a confined glass.*

W. L. Roach, Jr., Oregon, June, *The application of the exponential distribution to a truncated stochastic process.*

H. L. Rolf, Vanderbilt, June, minor in physics, *Free lattices generated by a set of chains.*

F. E. Romie, California at Los Angeles, June, minor in engineering, *Heat transfer to fluids flowing with velocity pulsations in a pipe.*

N. J. Rose, New York, May, *Minimum time solutions of differential equations with a discontinuing forcing term.*

Joan R. Rosenblatt, North Carolina, June, minor in economics, *On a class of non-parametric tests.*

G. C. Rota, Yale, June, *Extension theory of differential operators.*

A. R. Roy, Stanford, April, *On χ square statistics with variable intervals.*

J. S. Rustagi, Stanford, April, *On minimising and maximising a certain integral with statistical applications.*

Jerome Sacks, Cornell, September, *Asymptotic distribution of stochastic approximation procedures.*

E. M. Saleme, Illinois Institute of Technology, June, *Stress concentration around a circular inclusion in a semi-infinite elastic plate.*

S. C. Saunders, University of Washington, August, *Sequential and randomised distribution-free tolerance limits.*

N. W. Savage, Jr., California at Los Angeles, June, *Weak boundary components of an open Riemann surface.*

L. A. Schmittroth, Jr., Stanford, January, *Deformation classes of conformal mappings.*

D. E. W. Schumann, Virginia Polytechnic Institute, June, *The comparison of the sensitivities of experiments using different scales of measurement.*

Berthold Schweizer, Illinois Institute of Technology, June, *Jacobi series and the numerical solution of eigenvalue problems.*

R. C. Seber, State University of Iowa, February, *The development and organization of teaching materials in a collegiate mathematics program for students of the nonphysical sciences.*

C. F. Sebesta, Pittsburgh, February, *A quartic hypersurface invariant under the collineations of the simple group of order* 660 *in six variables.*

R. G. Segers, Purdue, January, minor in electrical engineering, *On the initial value problem for systems of partial differential equations with several time-like variables.*

J. A. Seiler, Brown, June, *Plastic deformation of a circular arch under dynamic loading.*

Lily H. Seshu, Illinois, October, minor in electrical engineering, *On the number of simultaneous representations of a given pair of integers as the sum of four integers and the sum of their squares.*

Arthur Shapiro, California at Berkeley, June, *Some conditions for the existence of similar regions.*

R. D. Sheffield, Tennessee, August, minor in physics, *On pseudo-inverses of linear transformations in Banach spaces.*

Abe Sklar, California Institute of Technology, June, minor in physics, *Summation formulas associated with a class of Dirichlet series.*

Paul Slepian, Brown, June, *Theory of Lebesgue area of continuous maps of 2-manifolds into n-space.*

A. H. Smith, University of Southern California, September, *Homology of spaces with operators.*

G. F. Smith, Brown, June, *The thermodynamic potential functions for anisotropic materials.*

L. A. Smith, McGill, October, *The ejection of K-electrons by Beta decay.*

R. T. Smith III, George Washington University, June, *A stochastic model for economic time series.*

Gustave Solomon, Massachusetts Institute of Technology, February, minor in music, *Noncommutative rings of valuation vectors.*

L. M. Sonneborn, California Institute of Technology, June, minor in physics, *Level sets on spheres.*

H. E. Speece, North Carolina, June, *Collineations in non-Riemannian spaces.*

R. A. Spinelli, New York, February, *Infinite dock of variable thickness in a running stream.*

Arthur Steger, California at Berkeley, January, *Idempotent matrices over commutative rings.*

Shlomo Sternberg, Johns Hopkins, November, *Some problems in discrete nonlinear transformations in one and two dimensions.*

Andrew Sterrett, Pittsburgh, June, *An efficient method for the detection of defective members of large populations.*

D. A. Storvick, Michigan, February, *The boundary behavior of* *meromorphic and pseudo-meromorphic functions.*

G. R. Strohl, Jr., Maryland, June, *Strongly cyclic Peano spaces.*

C. A. Swanson, California Institute of Technology, June, minor in physics, *Asymptotic expansions of characteristic values and functions of a second order ordinary linear differential operator.*

H. C. Sweeny, Virginia Polytechnic Institute, June, *Analyses of experiments with correlated observations and heterogeneous variances.*

E. J. Taft, Yale, June, *Invariant Wedderburn factors.*

J. G. C. Templeton, Princeton, *A test for detecting single cell disturbances in contingency tables.*

Aram Thomasian, California at Berkeley, September, *On the magnitude of the sum of error probabilities.*

H. F. Trotter, Princeton, June, *Convergence of semi-groups of operators.*

D. R. Truax, Stanford, *Decision problems for the multivariate exponential family.*

B. M. Wall, Texas, January, *Some convergence sets for a continued fraction.*

C. J. Wallen, St. Louis, June, minor in physics, *A generalized Hansen series.*

M. W. Weaver, Texas, June, *The application of cosets and correspondences in the theory of semigroups.*

Guido Weiss, Chicago, December, *On certain classes of functions spaces and on the interpolation of sublinear operators.*

Irving Weiss, Stanford, *Limiting distributions in some occupancy problems.*

Oscar Wesler, Stanford, January, *A modified minimax principle.*

J. W. Wilkinson, North Carolina, June, *Analysis of paired comparison designs with incomplete repetitions.*

L. K. Williams, California at Berkeley, June, *On separating transcendency bases.*

M. J. Willis, Purdue, January, minors in education and psychology, *Generalizations in a certain class of non-linear regression problems.*

J. W. Woll, Jr., Princeton, June, *Homogeneous stochastic processes.*

E. T. W. Wong, Rochester, June, *The singular ideal and the maximal right quotient rings of a left faithful ring.*

R. C. Wrede, Indiana, September, minor in physics, *"n" dimensional considerations of basic principles A and B of the unified theory of relativity.*

Martin Wright, Rice, June, *Asymptotic Dirichlet series in a strip.*

Mishael Zedek, Harvard, June, *On generalized Tchebycheff polynomials.*

R. E. Zindler, Michigan State, June, *The qualitative behavior of integral curves of systems, their differential equations near a singular point.*

The following doctorates were conferred in 1955, but were not included in the list in the preceding volume of this Bulletin (vol. 62, pp. 278–290):

Robert Ackerson, Alabama Polytechnic Institute, August, *Properties of solutions of certain matrix equations of the type* $AX+XB$.

F. B. Sloss, Northwestern, August, *A self-adjoint boundary value problem with end conditions involving the characteristic parameter.*

ERRATA

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In the list of persons elected to ordinary membership in the Society the following was listed with an incorrect title.

Professor Anthony Mardellis, Long Beach State College.

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THE APRIL MEETING IN NEW YORK

The five hundred thirty-third meeting of the American Mathematical Society was held on Thursday, Friday, and Saturday, April 4–6, 1957, at New York University. A Symposium on Orbit Theory (sponsored by the Society with the aid of the Office of Ordnance Research) was held on Thursday and Friday. About 550 persons attended, including 430 members of the Society.

The Symposium was divided into three sessions which met at 10:00 A.M. and 2:00 P.M. on Thursday, and at 10:00 A.M. on Friday. The first session was presided over by Professor J. B. Rosser. Participants in the Symposium were welcomed at this session by Colonel G. F. Leist, Commanding Officer of the Office of Ordnance Research of the U. S. Army. The following papers were presented: *Orbit stability in particle accelerators*, by Dr. E. D. Courant of the Brookhaven National Laboratory; *Motion of cosmic-ray particles in galactic magnetic fields*, by Professor Stanislaw Olbert of the Massachusetts Institute of Technology; *Störmer orbits*, by Dr. W. H. Bennett of the U. S. Naval Research Laboratory. Discussion was led by Professor Thomas Gold.

At the second session of the Symposium, the following papers were presented: *General theory of oblateness perturbations*, by Professor Paul Herget of the University of Cincinnati; *Fundamental problems in predicting positions of artificial earth satellites*, by Professor F. L. Whipple of Harvard University; *Cis-lunar orbits*, by Dr. K. A. Ehricke of the General Dynamics Corporation; *Satellite launching vehicle trajectories*, by Dr. J. W. Siry of the U. S. Naval Research Laboratory. The chairman for this session was Professor F. J. Murray, and the discussion was led by Dr. H. E. Newell.

The papers presented at the third session of the Symposium were: *Numerical determination of precise orbits*, by Professor W. J. Eckert of Columbia University; *Comments on general theories of planetary orbits*, by Professor Dirk Brouwer of Yale University; *Relativistic orbits in the central field of Birkhoff's theory of gravitation*, by Professor C. G. Fernández of the University of Mexico. Professor R. E. Langer was the chairman for this session, and the discussion was led by Professor Jurgen Moser.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor Bernard Friedman of New York University addressed the Society at 2:00 P.M. on Friday on *Boundary conditions at infinity and virtual eigenvalues*, and Professor R. V.

Kadison of Columbia University delivered an address, entitled *A survey of the theory of operator algebras*, on Saturday at 2:00 P.M. These sessions were presided over by Professors K. O. Friedrichs and E. R. Kolchin respectively.

Sessions for contributed papers were held at 3:15 P.M. on Friday, and at 10:00 A.M. and 3:15 P.M. on Saturday. Chairmen at these sessions were Professors Melvin Henriksen, Fritz John, R. E. Johnson, P. D. Lax, Wilhelm Magnus, Louis Nirenberg, Mary E. Rudin, H. N. Shapiro, John van Heijenoort.

The Institute of Mathematical Sciences sponsored a tea on Friday afternoon.

Abstracts of the contributed papers follow. Those with "*t*" after the abstract number were presented by title. In the case of joint papers, the name of the author who read the paper is followed by (p). Miss Wood-Schot was introduced by Dr. R. M. Davis, Professor Preston by Professor A. H. Clifford, Mr. Miranker by Professor Lipman Bers, Dr. D. J. Newman by Professor M. S. Klamkin, Mr. Bzoch by Professor Pasquale Porcelli, Mr. Foote by Professor K. O. Friedrichs, Mr. Wendroff by Dr. G. M. Wing.

ALGEBRA AND THEORY OF NUMBERS

397. Shreeram Abhyankar: *Coverings of algebraic curves.*

One thesis of this paper is that the possibility of the "splitting of a single branch point by itself" is the basic fact which distinguishes the global ramification theory in the modular case from that in the classical case. Some of the consequences of this *possibility* are: Let the ground field k be algebraically closed of characteristic $p\neq 0$, then we have: (1) Given any one-dimensional algebraic function field K/k there exists x in K such that $x=\infty$ is the only valuation of $k(x)/k$ ramified in K. (2) There exist unsolvable unramified coverings of the affine line—which is a commutative group variety. (3) For nontamely ramified extensions, in no possible sense is the monodromie group generated by loops around the branch points. (2) is only a very special consequence of a general pattern which leads to the conjecture that (ramification theory in characteristic $p\neq 0$) = (ramification theory in characteristic zero for the *corresponding* situation) + (the class of quasi p-groups). Also the recent result of Lang-Serre to the effect that for a curve there are only a finite number of unramified coverings of a given degree is generalized to tamely ramified extensions with assigned branch points. (Received February 6, 1957.)

398*t*. E. H. Batho: *An analogue of Cohen's theorem for noncommutative local rings.*

Let R be a complete noncommutative local algebra over a field F such that R/J is finite dimensional and separable over F, where J is the Jacobson radical of R, (for terminology see Bull. Amer. Math. Soc. Abstract 61-6-633), then there is a related extension (S, ϕ) of R such that S is quasicyclic (Hochschild, Proc. Amer. Math. Soc. vol. 53 (1947) pp. 369–377). This generalizes a well known result of I. S. Cohen for complete commutative local rings. (Received February 10, 1957.)

399*t*. Harvey Cohn: *Some fundamental regions for which forms of dimension minus four are Eisenstein series.*

Using the variation of domain approach of the earlier work [Trans. Amer. Math. Soc. vol. 82 (1956) pp. 117–127], the author shows that the title is a valid property of a large class of regions with symmetries and generators essentially like those of the region for Hecke's modular functions ($ss'=a$, $s'=s+k$). (Received January 9, 1957.)

400*t*. H. S. M. Coxeter: *Groups defined by two relations.*

By an argument due to R. Frucht and B. H. Neumann (Publ. Math. Debrecen vol. 4 (1956) p. 191), the general metacyclic group of order mn can be presented in the form $S^m=T^n$, $T^{-1}S^nT=S^{n+1}$, $(1+n^{-1})^n\equiv 1 \pmod{m}$. Other interesting groups having two generators and two relations are: $T^n=1$, $T^{-1}ST=S^r$ $(r^n\neq 1)$, of order $n\cdot|r^n-1|$; $S^m=T^n=(ST)^2$ $(k=2m+2n-mn>0)$, of order $8mn/k$; $S^m=T^2=(ST)^n$ $(k>0)$, of order $8mn(4n-k)/k^2$; $S^m=T^n$, $(ST)^2=1$ $(k>0)$, of order $16mn(m+n)/k^2$; $S^m=(ST)^n$, $T^2=1$ $(k>0)$, of order $16mn(m-n)/k^2$; $T^n=1$, $STS=TST$ $(n<6)$, of order $24n^2/(6-n)^2$. (Received February 21, 1957.)

401. I. B. Fleischer: *Maximality and ultracompleteness in normed modules.*

The results are in the spirit of P. Conrad, *Embedding theorems for abelian groups with valuations*, Amer. J. Math. vol. 75 (1953) pp. 1–29; the methods similar to K. A. H. Gravett, *Valued linear spaces* Quart. J. of Math. vol. 6 (1955) pp. 309–315. The norm of the module M is non-archimedean and takes values in a linearly ordered set. The *residue class* module corresponding to an element λ of this set is the quotient module of elements with norm not exceeding λ by those with norm less than λ. An extension of M is *immediate* if it has the same value set and residue class modules; M is *maximal* if it admits no immediate extension. The existence and nonuniqueness of an immediate maximal extension is shown. M is *ultracomplete* if it is complete in the (not necessarily separated) topology defined by the inverse images of any collection of residue class modules. The equivalence of maximality and ultracompleteness is established in certain cases. For vector spaces, ultracompleteness implies the extensibility of bounded homomorphisms (cf. A. W. Ingleton, *The Hahn-Banach theorem for non-archimedean valued fields*, Proc. Cambridge Philos. Soc. vol. 48 (1952) pp. 41–45). This can be used to prove the uniqueness of the immediate maximal extension as well as to obtain some structure theory. Applications to linearly ordered vector spaces over linearly ordered fields are given. (Received February 20, 1957.)

402*t*. Lawrence Goldman and A. P. Hillman: *On sets of linear differential equations with dependent solutions.*

Let $\mathfrak{V}$ consist of all linear differential operators $D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n$ with coefficients a_i in a differential field $\mathfrak{F}$ (of characteristic zero). $L_1,\cdots,L_r$ in $\mathfrak{V}$ are said to be dependent if there exist $y_1,\cdots,y_r$, not all zero, in an extension of $\mathfrak{F}$, such that $L_i(y_i)=0$ and $y_1+\cdots+y_r=0$. If the L_i are noncomposite, necessary and sufficient conditions for them to be dependent with no proper subset dependent is that they all have the same order n (i.e., degree in D) and that there exist $Q_1,\cdots,Q_r$ of order less than n in $\mathfrak{V}$ such that $Q_i(y_1)$ is a zero of $L_i(y)$ for every zero y_1 of $L_1(y)$ and $Q_1+\cdots+Q_r$ is the operator zero but no proper subset of the Q_i add up to zero. This result is obtained from Picard-Vessiot Theory. The following sufficient conditions for independence are also obtained: Let $A, B\in\mathfrak{V}$. Let $f_1,\cdots,f_r$ be distinct elements of $\mathfrak{F}$ and let $L_i = A + f_iB$ be noncomposite. If the f_i are constants and the order of A exceeds that of B or if $r = 3$ and the order of A exceeds that of B by at least 2, then the L_i are independent. (Received February 18, 1957.)

403. Malcolm Goldman: *A coset-like decomposition relative to certain rings of operators.*

Let M and $N \subset M$ be factors of type II_1 (Murray and von Neumann, Ann. of Math vol. 37 (1936)) on a Hilbert space H such that the coupling constant (loc. cit.) of M is 1 and that of N is $1/n$, n an integer. Let f be a vector such that $\mathrm{Tr}\,(A) = (Af, f)$ for $A \in M$. We call such a vector a trace vector. Since $C = 1$ for M it is known that there is a cyclic trace vector f for M. The author shows that there is a unitary $U \in M$ such that $U^n = 1$, the subspaces $[NU^kf]$ $k = 1, \cdots, n$ are orthogonal for $k \neq j$ and they span H. He proves this by an inductive construction of trace vectors $f_1, \cdots, f_n$ for N such that $(Mf_i, f_j) = 0$ $i \neq j$. Letting $f_0 = \omega f_1 + \omega^2 f_2 + \cdots + \omega^n f_n$ where ω is a primitive nth root of unity he shows that $f_0/\|f_0\|$ is a trace vector for M and is cyclic under M. Therefore it is known that there is a unitary U with $Uf = f_0$. Thus if N has a complete orthonormal set of cyclic trace vectors on $[Nf]$ then M has such a family on H. Also if $UNU^{-1} \subset N$ then every $A \in M$ is a sum $A_1U + A_2U^2 + \cdots + A_nU^n$, $A_k \in N$. However, there are cases where no unitary $V \in M$ has the property $VNV^{-1} \subset N$. (Received Feburary 18, 1957.)

404t. Bruno Harris: *The structure of certain Jordan algebras.*

Let $\mathfrak{A}$ be the algebra of all $n \times n$ matrices over a field F, M an element of $\mathfrak{A}$ with invariant factors $\delta_1, \cdots, \delta_r$, δ_{i+1} dividing δ_i. The structure of $\mathfrak{A}(M)$, the algebra of matrices in $\mathfrak{A}$ commuting with M, is known (Jacobson, *Lectures in abstract algebra*, Vol. II): if $(F[x])_r$ is the algebra of $r \times r$ matrices with elements polynomials in an indeterminate x, $\mathfrak{C}$ the subalgebra of $(F[x])_r$ of matrices $(\alpha) = (\alpha_{ij}(x))$ such that $(\delta)^{-1}(\alpha)'(\delta)$ is in $(F[x])_r$ where $(\alpha)'$ is the transpose of (α) and $(\delta) = \mathrm{diag}\,(\delta_1, \cdots, \delta_r)$, and $\mathfrak{R}$ the ideal of $\mathfrak{C}$ of matrices of the form $(\alpha)(\delta)$, then $\mathfrak{A}(M)$ is isomorphic to $\mathfrak{C}/\mathfrak{R}$. We prove: 1. If $\mathfrak{J}$ is the Jordan algebra of $n \times n$ hermitian matrices over a field F of characteristic not two, N a nilpotent element with invariant factors $\delta_1, \cdots, \delta_r$, δ_{i+1} dividing δ_i, then $\mathfrak{J}(N)$, the Jordan algebra of elements of $\mathfrak{J}$ commuting with N, is isomorphic to $\mathfrak{S}/\mathfrak{K}$, $\mathfrak{S}$ the elements (α) of $(F[x])_r$ satisfying $(\delta)^{-1}(\alpha)^*(\delta) = (\alpha)$, $(\alpha)^*$ the conjugate transpose of (α), and $\mathfrak{K} = \mathfrak{S} \cap \mathfrak{R}$. 2. If $\mathfrak{J}$ is the Jordan algebra of $n \times n$ symplectic-symmetric matrices, N nilpotent with invariant factors δ_i, then $\mathfrak{J}(N) = \mathfrak{T}/\mathfrak{L}$, $\mathfrak{T}$ the matrices with $(\delta)^{-1}(\alpha)'(\delta) = (\alpha)$, $(\delta) = \delta_1(e_{12} - e_{21}) + \delta_3(e_{34} - e_{43}) + \cdots + \delta_{r-1}(e_{r-1,r} - e_{r,r-1})$, $\mathfrak{L}$ the matrices $(\alpha)(\delta)$. (Received February 20, 1957.)

405. Bruno Harris: *On a formula of Frobenius.*

If M is an $n \times n$ matrix over a field F with invariant factors $\delta_1, \cdots, \delta_r$ of degrees $d_1, d_2, \cdots, d_r$, then the algebra of $n \times n$ matrices commuting with M has dimension $\sum_{k=0}^{r-1} (2k+1)d_{k+1}$ (Frobenius; see Jacobson, *Lectures in abstract algebra*, vol. II). We generalize this formula to any central simple Jordan algebra $\mathfrak{J}$ over a field of characteristic not 2. First invariant factor degrees are defined for an element of such an algebra generalizing the usual definition and having the usual properties. The centralizer of the subalgebra generated by an element x is the subalgebra of all y in $\mathfrak{J}$ such that $R_yR_z = R_zR_y$ for all z which are polynomials in x, R_y denoting the multiplication by y in $\mathfrak{J}$: this is a generalization of "commutativity of two elements," due to Jacobson. $\mathfrak{J}$ is said to have degree n if on extending the base field to its algebraic closure, n is the maximum number of orthogonal idempotents in a decomposition of the identity. Theorem: If $\mathfrak{J}$ is central simple of degree n and dimension $n+n(n-1)s/2$, then the centralizer of the subalgebra generated by x has dimension $\sum_{k=0}^{n-1}(sk+1)d_{k+1}$. If $n\geqq 3$ then $s=1$, 2, 4, 8 and refers to the algebras of symmetric, complex hermitian, quaternionic hermitian, and octonion hermitian matrices, respectively. (Received February 20, 1957.)

406*t*. L. A. Hostinsky: *On splitting criteria.*

Within the framework of a complete modular lattice two types of criteria for uniform splitting (the existence of a complement) are developed. One deals with the concept of an almost periodic endomorphism defined as follows: The endomorphism η of $p/0$ is almost periodic on $x/0$ if $x\eta\leqq x\leqq p$ and if for every $y\leqq x$ there exists a positive integer $j(y)$ such that $y=y\eta^{j}$. It is shown that such an η is a uniformly splitting endomorphism of $p/0$ if and only if η induces a uniformly splitting endomorphism of p/x. The other criterion concerns the existence of a set of elements which are uniformly split by η. Application of this theory of splitting gives a further connection between homomorphisms and direct decompositions. Methods of proof depend largely upon properties of complements and decomposition endomorphisms. The investigation is based upon and extends certain previous results obtained by the author. (Received February 4, 1957.)

407. Abraham Karrass and Donald Solitar (p): *On free products of groups.*

Results are obtained for free products, among which are generalizations of theorems known for free groups. If a subgroup of a free product has finitely many conjugates H_i, then the H_i have a nontrivial intersection. In any group G, the intersection of a finitely generated subgroup with a subgroup of finite index is finitely generated. Let F denote a free product of finitely many groups; each of which is finite or more generally a finite extension of a free group of finite rank. F is hopfian. If the extensions are prime power corresponding to the same prime, then the lower central series of F intersects in the identity. A finitely generated subgroup containing a nontrivial normal subgroup of F must be of finite index in F. As a consequence we have the equivalence of the following for a subgroup H of F: (i) H is of finite index; (ii) H is finitely generated and has finitely many conjugates; (iii) H is finitely generated and contains the X^{d}-verbal subgroup of F for some $d>0$, i.e. the subgroup generated by the dth powers of elements of F. (Sponsored by National Science Foundation grant no. NSFG-2796). (Received January 30, 1957.)

408. M. D. Marcus: *On subdeterminants of doubly stochastic matrices.*

Let A be an n-square doubly stochastic (d.s.) matrix of rank k and let $1\leqq r\leqq k$. Let D_r be the sum of the squares of all r-square minors of A. *Then* $D_r\leqq\binom{k}{r}$ *and the equality holds for some r if and only if* $A=PBQ$ *where P and Q are permutation matrices and B is the direct sum of k* d.s. *matrices of rank* 1 (i.e. all of whose elements are the same). From this it follows that if A is as above then *A has* $\binom{k}{r}$ *r-square minors of value* ± 1 *if and only if* $k=n$ *and A is a permutation matrix.* (Received February 7, 1957.)

409t. Elliott Mendelson: *Some algebraic properties of nonstandard models for number theory.*

Let F be a first-order formal system for arithmetic, having as axioms all sentences which are true for the ring Z of integers. It is a well-known consequence of the Skolem-Löwenheim theorem that F has nonstandard models, that is, models which are not isomorphic with Z. Each nonstandard model R is an ordered integral domain having Z as a proper subring. Call such an integral domain R a *Peano ring*, and its quotient field K a *Peano field*. Peano rings have all "elementary" arithmetic properties in common with Z. The standard decomposition theorems for elements or ideals fail in R. K is the quotient field of no other Peano ring. Z is integrally closed in R, and R in K. Every element in $K-Q$ (where Q is the set of rationals) is transcendental over Q, and the absolute transcendence degree of K is infinite, but K is not a pure transcendental extension of Q. For any infinite cardinal m, there is a Peano field of absolute transcendence degree m. Given any transcendence base V of the reals over Q, there is a Peano field containing $Q(V)$ as an ordered subfield. (Received February 6, 1957.)

410. A. R. Amir-Moez (p) and Alfred Horn: *Singular values of a matrix.*

If A is any matrix with complex values, define the geometric singular values of A to be the non-negative square roots of the eigenvalues of A^*A. The real arithmetic singular values of A are the eigenvalues of $(A+A^*)/2$, and the imaginary singular values of A are the eigenvalues of $(A-A^*)/2i$. Relations between these singular values and the eigenvalues of A are discussed. In particular, a necessary and sufficient condition that there exist a matrix with real arithmetic singular values $\alpha_1 \geq \cdots \geq \alpha_n$ and eigenvalues λ_i such that $R\lambda_1 \geq \cdots \geq R\lambda_n$ is that $R\lambda_1 + \cdots + R\lambda_k \leq \alpha_1 + \cdots + \alpha_k$ for $1 \leq k \leq n$, and $R\lambda_1 + \cdots + R\lambda_n = \alpha_1 + \cdots + \alpha_n$. A similar result holds for the imaginary singular values. If the geometric singular values of A are $\rho_1, \cdots, \rho_n$ and its real and imaginary arithmetic singular values are $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$, then $\rho_1^2 + \cdots + \rho_n^2 = \alpha_1^2 + \cdots + \alpha_n^2 + \beta_1^2 + \cdots + \beta_n^2$. (Received February 18, 1957.)

411. Morris Newman: *Some theorems about $p_r(n)$.*

A general congruence for the coefficients $p_r(n)$, defined by $\sum_{n=0}^{\infty} p_r(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^r$ is derived, from which the Ramanujan congruences for partitions modulo 5, 7, 11 follow as a corollary. In addition it is shown that for $r=2, 4, 6, 8, 10, 14, 26$ the products $\prod_{n=1}^{\infty} (1-x^n)^r$ are lacunary, in the sense that arbitrarily long strings of zero coefficients occur, arbitrarily many times. (Received February 7, 1957.)

412. G. B. Preston: *The homomorphism theorems for partial algebras.*

A (λ, μ)-relation over a set A, where λ, μ are ordinals is a set $R=R(\lambda, \mu)$ consisting of ordered pairs of elements (a, b) where $a \in A^\lambda$ and $b \in A^\mu$, and such that if (a, b), $(a, b') \in R$ then $b=b'$. A partial algebra X is an ordered pair $X=(A, B)$ where each element of B is, for some λ, μ, a (λ, μ)-relation over A. Let $a=(a_\nu)$, $b=(b_\nu) \in A^\tau$; then we define $a+b$ to be $((a_\nu, b_\nu)) \in (A \times A)^\tau$. The set $Q \subset A^2$ is a congruence over X if Q is an equivalence over A and if, for (a, b), $(c, d) \in R=R(\lambda, \mu) \in B$, $a+c \in Q^\lambda$ implies $b+d \in Q^\mu$. Let $f: x \to xf$ be a single-valued mapping of A into M; if $a=(a_\nu) \in A^\tau$ denote by af the sequence $(a_\nu f) \in M^\tau$. By a homomorphism of $X=(A, B)$ into $Y=(M, N)$ we mean a pair (f, g) of single-valued mappings $f: A \to M$ and $g: B \to N$ such that if $(a, b) \in R \in B$ then $(af, bf) \in Rg$. By considering a noncommutative product of congruences over subalgebras of X we establish analogues of the isomorphism theorems and of Zassenhaus' Lemma for groups. In contrast to the situation in full algebras (in the sense of B. H. Neumann, Proc. London Math. Soc. (3) vol. 4 (1954) pp. 138–153) there are several natural analogues of these theorems, all of which remain true in an arbitrary partial algebra. We give conditions under which generalized Jordan-Hölder Theorems are valid. (Received February 18, 1957.)

413*t*. Irving Reiner: *A theorem on continued fractions.*

Let $R=K[x]$ be the ring of polynomials in an indeterminate x over a field K. For $f_1, \cdots, f_n \in R$, write $P/Q \sim [f_1, \cdots, f_n]$ to indicate that P and Q are the formal numerator and denominator, respectively, of the simple continued fraction $f_1+1/(f_2+ \cdots +1/f_n)$. Let $f \to f^*$ be any K-linear additive homomorphism of R into itself which is the identity on K. It is shown that if $f_1, \cdots, f_n \in R$ are such that P and Q lie in K, then also $[f_1^*, \cdots, f_n^*] \sim P/Q$. This implies that any identity of the form $[f_1, \cdots, f_n]=0$ depends only upon the additive structure of R, and not upon its multiplicative structure. (Received December 19, 1956.)

414. Fred Supnick: *Lattice points in n-spheres.*

Let G be the greatest integer such that *every* (closed, solid) n-sphere with diameter $n^{1/2}$ contains at least G lattice points (of an n-dimensional rectangular cartesian coordinate system). Let k denote the greatest integer less than or equal to $n/4$. In Warncke and Supnick, *On the covering of E_n by spheres*, (Proc. Amer. Math. Soc. vol. 8 (1957) pp. 299–303) the following result (a by-product of the proof of the main theorem) was obtained: $G \geqq 2^k$. In this paper a lower bound $H=1+\sum_{j=1}^{n-8}\sum_{i=1}^{k} A_{i,j}$ for G, $H \geqq [4^k(3n+7-12k)-(3n-2)]/9 \geqq 2^{2k-1}$ for $(n \geqq 4)$ is obtained, $(A_{i,j})$ being defined as follows: $A_{1,j}=1$ for all (positive integral) j; $A_{i,j}=0$ for each $j \leqq 4(i-1)$, whereas $A_{i,j}=\sum_{h=1}^{j-1} A_{i-1,h}$ for each $j>4(i-1)$. (Received February 18, 1957.)

415. E. J. Taft: *Invariant Wedderburn factors.*

Let C be a class of (finite-dimensional) algebras defined by identities. Let $A \in C$. Denote by R the radical of A. Let A/R be separable. Then the Wedderburn principal theorem for C states that A may be written $A=S+R$, where S is a separable subalgebra of A, $S \cong A/R$. Let G be a finite group each of whose elements is an automorphism or antiautomorphism of A. Assume that the characteristic of the base field is not a divisor of the order of the group G. Then ideal-theoretic conditions are given on C which guarantee the existence of Wedderburn factors S which are invariant under the operators of G. The results apply for C the class of associative algebras, alternative algebras, Jordan algebras, or Lie algebras (the last over a field of characteristic zero). The question of uniqueness is answered for the special case of self-adjoint (i.e., invariant under an involution) Wedderburn factors of an associative algebra over a field of characteristic zero. Any two such factors are conjugate via an orthogonal element, i.e., an element x that $xx^*=1=x^*x$. (Received January 25, 1957.)

416*t*. E. J. Taft: *The Wedderburn principal theorem for Jordan algebras.*

A proof is given of the Wedderburn principal theorem for Jordan algebras of characteristic not two, thus generalizing the result of Penico for characteristic zero (cf. *The Wedderburn Principal Theorem for Jordan Algebras*, Trans. Amer. Math. Soc. vol.

70 (1951). The main tools used are a structure theorem of Jacobson, and the existence of self-adjoint Wedderburn factors in alternative algebras (see previous abstract). Penico's proof consists mainly of a separate consideration of the five types of split algebras. Here his result is used for the case of split algebras of degree two. The other four cases (classes A, B, C, E) may be considered simultaneously. They are of the form $H(D_n)$, $n \times n$ hermitian matrices with entries from an alternative algebra D with an involution, $n \geq 3$. Then self-adjoint Wedderburn factors of D give rise to Wedderburn factors of $H(D_n)$. A recent result of Jacobson shows that there are no new split algebras of characteristic $p \neq 2$. Hence Penico's proof, whose calculations involve only characteristic $\neq 2$, holds in this generality. The technique described here, however, sidetracks a considerable amount of calculation. (Received January 25, 1957.)

417*t*. R. L. Vaught: *Non-recursive-enumerability of the set of sentences true in all constructive models.*

Consider the first order logic without (or with) identity and with one (other), binary, predicate—at least. Let $\mathcal{K}_i$ ($i \leq 4$) be the class of all models $\langle A, R\rangle$ such that A and R are (0) arbitrary, (1) recursively enumerable (r.e.), (2) recursive, (3) primitive recursive, or (4) finite. Let V_i ($i \leq 4$) be the set of sentences true in all $\mathcal{K}_i$-models. Kreisel and Mostowski (cf. Fund. Math. vol. 42 (1955) p. 125) proved $V_0 \neq V_1$ (assuming more complicated predicates). Trahtenbrot (Doklady Akad. Nauk SSSR vol. 90 (1950) p. 569; ibid. vol. 88 (1953) p. 953; Izv. Akad. Nauk SSSR Ser. Mat. vol. 20 (1956) p. 569) showed that V_4 is not r.e. (cf. also, Kalmar, Acta Math. Acad. Sc. Hung. vol. 2 (1951) p. 125), and, moreover, that if $V_0 \subseteq X \subseteq V_4$, then X is not recursive. *Theorem* (*). *If* $V_1 \subseteq X \subseteq V_4$, *then* X *is not r.e.* Thus, the title result holds, if "constructive" has any meaning such that the class $\mathcal{K}$ of all constructive models satisfies $\mathcal{K}_4 \subseteq \mathcal{K} \subseteq \mathcal{K}_1$. Examples of sets X satisfying the hypothesis of (*) are V_1, V_2, and V_3, and, also, the sets W_i ($i = 1, 2$) of sentences valid in (1) all decidable theories, or (2) all finitely axiomatizable, complete theories. W_1 and W_2 were introduced by Szmielew-Tarski (Bull. Amer. Math. Soc. Abstract 55-11-577). That $V_1 \subseteq W_i$ results from the lemma: *Every consistent, decidable theory has a* $\mathcal{K}_2$*-model.* (For terminology, cf. Tarski, *Undecidable theories*, Amsterdam, 1953.) (Received March 11, 1957.)

418. D. W. Wall: *Algebras with unique minimal faithful representations.*

In this paper an algebra means a finite dimensional algebra with unit over an algebraically closed field. The generalized uniserial algebras form a subclass of the QF-3 algebras which are those with unique minimal faithful representations. Thrall (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 173–183) has shown that a QF-3 algebra can be defined as an algebra in which every primitive ideal is weakly subordinate to a sum of dominant ideals. Subclasses of the QF-3 algebras can be obtained by strengthening this definition by making one or more changes. Twelve subclasses of the QF-3 algebras are obtained in this manner and examples are given to show that these classes are distinct. Each of these classes contains the generalized uniserial algebras as a subclass. Necessary and sufficient conditions are given in order that an algebra in one class belong to a particular subclass. (Received February 21, 1957.)

419*t*. N. A. Wiegmann: *Some generalizations of Burnside's Theorem.*

Burnside's Theorem in the theory of group representations provides a necessary and sufficient condition that a semigroup of matrices over the complex field be irreducible; and this theorem has been generalized when the matrix elements lie in a division ring. When the elements are real quaternions, the following are among some results obtained: If $\mathfrak{A}$ is an irreducible semigroup of quaternion matrices, then any matrix M which commutes with every element of $\mathfrak{A}$ is of the form $P^{-1}(kI)P$ where k is complex and P is a nonsingular quaternion matrix. Let $\mathfrak{A}$ be a semigroup of quaternion matrices of degree n which is not similar to a complex set; if $\mathfrak{A}$ is irreducible, then $\mathfrak{A}$ has 1-rank n^2 and, conversely, if every semigroup similar to $\mathfrak{A}$ has 1-rank n^2, then $\mathfrak{A}$ is irreducible. (Received February 7, 1957.)

420*t*. K. G. Wolfson: *Ideals of the affine near-ring.*

Let $N(F, A)$ be the near ring of all affine transformations of the vector space A over the division ring F. Let S be the set of all constant transformations in $N(F, A)$. Denote the ring of all linear transformations of (F, A) by $T(F, A)$. For each ordinal $\nu \geq 0$, $T_\nu(F, A)$ is the set of all transformations in $T(F, A)$ whose range has dimension less than $\aleph_\nu$, and $T_{-1} = 0$. By an ideal is meant the kernel of a near-ring homomorphism. (Blackett, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 517–519.) It is shown (1) For each $\nu \geq -1$, $T_\nu + S$ is an ideal in $N(F, A)$ and every ideal has this form. [$T_\nu + S$ is the set of elements $t+s$, t in T_ν, s in S.] (2) $N/T_\nu + S$ is isomorphic to T/T_ν. Hence, every proper homomorphic image of $N(F, A)$ is a dense ring of linear transformations. If (F, A) has finite dimension, S is the only proper ideal, so that $T(F, A)$ is the only proper homomorphic image of $N(F, A)$. (Received February 20, 1957.)

421*t*. K. G. Wolfson: *A generalization of H^*-algebras.*

Let $\{A_i\}$ be a set of complex Banach algebras. By the l_p direct sum of the $\{A_i\}$ is meant the set of all vectors $a = \{a_i\}$, $a_i \in A_i$ for which $\sum_i \|a_i\|^p < \infty$ with the coordinatewise operations and obvious norm. By a full matrix p-algebra of order n (n is a finite or infinite cardinal, $1 \leq p \leq \infty$) is meant the set of all $n \times n$ matrices $a = (a_{ij})$ with complex entries for which $\sum_{i,j} |a_{ij}|^p < \infty$. If $2 < p \leq \infty$ then n can be finite only. Necessary and sufficient conditions are given that a Banach algebra be equivalent to an l_p direct sum of p-matrix algebras A_i (of order n_i) (same p in both places, n_i may be different in each summand). Use is made of the characterization of l_p spaces given by Bohnenblust (Duke Math. J. (1940) pp. 627–640). Such an algebra is dual (Kaplansky, Ann. of Math. (1948) pp. 689–701) if, and only if, all n_i are finite, or $p = 2$. (The latter is the H^*-algebra of Ambrose, Trans. Amer. Math. Soc. (1945) pp. 364–386.) However, a sufficiently weakened remnant of duality may be used in the characterization. It is shown that the L_p algebras of a compact group have such representations (with all n_i finite). (Received February 20, 1957.)

ANALYSIS

422. J. W. Brace: *The weak completion of Banach spaces.*

A Cauchy net for the weak topology on a Banach space X converges to an element of X for the weak topology if and only if the net has a weak limit when embedded in an adjoint space of X under the natural embedding. (Received February 21, 1957.)

423*t*. R. C. Bzoch: *Concerning the existence and properties of the Lane integral.*

The object of this paper is to develop some of the properties of an integral defined recently by R. E. Lane (cf. Proc. Amer. Math. Soc. vol. 6, pp. 392–401). Our principal result is: Theorem A. If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$, then a necessary and sufficient condition in order that the Lane integral $L\int_a^b f dg$ exist is that there exist a function f' on $[a, b]$ such that (i) the sigma-mean integral $\int_a^b f' dg$ exists, (ii) $f'=f$ on a subset G of $[a, b]$ such that $a \in G$ and $b \in G$, (iii) if $H=[a, b]-G$, then (f, g, H) is a singular graph, and (iv) if $x \in G$ and g is not continuous at x, then x is not exceptional point for f and g. An example is given to show that the theorem can not be extended to the case where f is unbounded. In the case of the necessity (iv) is redundant. The sufficiency part is similar to Lane's Theorem 3.3 (loc. cit.) but not equivalent to it. Also, theorems are developed on the structure of singular graphs and on the oscillatory properties of f on the set G. (Received January 3, 1957.)

424t. A. P. Calderón: *Uniqueness in the Cauchy problem for partial differential equations.*

Let $P(u)=0$ be an mth order partial differential equation with highest order coefficients in $C_{1+\alpha}$, $\alpha>0$, and remaining ones bounded and measurable. Assume that the characteristics of P are nonmultiple and let $u=u(x_1, \cdots, x_k) \in C_m$ be a solution of $P(u)=0$ in a domain with vanishing Cauchy data on a noncharacteristic manifold M. Then if $k \neq 3$ or if $m=2$, u vanishes identically in a neighborhood of M. Analogous result holds for systems provided $k=2$ or $k>4$. If the coefficients are assumed to be $m(r-1)+2$ times continuously differentiable, where r is the number of equations in the system, then the result holds also for $k=4$. The proof is based on a representation of a linear partial differential operator as a product of a fractional power of the Laplacian and a singular integral operator. The approximate functional calculus of singular integral operators allows then to reduce the equation to a form to which an inequality similar to the ones used by M. E. Heinz and N. Aronszajn can be applied. (Received February 5, 1957.)

425t. Y. W. Chen: *Reflection laws of differential equations of mixed type.*

Consider first the equation (*) $x^s u_{yy}+u_{xx}=0$ with positive exponent s. The following theorem holds: Let $u(x, y)$ be a solution of (*) of class C_2 in a domain $0<x<1$, $a<y<b$, with continuous first and mixed second derivatives on $x=0$. Let $u(0, y)=T_0(y)$ and $u_x(0, y)=T_1(y)$. Then, 1. T_0 and T_1 uniquely determines the solution $u(x, y)$; and 2. when one of the T_i, $i=0, 1$, is zero, the other one is necessarily an analytic function of y, and $u(x, y)$ can be extended to an analytic function of two complex variables. Secondly the same equation is considered with negative $s>-1$. Under the additional assumption that u_{xyy} is continuous for $0<x<1$ and on $x=0$, the above mentioned theorem is valid also in this case. The theorem generalizes the classical reflection principle of harmonic functions ($s=0$). (Received February 27, 1957.)

426. Avron Douglis: *On linear, hyperbolic, partial differential equations of second order.*

Consider Cauchy's problem for a linear, hyperbolic, partial differential equation of second order in m independent variables. Its solution for any $m>2$ is characterized by an integral equation of the form $u(P)=f(P)+{}_{C^P}\int K(P, Q)u(Q)dS_Q+{}_{D^P}\int J(P, Q) \cdot u(Q)dV_Q$, C^P being the part of the back characteristic conoid with vertex at P cut off by the initial surface and D^P its inside. The Cauchy data enter into the function $f(P)$. The kernels J and K are absolutely integrable over their domains, J vanishing for even m and K for odd m. The derivations of these formulas for even and for odd m run exactly parallel. (Received February 22, 1957.)

427*t*. Albert Edrei and W. H. J. Fuchs: *Deficient values which are also asymptotic*. II.

Continuing a previous investigation, the authors prove the following theorem. Let $f(z)$ be a meromorphic function of finite order λ, $1/2<\lambda<+\infty$, with $\delta(\infty)\geq 1-\gamma$, $\sum_{a\neq\infty}\delta(a)\geq 1-\gamma$. If γ is sufficiently small, there exists an integer p such that $|\lambda-p|<1/2$ and such that there are S $(1\leq S\leq p)$ finite deficient values which are also asymptotic. The deficiency of each of these values is at least a/p, where $a=a(\gamma, p)$ and $\lim_{\gamma\to 0} a(\gamma, p)=1$. The sum of the deficiencies of these S values is greater than a constant $b=b(\gamma, p)$ and $\lim_{\gamma\to 0} b(\gamma, p)=1$. (Received February 15, 1957.)

428. Charles Fox: *A composition theorem for general unitary transforms*.

The following theorem is proved: Let $k(a, x)$; $l(a, x)$ be a pair of kernels of a unitary transform (Chapter V, *Fourier transforms*, Bochner and Chandrasekharan) and let $p(a, x)$; $q(a, x)$ be another such pair of unitary kernels then $(\partial/\partial x)\int_0^\infty[k(x, y)]^- \cdot p(a, y)dy$ and $(\partial/\partial x)\int_0^\infty k(a, y)[p(x, y)]^- dy$ are also a pair of unitary kernels. (In these integrals $[f(x, y)]^-$ denotes the conjugate complex of $f(x, y)$.) It is also shown that these kernels can be related to $k(a, x)$ and $p(a, x)$ by means of the U and V operators introduced by Bochner (*Inversion formulae and unitary transforms*, Ann. of Math. vol. 55 (1934)). (Received February 7, 1957.)

429. F. W. Gehring: *The Fatou theorem for functions harmonic in a half space*.

It is supposed that $u(x, y, z)$ can be expressed as the difference of two functions each of which is both nonnegative and harmonic in $z>0$. Then $u(x, y, z)=kz+z/2\pi\iint_{-\infty}^{\infty}\{(x-\xi)^2+(y-R)^2+Z^2\}^{-3/2}d\mu(\xi, R)$ where μ is a measure defined over all Borel sets in the plane such that $\iint_{-\infty}^{\infty}(\xi^2+R^2+1)^{-3/2}|d\mu(\xi, R)|<\infty$. Say that $D\mu(0, 0)=A$ if $1/\pi r^2\int_0^{2\pi}\int_0^r d\mu\to A$ as $r\to 0+$. Similarly we say that $D^*\mu(0, 0)=A$ if for all $\alpha<\beta, 1/\pi r^2\int_\alpha^\beta\int_0^r d\mu\to((\beta-\alpha)/2\pi)A$ as $r\to 0+$ and if $|1/\pi r^2\int_\alpha^\beta\int_0^r d\mu|\leq M<\infty$ uniformly in α, β, for small r. The following results are analogues for theorems due to Fatou and Loomis. THEOREM 1. If $D\mu(0, 0)=A$, then $u(0, 0, z)\to A$ as $z\to 0+$. THEOREM 2. If $u(x, y, z)\geq 0$ and if $u(0, 0, z)\to A$ as $z\to 0+$, then $D\mu(0, 0)=A$. THEOREM 3. If $D^*\mu(0, 0)=A$, then $u(x, y, z)\to A$ as $(x, y, z)\to(0, 0, 0)$ along each ray in $z>0$. THEOREM 4. If $u(x, y, z)\geq 0$ and if $u(x, y, z)\to A$ as $(x, y, z)\to(0, 0, 0)$ along each ray in $z>0$, then $D^*\mu(0, 0)=A$. (Received February 21, 1957.)

430. Hugh Gordon: *Compatible topologies on Riesz spaces*. Preliminary report.

Let V be a Riesz space (i.e. a vector lattice). We call a topology on V compatible if: (1) it is compatible with the vector space structure of V; (2) the mapping $f\to f^+$ is continuous at 0; and (3) for each neighborhood U of 0 there is a neighborhood U' of 0 such that $f\in U'$, $g\in V$ and $0\leq g\leq f$ imply $g\in U$. These conditions imply the continuity of the lattice mappings $\vee$ and $\wedge$ of $V^*V\to V$. If V is a topological Riesz space (i.e. a Riesz space with a compatible topology), every continuous linear functional on V is relatively bounded, and the set V^* of all such functionals is a completely reticulated Riesz space. V^* is a topological Riesz space both under the topology defined by the set of all semi-norms $F\to|F|f$ (f positive in V) and under the strong topology. If M is a subset of the set V_b^* of all relatively bounded linear functionals on a Riesz space V, a necessary and sufficient condition for M to consist of all continuous linear functionals in a compatible topology is that: (1) M be a linear subspace of V_b^* and (2) $|F|\leqq|G|$, $F\in V_b^*$ and $G\in M$ imply $F\in M$. (Acknowledgment is made to the National Science Foundation under Contract NSF G 1981.) (Received February 19, 1957.)

431. L. J. Heider: *Measures on Boolean σ-algebras.*

Let $\mathfrak{A}$ be a Boolean σ-algebra. Let X_Δ denote Stone's zero-dimensional, compact Hausdorff space with element a of $\mathfrak{A}$ corresponding to open-closed subset Δ_a of X_Δ. Then $\mathfrak{A}$ as an σ-algebra is isomorphic to the algebra $\mathfrak{B}$ of Baire subsets of X_Δ, modulo the Baire sets of the first category. A measure ϕ on $\mathfrak{A}$ (and analogously for $\mathfrak{B}$) is a bounded real-valued function defined on $\mathfrak{A}$ with $\phi(a\vee b)=\phi(a)+\phi(b)$ for $a\wedge b=0$ in $\mathfrak{A}$. The usual distinctions of countably, finitely, and purely finitely additive measures are to be made. Then: the countably additive (Baire) measures on $\mathfrak{B}$ as restricted to the open-closed subsets Δ_a determine all measures on $\mathfrak{A}$. Each non-negative Baire measure ϕ on $\mathfrak{B}$ has a unique decomposition $\phi=\phi_G+\phi_P$ into a non-negative Baire measure ϕ_G identically zero on Baire sets of the first category, plus a Baire measure ϕ_P minimal on these sets in the sense that $0\leqq\psi\leqq\phi_P$ with ψ a Baire measure identically zero on the Baire sets of the first category implies that ψ is the zero measure. Under the correspondence $\Delta_a\leftrightarrow a$, this decomposition of Baire measures on $\mathfrak{B}$ determines a corresponding unique decomposition of measures on $\mathfrak{A}$ into sums of a countably additive measure and of a purely finitely additive measure on $\mathfrak{A}$. For any measure ϕ on $\mathfrak{A}$ this latter decomposition can be obtained directly and, indeed, either so as to obtain first the countably additive portion of the measure or so as to obtain first the purely finitely additive part of the measure. (Received February 13, 1957.)

432. W. P. Jurkat: *On relations between Borel's and Cesàro's methods of summation.*

Using the complex methods introduced in previous papers the following typical result is obtained: The two conditions (i) $f(z)=e^{-z}\sum_{n=0}^{\infty}s_nz^n/n!=o(1)$ as $z\to\infty$, $|z|-\mathrm{Re}\,z=O(1)$, (ii) $\sum_{k=0}^{n}s_{N+k}=o(N^{1/2})$ as $N\to\infty$, $n=O(N^{1/2})$ are equivalent for bounded sequences s_n. This theorem contains most of the Tauberian Theorems for Borel summability. Actually (ii) implies (i) without restriction, and the converse can be proved even under more general conditions as $\sum|s_k|^2=O(n^{1/2})$ summing from n to $n+n^{1/2}$. There are several generalizations and a discussion of the relationship to well known results of Hardy-Littlewood in this field. (Received February 18, 1957.)

433. M. S. Klamkin: *An application of the Gauss multiplication theorem.*

S. K. L. Rao has given a derivation of Legendre's duplication formula by means of Mellin transforms. On attempting to extend the method to a proof of the Gauss multiplication theorem, one is led to the evaluation of some interesting integrals involving modified Bessel functions. (Received February 18, 1957.)

434. Paul Koosis: *A completeness theorem.*

$m\neq 0$ is a measure with (finite) supporting interval I. Let $\{a_k\}$ be all the complex zeros of $\widehat{m}(\lambda)=\int e^{i\lambda x}dmx$, a_k occurring with mult. n_k+1. $X=\{x^n\exp(-ia_kx)\,|\,0\leq n\leq n_k;\, a_k\}$. Theorem: X is unif. complete on any interval of length $<$ that of I. Proof: If not, there is a meas. $p\neq 0$ of supp. int. $J\subset I$ prop., with $F(\lambda)=\widehat{p}(\lambda)/\widehat{m}(\lambda)$ entire, $\therefore$ of order ≤ 1. W.l.o.g., $I=[0, A]$, $J=[\delta, A-\delta]$, $\delta>0$. $e^{-iA\lambda}\widehat{p}(\lambda)$ is bdd. in $y\lambda\geq 0$. Prev. result of author (cf. §2.3 of paper in Comm. Pure Appl. Math. February, 1957) then shows that given $\epsilon>0$ there exists $c>0$ so that for $re^{i\beta}=\lambda-ic$ & $0<\beta<\pi$, β bdd. away from 0 & π, $|e^{-iA\lambda}F(\lambda)|$ is $O(\exp[\epsilon r/\sin\beta])$. $\therefore$ $F(\lambda)\rightarrow 0$ as $\lambda\rightarrow\infty$ in a suff. small sector ab't pos. imag. axis. Sim. in a sector ab't neg. imag. axis. Phragmén-Lindelof Th. shows then that $F(\lambda)$ is bdd. $\therefore$ $F(\lambda)=$ const & const is zero. $\therefore$ $p=0$, contr. This clears up an an obscurity in a paper of L. Schwartz (cf. p. 157, l. 7–10 & ftnote, Annales Fac. Sci. Toulouse, VI, 1942). (Received February 19, 1957.)

435. Walter Koppelman: *The Riemann-Hilbert problem for finite Riemann surfaces.* Preliminary report.

Suppose that M is a Riemann surface, topologically equivalent to a sphere with k handles and m holes, whose boundary ∂M has a Holder continuously turning tangent. Assume that Λ, γ are continuous functions on ∂M, where Λ is complex-valued, Hölder continuous, with $|\Lambda|=1$ and where γ is real. The boundary value problem Re $(\Lambda w)=\gamma$ for analytic function multiples w of a given divisor d in M has a solution if, and only if $\int_{\partial M}\bar{\Lambda}\gamma dV=0$ for every analytic differential multiple of $1/d$ which satisfies Re $(\bar{\Lambda}(dv/ds)(ds/dt))=0$ on ∂M. For $\gamma=0$, there exists a Riemann-Roch theorem which relates the dimensions of the two linear spaces of the functions and differentials involved. This is a generalization of part of a result obtained by I. N. Vekua (Mat. Sbornik, N.S. vol. 31 (73) (1952) pp. 217–314). The proof is first given for analytic boundaries, where the double F of M is used to reduce the problem to a Dirichlet problem. The Riemann-Roch theorem above is a consequence of the Riemann-Roch theorem for F. For nonanalytic boundaries, new uniformizers are introduced so that the boundary curves are analytic relative to the new co-ordinates. (Received November 29, 1956.)

436. Saul Kravetz: *On the geometry of Teichmueller manifolds and the structure of the mapping-class group*. III. Preliminary report.

The author continues his investigation [Bull. Amer. Math. Soc. 59 (1953) p. 541] of the geometry of Teichmueller manifolds. The negative curvature, as defined by H. Busemann ([1] Acta Math. 80 (1948) pp. 259–310), of the Teichmueller manifolds is proved. It follows [1, Theorem 4.11] that every finite group of self-isometries of a Teichmueller manifold has a fixed-point. This solves *Teichmueller's Fixed-Point Problem*, i.e. Does every finite subgroup of the mapping-class group acting on a Teichmueller manifold have a fixed-point? Teichmeuller noted [Abh. der Preuss. Akad. der Wiss., Math.-Naturw. Klasse (1939) No. 22, §149] that this fixed-point problem is equivalent to the *Hurwitz-Nielsen Realization Problem*, i.e. Given a finite subgroup of the mapping-class group of a compact topological surface, does there exist a complex-analytic structure on the surface (making it a Riemann surface) admitting a group of conformal self-mappings whose natural image in the mapping-class group is just the given finite subgroup of the mapping-class group? (Received February 22, 1957.)

437. Klaus Krickeberg: *Distributions and Lebesgue area.*

Let G be open in R_n, and write points of R_n as (x, y) with $x\in R_m$, $y\in R_k$, $m+k=n$.

Given a subset B of G, a function f on G and y in R_k, denote by B_y the set $\{x: (x, y) \in B\}$ and by f_y the function $f_y(x)=f(x, y)$. Consider a linear differential operator D_0 over R_m with constant coefficients, and the operator D over R_n defined by $(Df)(x, y) = (D_0 f_y)(x)$ if f is infinitely differentiable. Assuming f to be locally summable in G, the distribution Df in the sense of L. Schwartz is a measure μ if and only if $D_0 f_y$ is a measure μ_y for almost all y, and the total variation $|\mu_y|(B_y)$ is a summable function of y for every bounded Borel set $B \subseteq G$. In this case $\mu(B) = \int_{R_k} \mu_y(B_y) d\lambda_k(y)$ where λ_k denotes the k-dimensional Lebesgue measure. It follows that a measurable function f on G is locally summable and its derivatives $\partial f/\partial x_i$ are measures ν_i if and only if f is locally of bounded variation in the sense of Tonelli and Cesari (TC). The Lebesgue area of the nonparametric hypersurface $x_{n+1}=f(x_1, \cdots, x_n)$ which by a generalization of a theorem of Cesari is finite if and only if $\lambda_n(G) < +\infty$ and f is of bounded variation TC, is then given by $\int_G((d\lambda_n)^2+(d\nu_1)^2+ \cdots +(d\nu_n)^2)^{-1/2}$, i.e. the total variation of the vectormeasure $(\lambda_n, \nu_1, \cdots, \nu_n)$ over G. (Received February 18, 1957.)

438. P. D. Lax: *An asymptotic Huygens' principle for nonlinear equations*. Preliminary report.

Let $u(x, t)$ be a solution of a first order quasilinear hyperbolic system $u_t+A(u)u_x = 0$. Suppose that initially u is zero outside of a finite interval. If the number of variables is two, the Riemann invariants are constant along the characteristics and therefore the solution is zero at all points which cannot be connected by at least one characteristic to the interval containing the initial disturbance. This is an example of Huygens' Principle. For systems with a larger number of dependent variables, such a Huygens' Principle has an asymptotic validity in the initial amplitude. In the presence of shocks the result is valid up to terms of third order. (Received February 11, 1957.)

439t. Dorothy Maharam: *The realization of point-isomorphisms*.

It is shown that any set-isomorphism of a product of (perhaps uncountably many) normal measure spaces can be realized by a point-isomorphism. (The terminology is that of Halmos and von Neumann, Ann. of Math. vol. 43 (1942) pp. 332–350). (Received February 12, 1957.)

440t. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. I.

Let X, $\tilde{X}$ be a Banach space and its space of endomorphisms. The following spaces of functions of $t \in J = [0, +\infty)$ in X (or $\tilde{X}$) are considered: $\mathfrak{L}$, space of functions integrable in any finite interval; $\mathfrak{C}$ space of bounded continuous functions with norm $|||x||| = \sup \|x(t)\|$; $\mathfrak{C}_0$, $\mathfrak{A}$, $\mathfrak{P}_\omega \subset \mathfrak{C}$, subspaces of functions $\to 0$ when $t \to +\infty$, almost-periodic, limit-periodic of period ω, respectively; $\mathfrak{M}$ space of functions $\in \mathfrak{L}$ with $\sup \int_t^{t+1} \|x(\tau)\| d\tau < \infty$ and norm $|||x|||_{\mathfrak{M}}$ equal to that sup; $\mathfrak{L}^p$, $1 \leq p \leq \infty$, space of pth-power integrable or essentially bounded functions with the usual norm. Assume that the values for $t=0$ of the bounded solutions of (1): $\dot{x}+A(t)x=0$, $A \in \mathfrak{L}(\tilde{X})$, $x \in X$, form a subspace X_0 (which is not always the case) and that X is the direct sum of X_0 and another subspace X_1. Theorem 1: If (2): $\dot{x}+A(t)x=f(t)$ has at least one bounded solution for each $f \in \mathfrak{F}$, $\mathfrak{F}$ being any one of $\mathfrak{C}$, $\mathfrak{C}_0$, $\mathfrak{A}$, $\mathfrak{P}_\omega$, $\mathfrak{M}$, $\mathfrak{L}^p$ $(1 \leq p \leq \infty)$, there exists K depending only on A and $\mathfrak{F}$, such that the (unique) bounded solution of (2) with $x(0) \in X_1$ satisfies $|||x||| \leq K \cdot |||f|||_{\mathfrak{F}}$. Furthermore, if the assumption holds for any one of $\mathfrak{C}$, $\mathfrak{C}_0$, $\mathfrak{A}$, $\mathfrak{P}_\omega$ and if X_1 is reflexive it also holds for $\mathfrak{L}^\infty$. (Received January 28, 1957.)

441*t*. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. II.

With the notations of Part I, the following theorems are proved: 2. If for some p, $1<p\leqq\infty$, and each $f\in\mathfrak{L}^p$, (2) has at least one bounded solution, there is a constant $\mu>0$ and a function $M(t_0)$, $t_0\in J$, such that for any bounded solution of (1) $\|x(t)\|\leqq M(t_0)\cdot\exp\left[-\mu(t-t_0)^{1/q}\right]\cdot\|x(t_0)\|$, $t\geqq t_0$, $(1/p)+(1/q)=1$. 3. The same result holds for $f\in\mathfrak{C}$, $\mathfrak{C}_0$ (and, if X_1 is reflexive, for $f\in\mathfrak{A}$, $\mathfrak{P}_\omega$) with $q=1$. 4. If moreover $A\in\mathfrak{M}$, positive constants N, N', ν, ν' exist such that for $t\geqq t_0$: (i) any bounded solution of (2) satisfies $\|x(t)\|\leqq N\exp\left[-\nu(t-t_0)\right]\cdot\|x(t_0)\|$; (ii) any solution with $x(0)\in X_1$ satisfies $\|x(t)\|\geqq N'\exp\left[\nu'(t-t_0)\right]\cdot\|x(t_0)\|$; (iii) any pair of solutions belonging to these two different kinds stay angularly apart (in a specified sense). 5. Conversely, (i), (ii), (iii) imply that (2) has at least one bounded solution for every $f\in\mathfrak{M}$, a fortiori for $f\in\mathfrak{L}^p$ $(1\leqq p\leqq\infty)$, $\mathfrak{C}$, $\mathfrak{C}_0$, $\mathfrak{A}$, $\mathfrak{P}_\omega$; in this converse, (iii) may be replaced by $A\in\mathfrak{M}$. 6. If for every $f\in\mathfrak{L}^1$ (2) has at least one bounded solution, (i), (ii), (iii) hold with $\nu=\nu'=0$ (no assumption is made on A). 7. The converse of Theorem 6 holds. (Received January 28, 1957.)

442*t*. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. III.

With the notations of the preceding Parts, the following theorems are proved: 8. If $A\in\mathfrak{M}$ and if a scalar function $V(x, t)$ exists with an infinitely small upper bound (nothing is assumed concerning the sign of V) such that its total derivative V' along the solutions of (1) is negative definite, all bounded solutions of (1) satisfy (i) and their initial values $x(0)$ form a subspace X_0; if X_1' is another subspace on which $V(x, 0)\leqq 0$, all solutions of (1) with $x(0)\in X_1'$ satisfy (ii). 9. The same is true if instead of $V\leqq 0$ on X_1' we assume $X_0\cap X_1'=\{0\}$ and $\dim X_1'<\infty$ (counterexamples are given when these assumptions do not hold). 10. If the assumptions (i), (ii), (iii) are satisfied, given any even integer κ, functions $V_0(x, t)$, $V_1(x, t)$ exist which are homogeneous polynomials in x of degree κ, positive semidefinite and bounded $(0\leqq V_i\leqq S\|x\|^\kappa$, $i=0, 1)$, such that V_0+V_1 is positive definite $(V_0+V_1\geqq s\cdot\|x\|^\kappa$, $s>0)$ and $V=V_0-V_1$ has a negative definite V'. (Received January 28, 1957.)

443*t*. J. L. Massera and J. J. Schäffer: *Linear differential equations and functional analysis*. IV.

Previous results of Parts I–III yield the following applications: 11. If the assumptions of Theorem 1 are satisfied and if $h(x, t)$ is any (nonlinear) function defined for $t\in J$, $\|x\|\leqq a$, such that $h(0, t)\in\mathfrak{F}$ and that for any two x', $x''\in X$ one has $\|h(x', t)-h(x'', t)\|\leqq\gamma_1(t)\|x'-x''\|$; where the scalar function $\gamma_1(t)\in\mathfrak{F}$, and if the norms of $h(0, t)$ and γ_1 in $\mathfrak{F}$ are sufficiently small, then the equation (3): $\dot{x}+A(t)x=h(x, t)$ has a family of bounded solutions $x(t, \xi_0)$, where $\xi_0=x(0, \xi_0)$ takes any arbitrary value in a sphere $\xi_0\in X_0$, $\|\xi_0\|\leqq b$. 12. If the assumptions of Theorem 1 are satisfied with $\mathfrak{F}=\mathfrak{C}_0$, then all bounded solutions of (2) tend to 0 when $t\to+\infty$; the same is true of the solutions of (3) described in Theorem 11 if the assumptions of this theorem are satisfied. 13. If $A\in\mathfrak{A}[\mathfrak{P}_\omega]$, if the assumptions of Theorem 1 are satisfied with $\mathfrak{F}=\mathfrak{A}[\mathfrak{P}_\omega]$ and if X_1 is reflexive, then, for each $f\in\mathfrak{A}[\mathfrak{P}_\omega]$, equation (2) has one and only one almost-periodic [limit-periodic of period ω] solution; the same is true of (3) if h satisfies the assumptions of Theorem 11. 14. The properties stated before (particularly in Theorems 4 and 13) are rough in the following sense: the set of operators $A \in \mathfrak{F}$ (where $\mathfrak{F} = \mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{P}_\omega$ as the case may be) for which these properties hold is open in $\mathfrak{F}$. (Received January 28, 1957.)

444*t*. L. M. Milne-Thomson: *Gauss's theorem.*

By suitably defining a content-tensor for a parallelotope (generalized parallelpiped) in Euclidean space of n dimensions, Gauss's theorem for converting a volume into a surface integral is generalized to n-space. It then appears that the general theorem includes as special cases not only Gauss's but also Stokes's theorem, and the ordinary definite integral in one variable. (Received February 18, 1957.)

445. John Nash: *A priori Hölder continuity of solutions of parabolic and elliptic equations.*

Consider a heat equation $\nabla \cdot (C \cdot \nabla T) = \dot{T}$ where C is the (symmetric) conductivity tensor. C varies with position and time. Assume all eigenvalues c_ν of C everywhere obey $0 < c_1 \leqq c_\nu \leqq c_2$. Without continuity requirements on C, we have $|T(x_1, t) - T(x_2, t)| \leqq Bk|x_1 - x_2|^\alpha t^{-\alpha/2}$ if $|T| \leqq B$ everywhere when $t = 0$. The constants k and α depend only on c_1, c_2 and the dimension n of the space. For elliptic boundary value problems a Hölder exponent $\alpha/1+\alpha$ is derivable. The method depends on first estimating the moment $M(t) = \int rT\,dV$ of a fundamental solution, r being distance from its point of origin. This is done by interrelating M and $\dot{M}$ and the strictly increasing entropy $S(t) = -\int T \log T\,dv$ and $\dot{S}$. Result: $at^{1/2} \leqq M(t) \leqq bt^{1/2}$. Second, one shows that two adjacent fundamental solutions T_1 and T_2 overlap after a certain time so that $\int \min(T_1, T_2)\,dV \geqq \epsilon$ when $t \geqq \tau$. Third, one shows that $\int |T_1 - T_2|\,dV \to 0$ as $t \to \infty$ like $t^{-\alpha/2}$, which gives the result. These a priori estimates are of sufficient strength to apply to nonlinear equations. We believe the results and methods will open up areas of this field heretofore intractable, in particular the existence problems for compressible viscous fluid flows. (Received April 2, 1957.)

446*t*. O. G. Owens: *The explicit solution with given integral values of the reduced wave equation.*

The main motivation for this paper is in the sequence of steps which yields the Poisson integral solution of the classical boundary-value problem for Laplace's equation and the circular domain. The complete analogue of this process is developed for the two-dimensions reduced wave equation. The principal theorem obtained is the following: Assume that $u(x, y) \in C^2$ and satisfies the reduced wave equation, $u_{xx} + u_{yy} + u = 0$, for $x^2 + y^2 < \infty$. Furthermore, assume that uniformly in θ the integral $\int_0^\infty u(r, \theta)\,dr = F(\theta)$ $(x = r\cos\theta,\ y = r\sin\theta)$, $F(\theta)$ being of period 2π and fulfilling a uniform Hölder condition of order $\alpha > 1/2$. Then the integral-value problem has a unique solution with the integral representation $u(x,y) = (1/2\pi)\int_0^{2\pi} F(\phi)\cos r \sin(\phi - \theta)\,d\phi + (1/2\pi)^2 \int_0^{2\pi} d\phi\, F(\phi) \cdot \int_0^{\pi/2}([\sin r \sin(\tau + \phi - \theta) + \sin r \sin(\tau - \phi + \theta]\,d\tau/\sin\tau)$. In the proof of this theorem an integral summation formula is derived for the sum $E(r, \theta) = J_0(r)/2 + \sum_{k=1}^\infty J_k(r)\cos k\theta$; namely, $E(r, \theta) = \cos r \sin\theta/2 + (1/2\pi)\int_0^{\pi/2}([\sin r \sin(\tau + \theta) + \sin r \sin(\tau - \theta)]\,d\tau/\sin\tau)$. (Received February 18, 1957.)

447. Ivan Polonsky: *Pseudoanalytic functions with characteristic coefficients in L_p.*

Let F and G be bounded functions with strong L_p $(p > 2)$ and L_{p_1} $(1 < p_1 < 2)$ derivatives as defined by Friedrichs and Sobolev. The characteristic coefficients a, b, A, B (cf. Bers, Bull. Amer. Math. Soc. vol. 62 (1956) p. 293) of the generating pair

(F, G) are then in $L_p \cap L_{p_1}$ ("condition C"). In a bounded domain a function w is (F, G) pseudoanalytic if and only if it has strong derivatives satisfying the equation $w_{\bar{z}} = aw + b\bar{w}$. Also if a, b satisfy "condition C" (i) there exists a generating pair (F, G) with a, b as the first two characteristic coefficients. (ii) (F, G) has a successor with characteristic coefficients satisfying "condition C." (iii) A bounded continuous function w is (F, G) pseudoanalytic if and only if $(x) f = w + (1/\pi)\iint((aw + b\bar{w})/(\zeta - z))d\xi d\eta$ is analytic. The similarity principle of L. Bers is also proved under this weaker hypothesis and is then used to prove Taylor and Laurent expansion theorems for pseudoanalytic functions. By assuming that $a(z)|z|^{2(p-2)/p}$, $b(z)|z|^{2(p-2)/p}$ are also in L_p a theorem of Vekua about the existence of a solution of the equation (x) is extended to arbitrary domains. (Received February 19, 1957.)

448. R. T. Prosser: *Subalgebras of C^* algebras*. Preliminary report.

Let A be a C^* algebra with unit, and A' the dual space of bounded linear functionals on A. If B is a proper C^* subalgebra of A, also with unit, and $B^\perp$ the annihilator of B in A', then the Krein-Milman theorem implies that $B^\perp$, and hence B, is determined by the hermitian extreme points of its unit sphere. A careful analysis shows that every such extreme point may be uniquely resolved as half the difference of two positive extreme points (extreme states) of the unit sphere of A'. Thus every subalgebra of A, proper or not, is determined by the pairs of extreme states which it identifies. In particular, if two subalgebras identify the same pairs of extreme states, they must be the same; if a subalgebra separates all pairs of extreme points, then it must be A. Utilizing the correspondence between extreme states and maximal left ideals of A, we may rephrase these conclusions as follows: Every subalgebra of A is determined by its intersections with the maximal left ideals of A. In particular, if two subalgebras distinguish the same pairs of maximal left ideals, they must be the same; if a subalgebra distinguishes all pairs of maximal left ideals, it must be A. (Received February 20, 1957.)

449t. M. H. Protter: *Lower bounds for the first eigenvalue of elliptic equations.*

New estimates are obtained for the first eigenvalue of elliptic equations. In the simplest case of $u_{xx} + u_{yy} + \lambda^2 u = 0$ the expression $\iint_D[u_x^2 + u_y^2 + 2Puu_x + 2Quu_y + (P_x + Q_y - \lambda^2)u^2]dxdy$ vanishes for arbitrary $P = P(x, y)$, $Q = Q(x, y)$ if $u = 0$ on the boundary of D. If P and Q are selected so that the above integrand, considered as a form in u_x, u_y, and u, is positive definite for a range of λ, $0 \leq \lambda \leq \lambda_1$, then obviously λ_1, is a lower bound for the first eigenvalue. This can be done by taking P and Q as solutions of certain Riccati equations. The solution of these Riccati equations are in turn reduced to the solution of an eigenvalue problem for a linear second order ordinary differential equation. For some configurations, e.g. rhombus, this yields improved estimates over those obtained previously. The method can be extended by first applying Green's theorem to the expression $\iint aLu\,dxdy$ where A is an arbitrary positive function. (Received February 6, 1957.)

450t. Valdemars Punga: *On some property of a function satisfying Poisson's equation.*

It is known that if $\psi(x, y)$ is harmonic in a circular region with radius r, then the value of ψ at the center 0 is $\psi_0 = (1/2\pi)\int_0^{2\pi}\psi_r d\theta$, where ψ_r is the value of ψ on the circumference C. The finite-difference equivalent of this theorem can be stated in the following form: subdividing C into n equal arcs at points $1, 2, \cdots, n$, $\psi_0 = (1/n) \cdot \sum_{i=1}^{n} \psi_i + R(r^n)$, where error $R(r^n)$ is of order r^n. It is shown that if ψ satisfies Poisson's equation $\nabla^2\psi = f(x, y)$, then, for n odd ($n \geqq 3$): $\psi_0 = (1/n) \sum_{i=1}^{n} \psi_i - (r^2/4)f_0 - (r^4/64)\nabla^2 f_0 - \cdots - (r^{n-1}/2^{n-1}[((n-1)/2)!]^2)\nabla^{n-3} f_0 + R(r^n)$, where f_0 is the value of f at 0, and for n even ($n \geqq 4$): $\psi_0 = (1/n) \sum_{i=1}^{n} \psi_i - (r^2/4)f_0 - (r^4/64)\nabla^2 f_0 - \cdots - (r^{n-2}/2^{n-2}[((n-2)/2)!]^2)\Delta^{n-4} f_0 + R(r^n)$. (Received February 19, 1957.)

451t. R. A. Raimi: *On compactness in a space of operators*. Preliminary report.

Let H be an infinite dimensional Hilbert space, $L(H, H)$ its algebra of bounded linear transformations. Dixmier (Ann. Math. vol. 51 (1950)) discusses these topologies on L: L_s the "strong" topology of operators, L_s^* the topology transported from L_s by the mapping $T \leftrightarrow T^*$ (adjoint), and ^+L the common refinement of L_s and L_s^*. The present paper gives criteria for compact subsets in these topologies, and shows them to be distinct. E.g., A has compact closure in L_s^* if and only if it is equicontinuous as a set of mappings from H_k to H_k, where H_k is H fitted with the compact-open topology when considered as a set of functionals on H. A has compact closure in L_s if and only if $\{T^* | T \in A\}$ has the same equicontinuity property. Compactness in ^+L is equivalent to possession of both properties. It follows that in any of the three topologies, the closed convex circled hull of a compact set is again compact. Finally, since for norm-bounded subsets of $L(H, H)$ the "strong" topologies are equivalent to the "strongest" topologies (op. cit.), the same criteria apply there too. (Received February 22, 1957.)

452t. R. A. Raimi: *Mean values and Banach limits*, II.

In a previous paper (Bull. Amer. Math. Soc. Abstract 64-4-512) a class of Banach limits was obtained, using integral means, for translation-invariant Banach spaces E of real measurable bounded functions on $R(-\infty, \infty)$, and their extreme values computed. Here the line is replaced by any abelian group G, and convex combinations of the translation operators $T_s: f(x) \rightarrow f(x+s)$ replace the integral means. By the use of a fixed point theorem closely related to Eberlein's formulation of the mean ergodic theorem (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240), all Banach limits for the space of bounded functions on G are constructed, and their extreme values found to be lim sup and lim inf of $T_rf(0)$, where T_r is a convex combination of translation operators, and the set $\{T_r\}$ is made a net in the ordering given by Eberlein to a bounded abelian semigroup. It follows, via an example, that there are (nonuniformly) continuous functions on R which have Banach limits not given by the integral mean construction. Other kinds of invariant finitely additive measures can be found by the same method. (Received February 22, 1957.)

453t. I. F. Ritter: *The multiplicity function of an equation*.

The *multiplicity function* $\mu(z) = [1 - ff''(f')^{-2}]^{-1}$ is extremely helpful in solving the equation $f(z) = 0$. For any root r, Lim $\mu(z) = m$; $z \rightarrow r$, determines the multiplicity m of that root. When $m \neq 1$, or when roots with $m = 1$ are not widely separated, $|1 - \mu^{-1}(z)|$ will be, or is apt to be, far from zero, even if z is near r. This accounts for the unsatisfactory rate of convergence, under those conditions, of the Newton-Raphson iteration and of its modifications based on reversion of Taylor's series. The magnitude of $\mu(z)$ shows that simple roots need not be very close together to produce this retardation. With the guidance furnished by the behavior of $\mu(z)$, this trouble is overcome, and the iteration formula $\phi(z)=z-[\mu(z)+m]f(z)/2f'(z)$ can be exploited to provide *third* order convergence to any root independently of its multiplicity or of the proximity of other roots. When formal differentiation of $f'(z)$ is to be avoided, a Hermitian type interpolation formula can be used to approximate values of $f''(z)$. This hardly impairs the effectiveness of the proposed method for dealing with multiple or nearly coincident roots. (Received January 21, 1957.)

454*t*. G. C. Rota: *A spectral mapping theorem for the essential spectra of ordinary linear differential operators.*

An ordinary differential operator $\tau=a_n(t)(d/dt)^n+a_{n-1}(t)(d/dt)^{n-1}+\cdots+a_0(t)$ defined on an interval $I=(a, b)$ $(-\infty\leqq a<b\leqq\infty,\ a_n(t)>0,\ a_i$ of class $C^\infty,\ 0\leqq i\leqq n)$ acting on the set of functions with compact carriers in I defines a linear operator in each of the B-spaces $L_p(I)$ $(1\leqq p\leqq\infty)$ whose closure $T_0(\tau, p)$ exists. The adjoint of the operator $T_0(\tau, p)$ is calculated. The essential spectrum of τ in $L_p(I)$ is defined as the set of all complex numbers λ for which the range of the operator $\lambda I-T_0(\tau, p)$ is not closed. The essential spectrum of τ is a closed set and coincides with the essential spectrum of the operator $T_0(\tau^*, q)$, where τ^* is the Lagrange adjoint of τ and $1/p+1/q=1$. *Theorem*: Let A be the essential spectrum of τ in $L_p(I)$ and let $p(z)=b_nz^n+\cdots+b_0$ be a complex polynomial. Then the essential spectrum of the operator $p(\tau)=b_n\tau^n+\cdots+b_0$ is the set $\{p(\lambda)\mid\lambda\in A\}$. (Received January 29, 1957.)

455*t*. G. C. Rota: *Extension theory of non-self-adjoint differential operators in B-spaces.*

An extension theory for ordinary linear differential operators in the spaces $L_p(I)$ (I an arbitrary interval of the line, cf. the preceding abstract) can be constructed which generalizes several features of von Neumann's extension theory of symmetric unbounded operators in Hilbert space. Some of the main results are the following: Let C be a connected component of the complement of the essential spectrum of the differential operator τ in $L_p(I)$. For every complex number λ in C the dimension of the space of solutions of class $L_p(I)$ of the differential equation $(\lambda-\tau)f=0$ is constant; this dimension is the deficiency index of the component C. An extension T of the operator τ in $L_p(I)$ is a closed operator such that $T_0(\tau, p)\subset T\subset T_0(\tau^*, q)^*$ $(1/p+1/q=1)$. *Theorem*: Assume that the deficiency indices of all connected components of the essential spectrum of τ are equal. Then there exists an extension T of τ whose spectrum consists of the essential spectrum of τ and a set of eigenvalues all of whose limit points lie in the essential spectrum of τ. A suitably stated converse of the theorem is also true. Explicit representations of extensions and their resolvents are derived, as well as several explicit results on the location of eigenvalues. (Received January 29, 1957.)

456. Robert Schatten: *On the space of completely continuous operators.*

Let Θ be the space of all completely continuous operators on a Hilbert space $\mathfrak{H}$. Thus, its conjugate space Θ^* is the "trace-class" and Θ^{**} the space of all bounded operators on $\mathfrak{H}$. Put $\Theta^*=\Theta^{(1)}$ and $(\Theta^{(n-1)})^*=\Theta^{(n)}$ for $n>1$. Since the "natural" imbedding of Θ into $\Theta^{(2)}$ is a proper one, it follows that $\Theta\subset\Theta^{(2)}\subset\Theta^{(4)}\subset\cdots$ and $\Theta^{(1)}\subset\Theta^{(3)}\subset\Theta^{(5)}\subset\cdots$ where each inclusion is a proper one. It is shown that Θ is indeed a "beginning" that is, Θ is not the conjugate space of any Banach space. (Received November 28, 1956.)

457. Martin Schechter: *On estimating partial differential operators. II.*

Let G be a bounded domain in E^n with closure $\bar{G}$ and boundary B of class C^∞. At each point $x \in B$ assign a nonnegative integer $r(x) \leq m$ and denote by $C_{\infty r}(G)$ the set of all (complex) functions u of class C^∞ in $\bar{G}$ having all derivatives of order $< r(x)$ vanish on B near x for each point $x \in B$. Let (A_k) be a finite set of linear mth order partial differential operators with complex valued coefficients continuous in $\bar{G}$. Consider the complex vector roots common to the characteristic polynomials P_k of the A_k. At each point $x \in B$ let $q(x)$ be the greatest number of such roots (properly defined) with imaginary parts perpendicular to B at x and pointing in the same direction (inward or outward). Let $\|u\|$ denote the $L_2(G)$ norm of u and $D^m u$ its generic mth order derivative. Then there is a constant K such that $\|D^m u\|^2 \leq K(\sum \|A_k u\|^2 + \|u\|^2)$ for all $u \in C_{\infty r}(G)$ if and only if the P_k have no real vector roots in common and $q(x) \leq r(x)$ at each point $x \in B$. (Received February 1, 1957.)

458. Annette Sinclair: *A general solution for a class of approximation problems.*

This paper is concerned with a class of approximation problems of which Carleman's approximation theorem and the author's generalization of Runge's Theorem are examples. In proofs of theorems of this type a nested sequence of sets $\{R_i\}$, depending on a given set $S \subset R$, is constructed. Then a sequence of functions $\{r_i(z)\}$ is defined in such a way that each $r_i(z)$ meets specified conditions on $R_n \cap S$ and such that $\lim r_i(z)$ satisfies preassigned conditions on S and belongs to a designated class of functions defined on all of R. In this paper a theorem abstracting this type of problem is proved and is applied to obtain some new results on approximation by analytic functions. (Received February 18, 1957.)

459. Paul Weiss: *Sampling theorems associated with Sturm-Liouville systems.*

A class C of functions $f(t)$ of the continuous variable t, $-\infty < t < \infty$, is said to satisfy a sampling theorem if there exists a discrete set of values t_n and a kernel $K(t, t_n)$ with the property $K(t_m, t_n) = \delta_{mn}$ such that $f(t) = \sum_n K(t, t_n) f(t_n)$ for every function $f(t)$ of C. It is shown that a sampling theorem is associated with each positive Sturm-Liouville system (one with positive eigenvalues), the class C and the kernel K depending on the system. Let $u(x, t)$ be the solution of the Sturm-Liouville differential equation $(p(x)u_x)_x + (-q(x) + t^2\rho(x))u = 0$, $a \leq x \leq b$, $p(x) > 0$, $\rho(x) > 0$, $q(x) \geq 0$, satisfying the boundary conditions at $x = a$: $u(a, t) = \sin \alpha$, $p(a)u_x(a, t) = \cos \alpha$, $0 \leq \alpha \leq \pi/2$, so that $l_\alpha[u(a, t)] = \cos \alpha \cdot u(a, t) - \sin \alpha \cdot p(a) \cdot u_x(a, t) = 0$. Let t_n^2 be the set of eigenvalues and $u(x, t_n)$ be the set of corresponding eigenfunctions obtained by imposing the additional boundary condition at $x = b$: $l_\beta[u(b, t)] = \cos \beta \cdot u(b, t) + \sin \beta \cdot p(b) \cdot u_x(b, t) = 0$, $0 \leq \beta < \pi/2$. Let an even function $h(t)$ be called "representable" if it can be expressed in the form $h(t) = \int_a^b g(x)u(x, t)\rho(x)dx$, where $g(x)$ satisfies $l_\alpha[g(a)] = 0$, $l_\beta[g(b)] = 0$, and possesses a Fourier expansion in terms of the eigenfunctions $u(x, t_n)$. Denote by $f_e(t)$ and $f_o(t)$ respectively the even and odd parts of an arbitrary function $f(t)$. Then a sampling theorem holds for the class G of functions $f(t)$, for which $f_e(t)$ and $t \cdot f_o(t)$ are representable. It reads $f(t) = \sum_{n=-\infty}^{\infty}{}' K(t, t_n) f(t_n)$, where the kernel is given by $K(t, t_n) = (t_n/t)(1/(t - t_n))(l_\beta[u(b, t)]/l_\beta[u_t(b, t_n)])$ in the sum, $t_{-n} = t_n$, and the prime denotes that the term with $n = 0$ is to be omitted. (Received February 21, 1957.)

460. Burton Wendroff: *An integral inequality.*

The inequality $u(x) \leq C + \int_{x_0}^{x} v(s)u(s)ds$, with $u(x) \geq 0$, $v(x) \geq 0$ for $x \geq x_0$, has the solution $u(x) \leq C \exp\left[\int_{x_0}^{x} v(s)ds\right]$. A partial generalization to functions of two independent variables can be obtained; namely, let $u(x, y) \leq a(x) + b(y) + A\int_{x_0}^{x} u(r, y)dr + B\int_{y_0}^{y} u(x, s)ds + \int_{x_0}^{x}\int_{y_0}^{y} v(r, s)u(r, s)drds$ where (a) $u(x, y)$, $v(x, y)$ are continuous, and (b) $u(x, y) \geq 0$, $v(x, y) \geq 0$, $a(x) \geq 0$, $a'(x) \geq 0$, $b(y) \geq 0$, $b'(y) \geq 0$, for $x_0 \leq x$, $y_0 \leq y$. This is solved by showing that $u(x, y)$ is majorized by the solution of the corresponding equality, and then obtaining an estimate for that solution. The final result is $u(x, y) \leq (Q(x_0, y)Q(x, y_0)/Q(x_0, y_0)) \exp\left\{A(x-x_0) + B(y-y_0) + \int_{x_0}^{x}\int_{y_0}^{y}[AB + v(r, s)]drds\right\}$ where, $Q(x_0, y_0) = a(x_0) + b(x_0)$, $Q(x_0, y) = a(x_0) + b(y_0) + \int_{y_0}^{y} e^{-B(y'-y_0)}b'(y')dy'$, $Q(x, y_0) = a(x_0) + b(y_0) + \int_{x_0}^{x} e^{-A(x'-x_0)}a'(x')dx'$. (Received February 22, 1957.)

461. Albert Wilansky: *Some Tauberian lemmas.*

Consider conservative matrices $A = (a_{nk})$ with $a_{nk} = 0$ for $k > n$ called triangular. If also $a_{nn} \neq 0$, call A a triangle. Let $a_k = \lim_n a_{nk}$, $\|A\| = \sup_n \sum_k |a_{nk}|$. THEOREM 1. *If $a_{nn} = 1$ for all n and $\sum |a_k| > \|A\| - 2$ then $\|A^{-1}\| < \infty$, i.e. Ax convergent implies x convergent.* The special case in which A is regular is known. Examples show this best possible. There follows the known result: $\{x_n + \sum_{k=1}^{n-1} t_k x_k\}$ convergent, $\sum |t_k| < \infty$, implies x convergent. Call A right-finite if there exists m such that $a_{nk} = 0$ for $k > m$. If A, B are conservative triangles with $\|B^{-1}\| < \infty$ and $A - B$ right-finite, clearly $\|A^{-1}\| < \infty$. The result is extended by omitting "conservative." This leads (with a remark of K. Zeller) to THEOREM 2. *The Banach algebra U of triangular matrices A with $\|A^{1}\| < \infty$, is not semi-simple.* THEOREM 3. *There exist bounded sequences x, y, y divergent such that $x - Ay$ is divergent for every triangular conservative A.* This contrasts with the known result that there exists conservative A such that $x - Ay = 0$; and says the map $f: U \to m/c$ is not onto. Call A generalized conservative if Ax is in the linear closure in c of the columns and the row-sum of A for all $x \in c$; generalized coercive if the same for all $x \in m$. This allows extension of theory of conservative matrices to class of matrices of finite norm. (Received February 18, 1957.)

462. František Wolf: *The singular spectrum of partial differential operators.*

In the case of a partial differential elliptic operator A in the Hilbert space $L^2(D)$, suppose $\lambda \in \sigma_e(A)$ the essential spectrum of A. Then there exists a normalized noncompact sequence of $f_n \in L^2(D)$ such that $\lim (A - \lambda)f_n = 0$. If λ is not an eigenvalue of A, then f_n converges to zero uniformly in every interior domain and at all "regular" points near the boundary. The irregular points are those in whose every neighborhood the noncompactness persists. Those points cause the essential spectrum at λ. We may write $\lambda \in \sigma_e(x)$, if x is such a point. If $\lambda \notin \sigma_e(x)$, for all $x \in \bar{D}$, then $\lambda \notin \sigma_e(A)$. In the case of boundary conditions of local character, we may say that $\sigma_e(A)$ derives from certain specific irregular points on the boundary. It can be shown that points which are regular for the differential equation and in whose neighborhood are valid boundary conditions of most of the usual types, are regular points. (Received February 22, 1957.)

APPLIED MATHEMATICS

463. J. H. Blau: *Social choice functions and simple games.*

A (social) *choice function* is a mapping from n-ples of strong orderings of a set A

to reflexive connected relations on A. It is *binary* if on each subset of A it induces a unique choice function. If and only if the function is *neutral* (Blau, *Econometrica* vol. 25 (1957)), it may be identified with a simple game, designating the *dictatorial* sets (ibid.) as winning coalitions. A *bicyclic ballot* is an n-ple on a three-element subset of A, in which exactly two distinct strong orderings occur, these being cyclically equivalent. A choice function is *bicyclically transitive* if the social order is transitive for those ordered triples which occur as individual strong orderings in a bicyclic ballot. A choice function satisfies the *weak unanimity rule* (URR) if a unanimous vote $x>y$ implies $x\geq y$ socially. *Theorem*: Let A have more than two elements. A binary choice function with URR is a simple game if and only if it is monotonic (ibid.) and bicyclically transitive. Other results are obtained without URR or monotonicity, and with a reduced domain. (Received February 18, 1957.)

464. R. J. Duffin and E. P. Shelly (p): *Difference equations of polyharmonic type.*

Of concern are partial difference equations with constant coefficients. Fourier methods are used to study growth rate theorems and properties of the fundamental solution. A study is made of polynomial solutions and continuation theorems. As is well known, there are differential operators such as $r\cdot$grad which convert harmonic functions into harmonic functions. Analogous operators are developed for the theory of discrete harmonic functions. By the use of such operators, it is found possible to give a complete evaluation of the fundamental solution of the Laplace difference equation in three dimensions. If f is harmonic, then r^2f is biharmonic. Relationships of this type are extended to the discrete case, and such relationships leads to a complete evaluation of the fundamental solution of the discrete biharmonic equation in two dimensions. (Received February 18, 1957.)

465t. J. R. Foote: *Recursion relations for parabolic cylinder functions.*

Formulas are found permitting the raising or lowering of the index ν of the parabolic cylinder function $D_\nu(z)$ appearing in certain derivatives and indefinite integrals. The functions are various combinations of the D_ν and of exp $[-z^2/4]$, such as: $z^nD_\mu D_\nu$, $e^{-z^2/2}D_\mu D_\nu$, $e^{-nz^2/4}D_\mu D_\nu$, $e^{\pm z^2/4}D_\nu$, where we may have $n=0$, and/or $\mu=\nu$. When μ, ν, n are integers, the recursion relations allow reduction of the derivatives and integrals to algebraic calculations among similar functions and to integrals related to the error function. (Received February 19, 1957.)

466t. J. R. Foote: *An asymptotic method for free convection past a vertical plate.*

Recently Finston noted that there is a similarity transformation of the free convection equations which requires a variable plate temperature. The resulting pair of nonlinear equations may be solved in an iterative manner by a method recently detailed by Fettis which gives a system of linear equations with constant coefficients in this case. Three terms of the series are given for arbitrary values of three parameters. (Received February 19, 1957.)

467. Abolghassem Ghaffari: *On the solutions of compressible flow past a wedge.*

Applying the hodograph method (Temple & Yarwood: *Compressible flow past a* *wedge*, Report No. S.M.E. 3222) for the regions $q>q_1$ and $q>q_1$ (q is the incompressible fluid speed around the wedge and q_1 its value in the main stream), the solutions for the incompressible field of flow and the corresponding compressible field of flow around a wedge are obtained. It is found that the solution for the stream function ψ of a steady, irrotational compressible flow is either (I) $\psi = lq_1 \sum_{n=0}^{\infty} (\psi_m(\tau)/\psi_m(\tau_1)) \cdot \sin((n+1)/a)\theta, m=(n+1)/a$, for $q<q_1$, or (II) $\psi = lq_1 \sum_{n=0}^{\infty} (\psi_{-m}(\tau)/\psi_{-m}(\tau_1)) \sin(n/a), \theta$, $m=n/a$, for $q>q_1$, where l is a fraction of the length L of the sloping part of the wedge, $a=\alpha/\pi$, where 2α is the wedge angle $(0<a<1)$, and θ is the angle between the x-axis and the direction of flow. $\tau_1 = w_1^2/w_m^2$ is the value of $\tau = w^2/w_m^2$ (where w is the compressible fluid speed, w_m its maximum, and w_1 the value of w in the main stream) in the main stream, $\psi_m(\tau) = \tau^{m/2} F_m(a_m, b_m, c_m; \tau)$ where $F_m(\tau)$ indicates the hypergeometric function. $\psi_{-m}(\tau)$ is some suitable solution of the equation for $\psi_m(\tau)$ in the region $\tau_1<\tau<1$. It is shown that the solutions (I), (II), are continuous at $\tau=\tau_1$, and that the continuity of $\partial\psi/\partial\tau$ at $\tau=\tau_1$ implies $[\psi'_m(\tau_1)/\psi_m(\tau_1)]_{m=(n+1)/a} = [\psi'_{-m}(\tau_1)/\psi_{-m}(\tau_1)]_{m=(n+1)/a}$. It is found that $\psi_{-m}(\tau)$ is bounded at $\tau=1$, and that when $w_m \to \infty$, $\psi_{-m}(\tau)/\psi_{-m}(\tau_1) \to (\tau/\tau_1)^{-m/2}$. (Received February 20, 1957.)

468. M. A. Hyman: *Eigenvalues and eigenvectors of general matrices.*

This paper extends to general matrices (complex elements, no symmetry) the method previously presented (Cambridge, October, 1956) for finding eigenvalues and eigenvectors or real matrices. By $(N-1)(N-2)/2$ complex rotations (unitary transformations each involving only two coordinate directions) the general matrix A of order N is transformed into matrix B with all zeros above the first super-diagonal: $B=R^{-1}AR$. The eigenvalue problem for B is then $(B-\lambda I)x=0$, where x is a column vector with components $x_1, x_2, \cdots, x_N$. Setting (usually) $x_1=1$ and guessing λ, one calculates $x_2, x_3, \cdots, x_N$ and $F(\lambda)$ recursively; $F(\lambda)$ is the characteristic polynomial, evaluated for the trial λ. By successive approximation (see abstract below) a sequence of λ's (usually complex) is obtained which approaches an eigenvalue E of B: $F(E)=0$. Simultaneously, the vectors x approach the eigenvector ξ. The corresponding eigenvalue and eigenvector of A are E, $R\xi$. Other λ sequences can be constructed which converge to the other eigenvalues of B; the x-vector sequences generated as a by-product of evaluating $F(\lambda)$ converge to eigenvectors of B. (Received February 21, 1957.)

469*t*. M. A. Hyman: *Finding complex zeros by interpolation*, II.

Consider $F(\lambda)$, a general function of the complex variable λ, with zero E. We have previously shown (Cambridge, October, 1956) how to construct a sequence $\{\lambda_i\}$ converging to E. Use is made of $\{P_i^{(N+1)}(\lambda)\}$, a sequence of Nth degree polynomials $(N=1, 2, 3, \cdots)$, coinciding with $F(\lambda)$ at $N+1$ points near E, and approximating $F(\lambda)$ ever more closely in that neighborhood. These "interpolation" methods for finding zeros appear to have many advantages for use with automatic computing machines. We here emphasize the use of "2-point" and "3-point" methods, with which we have had good success in numerical experiments. Let the respective interpolating polynomials be $P_{i+1}^{(2)}(\lambda) = \alpha_i(\lambda-\bar{\lambda}_i)+\beta_i$ and $P_{i+1}^{(3)} = A_i(\lambda-\bar{\lambda}_i)^2 + B_i(\lambda-\bar{\lambda}_i)+C_i$, where $\bar{\lambda}_i$ is the best approximation to E after i steps, and let the next approximation be $\lambda_{i+1} = \bar{\lambda}_i - \delta_i$; here $\delta_i = \beta_i/\alpha_i$ (2-point) and $\delta_i = C_i/B_i$ or $\delta_i = C_i/B_i + (A_i/B_i)(C_i/B_i)^2$ (3-point). $\beta_i = C_i = F(\bar{\lambda}_i)$; α_i, B_i are, respectively, the 2-point and 3-point approximations to $F'(\bar{\lambda}_i)$, if this derivative exists. Neither 3-point method (the second converges more rapidly) requires taking a square root. If λ_{i+1} is not closer to E (measured by the norm $|F(\lambda)|^2$) than one of the points from which it is obtained, δ_i is multiplied by K $(0<K<1)$ and λ_{i+1} recomputed. Any other zeros of $F(\lambda)$ are obtained by seeking zeros for $F(\lambda)|\prod_{r=1}^{n}(\lambda-E_r)$, where $E_1, E_2, \cdots, E_n$ are the roots already obtained (suggestion by G. E. Forsythe). (Received February 21, 1957.)

470. G. S. S. Ludford and S. H. Schot (p): *On sonic limit lines in the hodograph method.*

Lines in the physical plane on which the Jacobian $J=\partial(x, y)/\partial(q, \theta)$ of the hodograph transformation vanishes are called *limit lines*, of which two types have been distinguished. In a recent paper H. Geiringer (Math. Zeit. vol. 63 (1956) pp. 514–524) has identified a new type of limit line which coincides with a nondegenerate segment of the sonic line. She has called this a *sonic limit line*. In the present paper certain statements concerning the occurrence and geometry of sonic limit lines are proved, and a method of constructing flows with such lines is given, together with an example. This flow has no source-like character (cf. F. Ringleb, ZAMM vol. 20 (1940) pp. 185–198) and also contains an example—believed to be the first—of Geiringer's sonic double limit point. (This work was supported by the Office of Ordnance Research, U. S. Army, under Contract DA-36-034-ORD-1486 with the University of Maryland.) (Received February 19, 1957.)

471. Willard Miranker: *The reduced wave equation in a medium with a variable index of refraction.*

In this paper we discuss solutions u of the equation $\Delta u+k^2(x)u=0$ where $x=(x_1, x_2, x_3)$ and where $\lim_{|x|\to\infty} k^2(x)=k^2$. First we consider the case in which u is defined and bounded in all of space. We show that the L^2 norm of u over a sphere does not tend to zero as the radius of the sphere tends to infinity unless $u=0$. This result is an extension of one of Rellich who proved the same statement for a solution of $\Delta u+k^2u=0$. From our result we are able to demonstrate the nonexistence of quadratically integrable u for a suitable class of $k^2(x)$ which includes the case $k^2(x)=k^2=$constant. Then we consider the exterior boundary value problem for $\Delta u+k^2(x)u=0$ subject to the radiation condition. We show that if $u\neq 0$ there exists a solution $v\neq 0$ of $\Delta v+k^2v=0$ such that $\lim_{|x|\to\infty}|u-v|=0$. From this osculation theorem we define the validity of the Rellich estimate for v and from this in turn the uniqueness theorem for the exterior boundary value problem for v. (Received February 27, 1957.)

472t. C. S. Morawetz: *Nonexistence of continuous transonic flows past profiles*, III. Preliminary report.

Suppose that for a given pressure-density relation there exists a smooth transonic flow past a profile with a subsonic speed q_∞ at infinity. By changing slightly the pressure-density relation for supersonic speeds a whole manifold of such profiles with smooth flows having the same speed at infinity can be shown to exist. For this set of flows the perturbation problem, which corresponds to finding a flow for a small change $(q_\infty+\delta q_\infty)$ of the speed at infinity, has no continuous solution. (Received February 20, 1957.)

473. L. E. Payne (p) and H. F. Weinberger: *Bounds for the displacement of free and supported plates.*

A method is given for finding arbitrarily close upper and lower bounds for the normal displacement at an interior point of a thin convex plate when the moment and either the shear or the normal displacement are prescribed on the boundary. The given quantities are approximated in a square integral norm by a linear combination of biharmonic functions, and the error in displacement is estimated in terms of this norm. The method is similar to that used by the authors for estimating the displacement of membranes and clamped plates (J. Math. and Phys. vol. 33 (1955) pp. 291–307). (Received February 21, 1957.)

474. I. F. Ritter: *Computation of the exact solution of a system of linear equations.*

If the numerical matrices A and R have integer elements, the solution X of the equation $AX=R$ is given by ratios of integers. A systematic way of computing these integers in sections of s digits on a calculator with place capacity s is described in form of a modification of the error correction in Bull. Amer. Math. Soc. Abstract 62-3-349. The desirability of obtaining the exact inverse of a matrix is indicated by J. Barkley Rosser [Jour. Res. Nat. Bur. Stand. vol. 49 (1952) p. 349], who also demonstrates a feasible scheme for computing it. The compact method of approximation by sections proposed here was found less laborious than Rosser's scheme, but its practical significance rests on its ability to avoid all those difficulties that one encounters in computing *decimal* approximations to X for a singular or ill-conditioned matrix A. (Received February 20, 1957.)

475. Charles Saltzer: *Multiple-valued solutions of the Laplace difference equation.*

By the use of the difference analogues of the Cauchy-Riemann equations multiple-valued solutions of the Laplace difference equation are defined and computed from the Green's function for the Laplace difference equation. The solution of the Dirichlet problem for the exterior of a finite slit for the case where the boundary values on opposite sides of the slit are different is discussed. (Received February 20, 1957.)

476. H. E. Salzer: *Note on multivariate interpolation for unequally spaced arguments, with an application to double summation.*

In two earlier articles the author expressed the multiple Gregory-Newton interpolation formula for $f(x, y, \cdots, z)$ in terms of mixed advancing differences as far as the nth order, in the simpler form $\sum\sum\cdots\sum_{i+j+\cdots+k=0,1,\cdots,n} C_{ij\cdots k}(x, y, \cdots, z)\cdot f(x_i, y_j, \cdots, z_k)$, where the function $C_{ij\cdots k}(x, y, \cdots, z)\equiv C_{ij\cdots k}$, the coefficient of the tabular entry $f(x_i, y_j, \cdots, z_k)$ was given explicitly, and even tabulated. This present note gives the generalization to where the independent variables are unequally spaced, by expressing that general mixed divided difference formula approximation to $f(x, y, \cdots, z)$ which is exact for an m-ary polynomial of total degree n, directly in terms of the tabular entries $f(x_i, y_j, \cdots, z_k)$. The new formula avoids the labor in the computation of the mixed divided differences $[x_0\cdots x_r; y_0\cdots y_s; \cdots; z_0\cdots z_u]$ and is a simpler looking expression for the m-ary n-*ic* approximation to $f(x, y, \cdots, z)$. The main result is (1) $C_{ij\cdots k}=(x-x_i)(y-y_j)\cdots(z-z_k)\times\sum_{u=k}^{n-i-j-\cdots} L_k^{(u+1)}(z)/(z-z_u)\cdots\sum_{s=j}^{n-i-\cdots-k} L_j^{(s+1)}(y)/(y-y_s)\sum_{r=i}^{n-j-\cdots-k} L_i^{(r+1)}\cdot(x)/(x-x_r)$, where $L_j^{(s+1)}(y)$, for example, is the $(s+1)$-point Lagrangian coefficient at y_j for points $y_0, y_1, \cdots, y_s$. The coefficients $C_{ij\cdots k}(x, y, \cdots, z)$ satisfy $\sum\sum\cdots\sum_{i+j+\cdots+k=0,1,\cdots,n}(C_{ij\cdots k}(x, y, \cdots, z)=1$, $C_{ij\cdots k}(x_f, y_g, \cdots, z_h)=\delta_f^i\delta_g^j\cdots\delta_h^k$, and $\sum\sum\cdots\sum_{i+j+\cdots+k=0,1,\cdots,n} C_{ij\cdots k}(x, y, \cdots, z)x_i^p y_j^q\cdots z_k^t$

$=x^0y^p\cdots z^q$ whenever $0+p+\cdots+q\leq n$, which suggests the name "n-ary Lagrangian" coefficients. From (1), new formulas were derived for determining the limits of double sequences $S_{p,q}$ as $p, q\rightarrow\infty$, the arguments $x=1/p$ and $y=1/q$ being unequally spaced. (Received March 21, 1957.)

477. Joanna Wood-Schot: *On the numerical solution of Poisson's equation in an elliptic region.*

One difficulty in the numerical solution of elliptic differential equations in simple closed regions is the treatment of points near curved boundaries. This problem is investigated for Poisson's equation in an ellipse with given boundary conditions. Since second order difference approximations are used at all interior points, truncation errors are introduced only at mesh boundary points, and hence their effect is studied by comparison with the analytic solution. The equation was solved on a high-speed computer using two iterative methods—the Gauss-Seidel method and the Successive Overrelaxation method (as developed by D. M. Young). The rate of convergence of the latter method is compared with that of both the Jacobi and the Gauss-Seidel methods. An error analysis of the results is given. (Received February 19, 1957.)

GEOMETRY

478. E. N. Gilbert (p) and H. O. Pollak: *A maximum-minimum problem connected with trees.*

Given N points in the plane, a *minimal tree* is a connected linear graph of shortest total line length having these N points as its vertices. If the N points are allowed to range over a closed bounded region R the lengths of minimal trees will somewhere attain a maximum, say $f(R, N)$. There exist numbers $K(R)$ and $C(R)$ such that $f(R, N)\leq K(R)(N-1)^{1/2}$ and $C(R)(N-1)^{1/2}\leq f(R, N)$. The upper bound is obtained using the following geometric property of minimal trees. If L is a line of a minimal tree let $D(L)$ denote the diamond-shaped figure which is the union of the two isosceles triangles having base line L and 30° base angles; then two such diamonds $D(L_1)$, $D(L_2)$ can intersect only at boundary points. The lower bound follows either by arranging N points in a regular lattice or by estimating the mean length of a minimal tree when N points are placed in R at random. If R is the unit square, suitable constants are $K(R)=2.34$, $C(R)=1$. (Received February 21, 1957.)

479. Donald Greenspan: *On two nonequivalent approaches to the study of vertices in E^3.*

In an attempt to generalize to E^n and to generalize topologically theorems on vertices, one must decide upon a definition of a vertex. For E^3, two possibilities readily suggest themselves. The first utilizes the concept of a stationary value of the radius of the osculating sphere, while the second makes use of the ideas of order of contact of a curve with an osculating sphere. It is shown that these are not equivalent and that one kind of vertex is a special type of the other kind. An example is provided to show that the radius of an osculating sphere may actually attain an extreme value without the curve and sphere having more than four consecutive points in common. (Received February 11, 1957.)

480. Melvin Hausner: *Differential representations of homotopy.*

Differentials $\alpha=\sum A_i dx^i$ are studied on a manifold M. The A_i are elements of some associative algebra A. It is then possible to form the product integral $\int_C(1+\alpha)$, the value (depending on the path of integration) an element of A. This integral maps paths C into A, preserving multiplication. Analytic conditions on α are derived in order that this map depend only on the homotopy class of the closed curve C. Thus, α is a representation of the homotopy group of M into A, if and only if $d\alpha+\alpha\wedge\alpha=0$, or equivalently, $\partial A_j/\partial x^i-\partial A_i/\partial x_j+A_iA_j-A_jA_i=0$. α is a trivial representation if and only if $\alpha=F^{-1}dF$, F an A-valued function (with inverse) on M. These are the analogues of exact and bounding differentials. In this way, representations of the homotopy group (into A) are obtained, and these groups—and homology groups over the integers—may be studied by means of differentials. Various examples are considered. (Received February 14, 1957.)

481. N. D. Kazarinoff: *D. K. Kazarinoff's inequality for tetrahedra.*

Let S be a tetrahedron and P be a point not exterior to S. Let the distances from P to the vertices and faces of S be denoted R_i and r_i, respectively. An analogue of the Erdös-Mordell inequality for triangles is established. THEOREM: For any tetrahedron whose circumcenter is not an exterior point, $(\sum R_i)/\sum r_i>2\cdot 2^{1/2}$, and $2\cdot 2^{1/2}$ is the greatest lower bound. Let the vertices of S be i, j, k, and l, (ij) be the length of the edge joining i and j, H_i be the length of the altitude through i, R be the radius of the circumsphere with center O, and r_i be the distance from P to the face opposite i. D. K. Kazarinoff proved the following lemma: $\sum R_i\geq\sum[(ij)^2+(ik)^2+(il)^2]r_i/2RH_i$; equality holds if and only if P and O coincide. Using a method of successive minimization, one is able to show that the minimum value of the coefficient of an r in the last relation is $2\cdot 2^{1/2}$ in the case considered. The minimum is taken on only in the degenerate case where two vertices are opposite ends of a diagonal of a square, the other two vertices are at another corner, and P is the midpoint of the diagonal. (Received March 5, 1957.)

482. Wilhelm Klingenberg: *Isometry of Riemannian submanifolds.* I.

Let M^* be a Riemannian manifold (referred to as big manifold) of dimension d^* and let M be a submanifold of M^* (referred to as small manifold) of dimension d. Everything shall be of class C^∞. From the bundle $F^*(M^*)$ of orthonormal frames over M^* one has the induced bundle $F^*(M)$ of big frames over M. Furthermore one has, from the inclusion map $F^*(M)\rightarrow F^*(M^*)$, on $F^*(M)$ the induced forms ω_i and $\omega_{ij}=-\omega_{ji}$ $(i,j=1,\cdots,d^*)$. Here the ω_i determine the situation of the small tangent space at M in the big frames and the ω_{ij} determine the horizontal space in the tangent space at $F^*(M)$, i.e., they determine a connection in $F^*(M)$. Consider a second manifold $'M^*$ of dimension d^* and herein a submanifold $'M$ of dimension d. Again one has the induced bundle $'F^*('M)$ of big frames over $'M$ and the forms $'\omega_i$ and $'\omega_{ij}$ on the this bundle. By an isometry of $F^*(M)$ onto $'F^*('M)$ one means a 1:1 diffeomorphic bundle map Φ, which is compatible with the Riemannian structure of these bundles, i.e., $'\omega_i\circ d\Phi=\omega_i$ and $'\omega_{ij}\circ d\Phi=\omega_{ij}$. The object of the paper is to characterize the existence of such an isometry in a way that contains for the special case $M=M^*$ and $'M='M^*$ a recent theorem of Ambrose (Ann. of Math. vol. 64). (Received February 19, 1957.)

483t. Wilhelm Klingenberg: *Isometry of Riemannian submanifolds.* II.

(For notations see the preceding abstract.) M and $'M$ are assumed to be complete and simply connected. Fix a point m in M and a big frame at m, of which the first d vectors are in the small tangent space M_m of M at m. In the same way fix a point $'m$ in $'M$ and a big frame at $'m$. This determines a bundle map J of the trivial principal fibre bundle $M_m \times O(d^*)$ onto the bundle $'M_{'m} \times O(d^*)$ ($O(d^*)$ is the structure group of $F^*(M)$). J induces a correspondence between the singly broken geodesics on M, emanating from m, and the singly broken geodesics on $'M$, emanating from $'m$, cf. Ambrose l.c. Using the parallel translation of the big frames along those singly broken geodesics by means of the big connection one gets a correspondence between the fibres of $F^*(M)$ and the fibres of $'F^*('M)$. If one restricts this correspondence to sufficient small neighborhoods of the endpoints of corresponding geodesic segments, emanating from m respectively from $'m$, then this correspondence becomes a 1:1 bundle map, to which we refer as local bundle maps. The main theorem states: If these local bundle maps are isometric, then there is an unique (global) isometry Φ from $F^*(M)$ onto $'F^*('M)$, which maps the fixed big frame at m into the fixed big frame at $'m$. Locally, Φ coincides with the given isometric local bundle maps. (Received February 19, 1957.)

484. T. S. Klotz: *On G. Bol's proof of Carathéodory's conjecture.*

Carathéodory conjectured that there exist at least two umbilic points on every closed surface of genus zero (with continuously turning tangent plane and continuous Gaussian curvature). This conjecture has been proved twice for the case of analytic surfaces, first by H. Hamburger (see Ann. of Math. vol. 41 (1940) pp. 63–86; Acta Math. vol. 73 (1941) pp. 174–332) and next by G. Bol (see Math. Zeit. vol. 49 (1943–1944) pp. 389–410). There is a gap in Bol's paper and arguments have been found to fill the gap. The new arguments involve a careful application of lemmas already proved in Bol's paper. The corrected version of Bol's proof is still considerably shorter than Hamburger's. (Received February 19, 1957.)

485. Z. A. Melzak: *Convex hulls of a class of closed space curves.* Preliminary report.

Let C be a rectifiable closed curve or a rectifiable open arc in E_n and let $\overline{C}$ denote its convex hull. Let $L(C)$ and $V(\overline{C})$ denote, respectively, the length of C and the volume of $\overline{C}$. Let (c, n) (or (o, n)) stand for the problem of maximizing $V(\overline{C})$ subject to the condition $L(C) = \text{constant}$, for a closed curve C (or an open arc C). The problems $(c, 2n)$, $(o, 2n)$ and $(o, 2n+1)$ have been treated and solved under certain restrictions on C (Schoenberg, Acta Math., 1954; Egervary, 1949, Publ. Math. Debrecen). Here $(c, 3)$, posed by Bonnesen and Fenchel in 1934, is treated and solved under certain (rather stringent) conditions on C. By means of elementary symmetrization techniques it is shown that $(c, 3)$ is equivalent to an isoperimetric problem whose Euler-Lagrange equations are: $z' = xy$, $x'' = -xy^2$, $y'' = -yx^2$; here the independent variable is the arc-length and x, y, z are Cartesian coordinates of C. This work was supported by the Office of Naval Research. (Received February 14, 1957.)

486. D. J. Newman: *Length preserving transformations.*

An arbitrary mapping of the plane into itself is given. We prove: If this mapping preserves the distance 1, then it preserves all distances. This is a generalization of an old Putnam problem, viz.: If the mapping preserves all rational distances, then it preserves all distances. (Received January 14, 1957.)

487t. H. W. E. Schwerdtfeger: *Remarks on Moebius transformations.*

Let $Z=H(z)=(az+b)/(cz+d)$, $c\neq 0$. It is known that its fixed points and the poles α, $\beta(H(\alpha)=\infty$, $H(\infty)=\beta)$ are the pairs of opposite vertices of a parallelogram (cf. E. Jacobsthal, *Über die Klasseninvariante ähnlicher linearer Abbildungen* II, Kon. Norske Vid. Selskabs Forhandlinger vol. 26 (1953) pp. 10–15) which may degenerate into a segment. The straight lines through one of the fixed points, carrying a pair of sides of this "characteristic" parallelogram of H are mapped onto one another by H; this mapping is a perspectivity having the other fixed point as centre. Similar Moebius transformations H and SHS^{-1} possess similar characteristic parallelograms and conversely. Every transformation K similar to H can be represented in the form $K=THT^{-1}$ where $z^*=T(z)=uz+v$ is an integral Moebius transformation. This T is unique except when H is an involution in which case there are two different T. (Received February 8, 1957.)

STATISTICS AND PROBABILITY

488t. T. E. Harris: *The random functions of cosmic-ray cascades.*

Consider the formulation for electron-photon cascades in, e.g., Jánossy's *Cosmic rays*, 2d ed., Oxford, 1950, pp. 206–208. Since Jánossy's $\mathfrak{H}(w, w')$ is a function of w'/w, we write $(w'/w)\mathfrak{H}(w, w')=K(w'/w)$. We use β rather than o for the collision-loss parameter. Let $N(y, t)$ be the number of electrons of energy $>y$ at thickness t; $f_i(s, y, t)=E(s^N)$ if the initial particle of energy 1 is of type $i=1$ (photon) or 2 (electron); electrons of zero energy are not counted. (1) If $\beta>0$, then $E\,N^2(0, t)<\infty$ (it was known that $E\,N(0, t)<\infty$), but with probability 1 $N(0, t)$ reaches ∞ in every t-interval. (2) If $\beta=0$, the total energy of all electrons converges in mean square and in probability to a constant as $t\rightarrow\infty$ (it was known that the expectation converges). (3) If $\beta=0$, the f_i are shown to have the necessary smoothness properties to satisfy (λ being a certain constant) $\partial f_1(s, y, t)/\partial t=-\lambda f_1(s, y, t)+\lambda\int_0^1 f_2(s, y/u, t)f_2(s, y/(1-u), t)\cdot uK(1/u)du$, $\partial f_2(s, y, t)/\partial t=\int_0^1\{f_1(s, y/u, t)f_2(s, y/(1-u), t)-f_2(s, y, t)\}\cdot K(u)du$. Although, starting with an electron, there is no first photon emission, it is also possible to get integral equations for the f_i by a modification of the "regeneration-point" method. (Received February 19, 1957.)

489. Bayard Rankin: *Computable probability spaces.*

Let $S=(a_1, a_2, \cdots)$ be uniformly dense in the unit interval I. Let X be a real valued Riemann integrable function on I with values $X(\alpha)$, $\alpha\in I$. Define $Ef(X)=\lim (1/N)\sum_1^N f[X(a_k)]$. For real continuous f, $0\leqq f\leqq 1$, with real domain $f(X)$ is Riemann and $Ef(X)$ equals the Riemann integral of $f(X)$ over I. X restricted to the space S is defined as a random variable with sample values $X(a_k)$ and mathematical expectation EX. The random events obtainable through X are defined as $f(X)$ with corresponding probabilities $Ef(X)$ for all f given above. In this context a modified weak law of large numbers holds for any strictly stationary and metrically transitive Riemann integrable sequence $X_1, X_2, \cdots$ that is defined on I and restricted to S. Namely, for any event $f(X)$ with $f(\beta)=0$, $Ef((1/N)\sum_1^N X_k)\rightarrow 0$ for β in a neighborhood of EX_k. S can be chosen so that each a_k is computable in the Turing sense and there exist X_k satisfying the above law all of whose sample values are computable. The same is true when "strictly stationary and metrically transitive" is strengthened to "statistically independent and identically distributed." (Received February 20, 1957.)

TOPOLOGY

490. J. J. Andrews: *A characteristic of light open maps of locally Euclidean spaces into locally Euclidean spaces.*

A closed subset X of Euclidean n-space R^n is finitely nonseparating if and only if $R^n - X$ has no bounded components. A map f of R^n into R^m is pseudo-monotone if $f^{-1}(X)$ is finitely nonseparating for every finitely nonseparating set X. A map f of a locally Euclidean space X into a locally Euclidean space Y is locally pseudo-monotone if for every open set U in Y homeomorphic to $R^m (k\colon U \rightarrow R^m)$ and for every open set V in X homeomorphic to R^n $(g\colon R^n \rightarrow V)$ such that $f\colon V \rightarrow U$, then kfg is pseudo-monotone. Theorem: A light map of a locally Euclidean space into a locally Euclidean space is open if and only if f is locally pseudo-monotone. (Received March 7, 1957.)

491. T. R. Brahana: *Axioms for local homology.*

In this paper the axioms of Eilenberg and Steenrod, (Foundations of Alg. Top.), are adapted to fit local homology theory, and a uniqueness theorem for local homology on locally triangulable spaces is obtained. Local homology theory differs from homology theory in two main aspects; the local homology groups are invariants of local character, and homotopic maps do not, in general, induce the same homomorphisms of local homology groups. In outline, the development is as follows. Local categories and local homotopies are defined. The Axioms 1–5 of Eilenberg and Steenrod are used without change, 6, 7 modified slightly, and an Axiom 8 (Local) providing for local character of invariants added. A Local Simplicial Approximation Theorem is proved. In the development of homology of a simplicial pair, retraction to a unit interval replaces retraction to a point. Finally a Main Isomorphism Theorem and a Uniqueness Theorem analogous to that of Eilenberg and Steenrod, (l.c., p. 100), are proved. It is a consequence of this theorem that local homology theories such as those of van Kampen, (Thesis, Den Haag), Seifert and Threlfall, (Lehr. der Top., p. 121), Alexandroff, (Annals of Math. vol. 36, p. 1), Wilder, (Top. of Manifolds, p. 191), White, (Canadian Journal of Mathematics vol. 4, p. 329), and others, lead to the same invariants in locally triangulable spaces. (Received February 11, 1957.)

492. M. L. Curtis and M. K. Fort, Jr. (p): *Homotopy groups of one-dimensional spaces.*

Let X be a locally connected continuum whose one-dimensional integral singular homology group is a torsion group, and let S be a one-dimensional separable metric space. It is shown that any map $f\colon X \rightarrow S$ is homotopic to a constant map. From this result it follows immediately that $\pi_k(S) = 0$ for all $k > 1$. If S is also arcwise connected and $\pi_1(S)$ is a free finitely-generated group, then all singular homology groups $H_k(S, Z) = 0$, $k > 1$. (Received February 13, 1957.)

493. M. L. Curtis (p) and M. K. Fort, Jr.: *Certain subgroups of the homotopy groups.*

If $F_n(X, x_0)$ is the set of all maps of (S^n, y_0) into (X, x_0), then there is a natural map $\phi\colon F_n(X, x_0) \rightarrow \pi_n(X, x_0)$. A subset S of $F_n(X, x_0)$ is called a *G-class* if $\phi(S)$ is a subgroup of $\pi_n(X, x_0)$. A map $f\colon A \rightarrow B$ is *k-light* if $\dim f^{-1}(b) \leq k$ except for a finite number of b in B. A map $f\colon A \rightarrow B$ is *k-monotone* if $H_k(f^{-1}(b))$ is finitely generated for each b in B. The set D of k-light elements of $F_n(X, x_0)$ is a G-class and the subgroup

$\phi(D)$ is denoted by $D_n^k(X, x_0)$. The set M of k-monotone elements of $F_n(X, x_0)$ is a G-class (with, for example, Cech homology over a field) and the subgroup $\phi(M)$ is denoted by $M_n^k(X, x_0)$. There are simple examples in which these subgroups are proper, and, of course, these subgroups are not homotopy-type invariants. If X is a finite polyhedron, then $M_n^k(X, x_0) = \pi_n(X, x_0)$ for all n, k. If dim $X \leqq m$, then $D_n^k(X, x_0) = 0$ for $n > m+k$. If X is an m-manifold, then $D_n^k(X, x_0) = \pi_n(X, x_0)$ for $n \leqq m+k$. (Received February 13, 1957.)

494*t*. I. S. Gál: *On completely (m, n)-compact spaces.*

A topological space X is called (m, n)-compact if for every open cover $\{O_i\}$ $(i \in I)$ of X whose cardinality card $I \leqq n$ one can select a subcover $\{O_{i_j}\}$ $(j \in J)$ such that card $J \leqq m$. A space X is called completely (m, n)-compact if every subspace of X is (m, n)-compact. Theorem: *X is completely (m, n)-compact if and only if any one of the following statements holds in X*: (i) *Every subset $S \subseteq X$ satisfying $m \leqq$* card $S \leqq n$ *contains an m-accumulation point.* (ii) *If $S \subseteq X$ and* card $S \leqq n$ *and if there is no nonvoid subset in S which is m-dense in itself then* card $S \leqq m$. (iii) *If $\{O_i\}$ $(i \in I)$ is a well ordered increasing family of distinct open sets in X and if* card $I \leqq n$ *then* card $I \leqq m$. If the Cartesian product of infinitely many factors is completely (m, n)-compact then every subproduct of finitely many factors is completely (m, n)-compact. We have the following converse: Theorem: *Suppose that every product space of finitely many factors selected from a family $\{X_i\}$ $(i \in I)$ of topological spaces is completely (m, ∞)-compact for some m satisfying* card $I \leqq m$. *Then the topological product $X = \prod_{i \in I} X_i$ is also (m, ∞)-compact.* (Received February 20, 1957.)

495*t*. Bert Mendelson: *Homology of symmetric products of spheres.* Preliminary report.

Let ${}_kS^n$ denote the k-fold symmetric product of an n-sphere. ${}_kS^n$ is given the structure of a CW complex such that the sequences of integers $(i_1, \cdots, i_s)$ with $0 \leqq s \leqq k-1$ and $2 \leqq i_j \leqq n$ are in one-to-one correspondence with the $(n+i_1+\cdots+i_s)$-cells, denoted by $[n; i_1, \cdots, i_s]$ of ${}_kS^n$. The homology boundary of $[n; i_1, \cdots, i_s]$ is computed. If ${}_kS^n$ is embedded in ${}_{k+1}S^n$, the chain complex based on the cells $[n; i_1, \cdots, i_s]$, $0 \leqq s \leqq k$, is the direct sum of the chain complex of ${}_kS^n$ and the chain complex based on the cells $[n; i_1, \cdots, i_k]$ of $({}_{k+1}S^n, {}_kS^n)$. Thus the integral homology of ${}_{k+1}S^n$ is given by $H({}_{k+1}S^n) = H({}_kS^n) + H({}_{k+1}S^n, {}_kS^n)$ (direct), a special case of a result of Steenrod's. Let $J(k+1; n; 2r)$, $r = 1, 2, \cdots, [n/2]$ be the chain group generated by the cells $[n; i_1, \cdots, i_k]$ in which either $2r$ or $2r+1$ occurs among the set $\{i_1, \cdots, i_k\}$, but no smaller integer occurs. It is shown that $J(k+1; n; 2r)$ is a subcomplex and that the chain complex of $({}_{k+1}S^n, {}_kS^n)$ is the direct sum $\sum J(k+1; n; 2r)$, and that furthermore $J(k+1; n; 2r)$ is chain isomorphic to $J(k+1; n+2; 2(r+1))$, by an isomorphism raising dimension by $2(k+1)$. Thus, r running from 0 to $[n/2]-1$, $H({}_{k+1}S^n, {}_kS^n) \approx \sum H(J(k+1; n-2r; 2))$. An inductive procedure is devised for computing $H(J(4; n; 2))$ for all n, thus determining $H({}_4S^n, {}_3S^n)$. Also generating cycles are explicitly arrayed leading to calculations of reduced products. (Received December 13, 1956.)

496. E. A. Michael: *Continuous extensions and selections for paraconvex sets.*

Let E be a Banach space, with distance ρ. A set $A \subset E$ is called (n-point) paraconvex if there exists an $\alpha < 1/2$ such that, for any $S \subset A$ (containing $\leqq n$ points) and any $x \in \operatorname{conv} S$, $\rho(x, A) \leqq \alpha(\operatorname{diam} S)$. Examples: Every convex set is paraconvex, and so is a proper closed subarc of a semi-circle or the letters X, Y, or Z in the euclidean plane; a regular pentagon is 2-point paraconvex but not 3-point paraconvex. Theorem: If $\mathcal{S}$ is a family of nonempty, closed (n-point) paraconvex sets in E for which the same α works, and if X is a paracompact space (of Lebesgue dimension $\leqq n-1$), then every lower semi-continuous $\phi: X \rightarrow \mathcal{S}$ admits a selection. (This generalizes Theorem 3.2″ of the author's paper [Ann. of Math. vol. 63 (1956) pp. 361–382], where terminology is explained.) Corollary: If $A \subset E$ is closed and (n-point) paraconvex, then A is an extension space for all metric, and hence all collectionwise normal, spaces (of Lebesgue dimension $\leqq n-1$). (Received February 6, 1957.)

497. J. P. Roth: *On the existence of a continuum of critical points.*

Let f be a map of a Hausdorff space S into the reals such that $S_c = \{x \mid f(x) \leqq c\}$ is compact. Let u and v be nonzero elements of Čech homology groups $H_r(S)$ and $H_{r+s}(S)$, $s > 0$, with the same critical value c [Bull. Amer. Math. Soc. Abstract 60-2-331]. Let u_* be a cocycle dual to u under an orthogonal pairing [Wilder, *Topology of manifolds*, American Mathematical Society Colloquium Publications, no. 32, 1949.] If the cap product $u_* \cap v$ is not zero, then there is an s-dimensional set of critical points at level c. Corollaries are generalizations of the Lyusternik-Schnirelmann theorems of the existence of critical sets of differentiable maps of projective n-space and the n-torus. The proofs do not utilize the notion of category. (Received February 19, 1957.)

498. F. J. Wagner: *Almost homogeneous spaces.*

Let X be a topological space, and G a topological group such that, corresponding to every $g \in G$, there exists a homeomorphism (also denoted by g) satisfying the following conditions: (1) if $g, h \in G$, then $g(h(x)) = (gh)(x)$ for every $x \in X$; (2) for every $g \neq e$ of G, there exists an $x \in X$ such that $g(x) \neq x$; (3) for every $x \in X$, the set $\{g(x) \mid g \in G\}$ is dense in X. Then X is called an almost homogeneous space (relative to G). Let $X = \bigcup C(\alpha)$ be a decomposition of X into its components. For every $g \in G$, $g(C(\alpha)) = C(g(\alpha))$ is a component of X, for every α. $H = \{g \in G \mid g(\alpha) = \alpha \text{ for every } \alpha\}$ is a closed normal subgroup of G. G/H is a group of permutations of the α's. $N(\alpha) = \{g \in H \mid g(x) = x \text{ for every } x \in C(\alpha)\}$ is a closed normal subgroup of H, and H is isomorphic to a subgroup of the direct product of the groups $H(\alpha) = H/N(\alpha)$. If for every α, α' there is a $g \in G$ such that $g(\alpha) = \alpha'$ (e.g. if X is locally connected), then all the components $C(\alpha)$ are of course homeomorphic and all the groups $H(\alpha)$ are isomorphic. (Received February 21, 1957.)

499. G. W. Whitehead: *Group extensions defined by cohomology classes.*

Let X be a free graded chain complex. Then each $u \in H^q(X; G)$ defines canonically a homomorphism $u': H_q(X) \rightarrow G$, as well as an element $u'' \in \operatorname{Ext}(H_{q-1}(X), \operatorname{Cok} u')$. If (X, A) is a pair, $i_q: H_q(A) \rightarrow H_q(X)$ is the injection, and $u \in H^q(X, H_q(X))$ is an element such that $u' =$ identity, then the extension $0 \rightarrow \operatorname{Cok} i_q \rightarrow H_q(X, A) \rightarrow \operatorname{Ker} i_{q-1} \rightarrow 0$ is determined by the injection into $\operatorname{Ext}(\operatorname{Ker} i_{q-1}, \operatorname{Cok} i_q)$ of the element $(i_q{}^* u)''$. Let X be an $(n-1)$-connected space, let $u \in H^n(X, \pi_n(X))$ be the first obstruction to contracting X to a point, and let $v \in H^{n+2}(X; \Gamma_{n+1}(X))$ be the Pontryagin (Steenrod) square of u if $n = 2$ $(n > 2)$; then, in the exact sequence $H_{n+2}(X) \xrightarrow{b} \Gamma_{n+1}(X) \rightarrow \pi_{n+1}(X)$

$\rightarrow H_{n+1}(X)\rightarrow 0$ of J. H. C. Whitehead, we have $v' = b$, while the extension $0\rightarrow \text{Cok } b \rightarrow \pi_{n+1}(X)\rightarrow H_{n+1}(X)\rightarrow 0$ is defined by v''. Let X be a space with just two nonvanishing homotopy groups; then a formal description of the homology groups of X in the stable range can be given in terms of u', u'', where u ranges over a set of cohomology classes which can be expressed by the action of iterated Steenrod operations on the Postnokov invariant of X. (Received March 28, 1957.)

R. D. SCHAFER,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The five hundred and thirty-fourth meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 19–20, 1957. Sessions began at 10:00 A.M. on Friday and the Meeting concluded with a session on analysis and two special sessions for late papers at 10:00 A.M. on Saturday. There was a total of 238 registrations. Among these were 205 members of the Society.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited Professor René Thom of the University of Chicago and the University of Strasbourg to address the Society. Professor Thom spoke on the topic *Topology of differentiable function spaces*. Professor E. H. Spanier presided at the session.

The Council of the Society met at 5 P.M. on Friday, April 19.

The Secretary announced the election of the following two hundred seventy-three persons to ordinary membership in the Society:

Mr. S. O. Albert, United States Army;
Mr. W. O. Alexander, Jr., University of Corpus Christi;
Mr. J. H. Ames, Bankers Life Insurance Company of Nebraska;
Mr. T. L. Anderson, Federal Life Insurance Company, Chicago, Illinois;
Mr. A. L. Armstead, National Advisory Committee on Aeronautics, Cleveland, Ohio;
Mr. G. L. Armstrong, Convair, San Diego, California;
Miss Thelma I. Arnette, Oak Ridge National Laboratory;
Mr. J. R. Ashcraft, Remington Rand UNIVAC Corp., St. Paul, Minnesota;
Mr. Samuel Asofsky, National Jewish Welfare Board, New York, New York;
Mrs. Joy H. Atkin, Bell Telephone Laboratories, Inc., Murray Hill, N. J.;
Mr. R. W. Bains, United Gas Research Laboratory, Shreveport, Louisiana;
Dr. J. G. Baker, Harvard College Observatory;
Mr. Edward Barber, Johns Hopkins University;
Mr. C. W. Barnes, University of North Carolina;
Mr. Alexander Basil, Airborne Instruments Laboratory, Mineola, N. Y.;
Dr. Frances B. Bauer, Reeves Instrument Corp., New York, N. Y.;
Mr. L. D. Baumert, Douglas Aircraft Company, Long Beach, California;
Dr. V. E. Benes, Bell Telephone Laboratories, Inc., Murray Hill, New Jersey;
Mr. M. C. Benson, Radio Corporation of America, Camden, New Jersey;
Professor W. H. Benson, Dickinson College;
Mr. R. D. Berlin, General Electric Co., Syracuse, N. Y.;
M. A. Bershad, U. S. Bureau of the Census, Washington, D. C.;
Miss Barbara B. Betts, D. C. Heath and Company, Boston, Mass.;
Mr. F. T. Birtel, USN Nuclear Power School, New London, Conn.;
Mr. D. L. Blackhurst, Mellon National Bank & Trust Co., Pittsburgh, Pennsylvania;
Professor C. R. Blyth, University of Illinois;
Dr. B. P. Bogert, Bell Telephone Laboratories, Inc., Murray Hill, N. J.;
Dr. D. M. Boodman, Operations Evaluation Group, Massachusetts Institute of Technology, Navy Dept., Washington, D. C.;

Mr. J. A. Bond, Texas Technological College;
Mrs. Janez Y. Bordeaux, Ramo-Wooldridge Corporation, Los Angeles, California;
Mr. T. A. Bordeaux, G. O. Noville and Associates, Santa Monica, California;
Mr. H. A. Bott, Minneapolis-Honeywell Regulator Co., Morton Grove, Ill.;
Mr. R. L. Boyell, Ramo-Wooldridge Corporation, Los Angeles, California;
Mr. S. R. Boyle, Hughes Aircraft Company, Culver City, California;
Professor K. H. Bracewell, Hamline University;
Mr. T. F. Bridgland, Jr., University of Florida;
Mr. R. O. Brooks, Northeastern University;
Mr. Andrew Browder, Massachusetts Institute of Technology;
Mr. W. H. Burgin, Jr., Dartmouth College;
Mr. J. F. Burke, Rochester, New York;
Professor Emeritus G. P. Bush, American University;
Mr. T. W. Chellis, North American Aviation, Bellflower, California;
Miss Ermine A. Christian, Naval Ordnance Laboratory, Silver Spring, Maryland;
Mr. Chia-kun Chu, Institute of Mathematical Sciences, New York University;
Mr. C. J. Cillay, Alamo, Texas;
Mr. M. L. Clinnick, University of California Radiation Laboratory, Livermore, California;
Mr. R. G. Clow, University of California, Berkeley;
Professor R. W. Clower, State College of Washington;
Mr. J. E. Coachman, Prudential Insurance Company of America, Newark, New Jersey;
Col. H. M. Cochran, U. S. A., Retired, Atherton, California;
Mr. W. J. Cody, Jr., University of Oklahoma;
Mr. E. L. Cohen, Harvard University;
Mr. R. C. Collmer, Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts;
Dr. J. M. Cook, Argonne National Laboratory, Lemont, Illinois;
Miss Doris F. Corley, Los Alamos Scientific Laboratory, University of California;
Mr. W. A. Cowan, University of Adelaide;
Mr. C. W. Cox, American Bosch Arma Corp., Garden City, N. Y.;
Mr. J. R. Cox, Robert W. Lowry, Inc., Harrisburg, Pennsylvania;
Mr. G. E. Crane, Westinghouse Electric Corp., Pittsburgh, Pennsylvania;
Mr. W. C. Crehl, Holloman Air Development Center, Holloman Air Force Base, New Mexico;
Mr. D. F. Criley, International Business Machines Corp., Glendale, California;
Mr. W. S. Currie, The Texas Company, Bellaire, Texas;
Mr. Vassili Daiev, Sea Cliff, New York;
Dr. R. H. Davis, Bell Telephone Laboratories, Inc., New York, New York;
Mr. J. D. Dickinson, Chance Vaught Aircraft, Incorporated, Dallas, Texas;
Mr. R. E. Dowd, Grumman Aircraft Engineering Corp., Bethpage, New York;
Professor B. S. Dreben, Harvard University;
Mr. C. D. Dunn, Convair, Fort Worth, Texas;
Mr. E. S. Eby, University of Illinois;
Miss Ruth B. Eddy, D. C. Heath and Company, Boston, Massachusetts;
Mr. E. A. Enrione, University of Miami, Miami, Florida;
Mr. H. M. Farkas, Brown University;
Mr. M. H. Farrant, Coates, Herfurth & England, San Francisco, California;
Mr. C. T. Fike, University of North Carolina;

Dr. G. B. Findley, U. S. Navy Mine Defense Laboratory, Panama City, Florida;
Mr. R. J. Fiore, Bethlehem Steel Company, Lackawanna, New York;
Professor F. B. Fitch, Yale University;
Mr. H. E. Flesner, North American Aviation, Inc., Downey, California;
Mr. J. F. Foley, Baltimore Gas and Electric Company, Baltimore, Md.;
Mr. J. M. Frank, Convair, Pomona, California;
1st Lt. D. A. Franks, United States Army;
Mr. J. R. Franks, U. S. Naval Air Missile Test Center, Point Mugu, California;
Mr. F. J. Gagliuso, Paul Revere Life Insurance Co., Worcester, Massachusetts;
Mr. K. S. Gale, Armour Research Foundation, Chicago, Illinois;
Miss Rheba E. Galloway, Arlington County Public Schools, Arlington, Virginia;
Dr. J. L. Gammel, Los Alamos Scientific Laboratory, Los Alamos, New Mexico;
Professor S. E. Ganis, Ohio Wesleyan University;
Dr. J. J. Gilvarry, Allis-Chalmers Co., Milwaukee, Wisconsin;
Mr. W. G. Goeller, New York Telephone Co., New York, New York;
Mr. Barry Gordon, Equitable Life Assurance Society, New York, New York;
Dr. M. J. Gottlieb, Market Facts, Inc., Chicago, Illinois;
Mr. H. W. Gould, University of Virginia;
Mr. W. H. Gregory, Sacramento Junior College;
Mr. E. D. P. Gross, Jr., The Underwood Corp., New York, New York;
Mr. Nathaniel Grossman, California Institute of Technology;
Dr. A. A. Groth, A. S. Hansen Consulting Actuaries, Chicago, Illinois;
Mr. J. S. Guastavino, Remington Rand Div., Sperry Rand Corp., New York, New York;
Mr. C. A. Haase, Nelson and Warren, Kansas City, Missouri;
Reverend M. A. Hanhauser, Siena College;
Mr. F. L. Hardy, Emory University;
Professor M. L. Harris, Philander Smith College;
Professor C. H. Heinke, Capital University;
Mr. J. J. Henrick, Convair, Ft. Worth, Texas;
Mr. Bert Henry, NACA, Lewis Flight Propulsion Laboratory, Cleveland, Ohio;
Professor A. F. Herbst, La Verne College;
Mr. P. R. Hildebrandt, RAND Corporation, Lexington, Massachusetts;
Mr. J. V. Holberton, U. S. Naval Bureau of Ships, Washington, D. C.;
Mrs. Dorothy B. Horning, Standard Oil Company, Cleveland, Ohio;
Mr. R. R. Hudson, International Business Machines Corp., Oklahoma City, Oklahoma;
Dr. J. L. Hult, RAND Corporation, Santa Monica, California;
Mr. R. M. Humphrey, Presto Recording Corporation, Paramus, New Jersey;
Miss Martha A. Jennison, U. S. Naval Proving Ground, Dahlgren, Virginia;
Dr. M. V. Johns, Jr., Stanford University;
Mr. M. S. Johnson, Melpar, Incorporated, Falls Church, Virginia;
Dr. E. E. Jones, Minneapolis-Honeywell Research Center, Hopkins, Minn.;
Mr. W. D. Kahn, U. S. Army Map Service, Washington, D. C.;
Dr. Jerome Karle, Naval Research Laboratory, Washington, D. C.;
Mr. Robert Katz, Tufts University;
Mr. Ellwood Kauffman, Electronic Associates, Inc., Princeton, New Jersey;
Mr. B. S. Kawar, Remington Rand UNIVAC, St. Paul, Minnesota;
Mr. R. C. Keating, Arthur Stedry Hansen Consulting Actuaries, Lake Bluff, Illinois;
Lt. Col. L. N. King, U. S. Marine Corps, Washington, D. C.;

Mr. E. L. Kirkpatrick Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland;
Mr. T. H. Kirkpatrick, Paul Revere Life Insurance Company, Worcester, Massachusetts;
Mr. B. S. Kleinman, Columbia University;
Mrs. Mary E. Knight, U. S. Naval Aviation Supply Office, Philadelphia, Pennsylvania;
Dr. E. C. Koenig, Allis-Chalmers Manufacturing Company, Milwaukee, Wisconsin;
Mr. H. W. Kreiner, Operations Evaluation Group, Massachusetts Institute of Technology, Alexandria, Virginia;
Mr. A. M. Kunis, Mount Vernon Life Insurance Company of New York, Mount Vernon, New York;
Mr. Ali Kyrala, Lessells and Associates, Inc., Boston, Massachusetts;
Mr. J. B. Lackey, United States Army;
Assistant Professor Emeritus Nellie T. Landblom, Colorado Agricultural and Mechanical College;
Mr. Harold Levenstein, The W. L. Maxson Corp., New York, New York;
Mr. D. M. Levy, S. Levy and Sons, Suffolk, Virginia;
Mr. J. E. Levy, Washington Engineering Services Company, Bethesda, Maryland;
Mr. Victor Lewicke, U. S. Army Map Service, Washington, D. C.;
Mr. T. L. Libby, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio;
Mr. O. A. de Lima, Roger Smith Hotels Corporation, New York, New York;
Mr. G. E. Lindquist, United Air Lines, Chicago, Illinois;
Mr. Leo Liolios, Bastian-Blessing Company, Chicago, Illinois;
Professor J. M. Long, College of William and Mary, Norfolk, Virginia;
Mr. W. M. Lowney, University of Notre Dame;
Lt. R. E. Lynch, U. S. Air Force, Wright Patterson Air Force Base, Dayton, Ohio;
Mr. J. F. MacLean, Bankers Life Insurance Company of Nebraska, Lincoln, Nebraska;
Captain A. T. Magnell, United States Navy;
Professor J. G. Marica, Humboldt State College;
Dr. Sidney Marks, General Electric Company, Richland, Washington;
Dr. E. L. Marshall, Lafayette Life Insurance Company, Lafayette, Indiana;
Mr. N. F. G. Martin, Iowa State College;
Mr. Johann Martinek, Reed Research Incorporated, Washington, D. C.;
Mr. R. M. Mason, Naval Research Laboratory, Washington, D. C.;
Mr. W. C. Mason, Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts;
Mr. M. D. McCord, University of Tennessee;
Mr. J. S. McCoy, Metropolitan Life Insurance Company, New York, New York;
Mr. R. E. McCrocklin, Terre Haute, Indiana;
Mr. D. C. McCune, Jones and Laughlin Steel Corporation, Pittsburgh, Pennsylvania;
Mr. R. B. McNab, Railroad Retirement Board, Chicago, Illinois;
Mr. J. C. McPherson, International Business Machines Corporation, New York, New York;
Dr. V. V. McRae, Johns Hopkins University;
Mr. E. A. Means, Means Laboratories, Incorporated, Wichita, Kansas;
Mr. Harold Mechanic, Nuclear Development Corporation, White Plains, New York;
Mr. R. L. Meyer, Jr., UARCO, Incorporated, Chicago, Illinois;
Mr. J. R. Miller, Ramseyer & Miller, Incorporated, New York, New York;

Mr. R. A. Moore, Northrop Aircraft, Incorporated, Hawthorne, California;
Mr. R. E. Moore, Missile Systems Div., Lockheed Aircraft Corp., Palo Alto, California;
Mr. C. S. Morell, American Machine & Foundry Company, Boston, Massachusetts;
Mr. H. D. Morgan, Eugene M. Klein and Associates, Cleveland, Ohio;
Mr. W. S. Morris, Talmage and Company, New York, New York;
Professor D. E. Moser, University of Massachusetts;
Mr. R. A. Mugele, Shell Development Company, Emeryville, California;
Mr. J. R. Muller, Sperry Rand Corporation, Long Island, New York;
Mr. A. A. Mullin, Massachusetts Institute of Technology;
Mr. A. E. Murray, Bausch and Lomb Optical Company, Rochester, New York;
Professor Peter Musen, University of Cincinnati;
Mr. G. F. Nardin, Jr., Armco Steel Corporation, Middletown, Ohio;
Mr. J. O. Neilson, RAND Corporation, Lexington, Massachusetts;
Professor R. J. Nelson, Case Institute of Technology;
Dr. Anil Nerode, University of Chicago;
Mr. P. E. Ney, Columbia University;
Professor E. D. Nichols, Florida State University;
Mr. L. F. Nichols, Picatinny Arsenal, Dover, New Jersey;
Mr. R. J. O'Keefe, Arthur D. Little, Inc., Cambridge, Massachusetts;
Dr. R. D. O'Neal, Bendix Aviation Corporation, Ann Arbor, Michigan;
Dr. R. K. Osborn, Oak Ridge National Laboratory;
Mr. W. L. Ostrowski, American Chemical Society, Washington, D. C.;
Mr. R. W. Paul, Jr., University of California, Radiation Laboratory, Livermore, California;
Mr. C. R. Paulson, ERCO Division, ACF Industries, Incorporated, Riverdale, Maryland;
Mr. Sidney Penner, Illinois Institute of Technology;
Mr. F. S. Perryman, Royal Insurance Company, Ltd., New York, New York;
Mr. B. I. Pesin, Los Angeles City Schools, North Hollywood, California;
Mr. C. F. Pinzka, Xavier University;
Mr. M. F. Pollack, Stockton, California;
Mr. K. F. Powell, International Business Machines Corp., Cleveland, Ohio;
Professor Carlo Pucci, University of Maryland;
Professor E. K. Rabe, University of Cincinnati Observatory;
Mr. George Rabinowitz, New York University;
Dr. J. H. Ramser, Atlantic Refining Company, Philadelphia, Pennsylvania;
Dr. R. B. Randels, Corning Glass Works, Corning, New York;
Professor Philburn Ratoosh, Ohio State University;
Mr. G. A. Reilly, DATA Processing, San Antonio, Texas;
Mrs. Dorothea J. Rhea, Bryn Mawr College;
Pvt. R. A. Rieger, United States Army, Fort Gordon, Georgia;
Dr. G. M. Roe, General Electric Research Laboratory, Schenectady, New York;
Mr. Azriel Rosenfeld, Ford Instrument Company, New York, New York;
Mr. Frank Rosett, Vickers, Incorporated, Detroit, Michigan;
Mr. D. J. Ross, Operations Research, Incorporated, Silver Spring, Maryland;
Mr. David Rothman, Rocketdyne, North American Aviation, Canoga Park, California;
Mr. R. E. Rowley, General Electric Company, Richland, Washington;
Reverend B. M. Russell, Gannon College;

Mr. A. A. Sagle, University of Washington;
Mr. Daihachiro Sato, University of California, Los Angeles;
Mr. Murray Sawits, Teleregister Corporation, Stamford, Connecticut;
Lt. Col. C. J. Schauers, 6th Army Signal Section, San Francisco, California;
Mr. I. S. Schechtman, Philco Corporation, Philadelphia, Pennsylvania;
Mr. E. D. Schell, Remington Rand UNIVAC Corporation, New York, New York;
Mr. H. E. Schnur, Texas Company, Houston, Texas;
Mr. R. L. Schwaller, Marquette University;
Mr. A. A. Schwartz, U. S. Railroad Retirement Board, New York, New York;
Mr. S. J. Scott, Vitro Corporation of America, Eglin Air Force Base, Florida;
Mr. J. F. Scott-Thomas, University of Toronto;
Mr. A. N. Seares, Remington Rand, New York, New York;
Professor C. F. Sebesta, Duquesne University;
Mr. L. A. Segel, Massachusetts Institute of Technology;
Mr. H. Manvel Semarne, Douglas Aircraft Company, Inc., Santa Monica, California;
Mr. Robert Serrell, Radio Corporation of America, RCA Laboratories, Princeton, New Jersey;
Mr. Harry Sherman, Reeves Instrument Corp., New York, New York;
Mr. T. R. Sherrow, Sinclair Oil and Gas Company, Tulsa, Oklahoma;
Dr. R. L. Shuey, General Electric Company, Schenectady, New York;
Mr. R. W. Sielaff, Northern Trust Company, Chicago, Illinois;
Mr. R. L. Sisson, Canning, Sisson and Associates, Los Angeles, California;
Mr. T. D. Sloan, New York Life Insurance Company, New York, New York;
Mr. H. L. Slonecker, Jr., North American Aviation, Incorporated, Columbus, Ohio;
Mr. C. A. Smith, City Schools, St. Louis, Missouri;
Miss Jean F. Smolak, E. R. Squibb and Sons, New Brunswick, New Jersey;
Dr. A. P. Speiser, International Business Machines Research Laboratory, Adliswil-Zurich, Switzerland;
Mr. R. F. Steinhart, International Business Machines Corporation, Newark, New Jersey;
Miss Fern G. Stenwick, David Taylor Model Basin, Carderock, Maryland;
Mr. W. F. Stephan, International Business Machines Corporation, Kingston, New York;
Mr. Erick Stern, ARMA Corporation, Garden City, New York;
Mr. A. I. Sternhell, Metropolitan Life Insurance Co., New York, New York;
Dr. S. A. Stone, Bradford Durfee Technical Institute;
Mr. A. J. Strecok, Argonne National Laboratory, Lemont, Illinois;
Mr. D. A. Swick, U. S. Naval Research Laboratory, Washington, D. C.;
Professor W. B. Swift, University of Wisconsin;
Mr. Irvin Tarnove, Ramo-Wooldridge Corporation, Los Angeles, California;
Mr. J. R. Teller, Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland;
Miss Eileen J. Theisen, North American Aviation, Incorporated, Canoga Park, California;
Mr. A. T. Thomas, Brown-Forman Distillers Corporation, Louisville, Kentucky;
Mr. J. D. Thomas, University of Oklahoma;
Mr. R. E. Thomas, North American Aviation, Inc., Columbus, Ohio;
Mr. D. G. Thoroman, International Business Machines Corporation, Dayton, Ohio;
Mr. J. K. Thurber, New York University;
Mr. D. B. J. Tomiuk, Catholic University of America;

Professor Cosimo Torre, Syracuse University;
Mr. A. L. Tritter, Lincoln Laboratory, Massachusetts Institute of Technology;
Mr. R. E. Tuck, Hughes Aircraft Company, Los Angeles, California;
Dr. Jaroslav Tuzar, Salerno-Megowen Biscuit Company, Chicago, Illinois;
Mr. S. A. Tyler, Argonne National Laboratory, Lemont, Illinois;
Mr. J. B. Tysver, University of Michigan;
Miss Theresa D. Vasques, Convair, San Diego, California;
Mr. J. B. Vieaux, Agricultural and Mechanical College of Texas;
Mr. R. M. Vogt, University of Illinois;
Mr. F. J. Wall, Sperry Rand Corporation, St. Paul, Minnesota;
Mr. R. C. Warth, Bendix Aviation Corporation, Eatontown, New Jersey;
Mr. Frederick Way, III, Case Institute of Technology;
Mr. M. R. Weinstein, Massachusetts Institute of Technology;
Mr. Burton Wendroff, Los Alamos Scientific Laboratory, University of California;
Mr. O. J. White, Ford Motor Company, Dearborn, Michigan;
Professor F. L. Wolf, Carleton College;
Mr. A. W. Yonda, Temple University;
Mr. P.A. Zaphyr, Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania;
Mr. E. C. Zelger, Guardian Life Insurance Company of America, New York, New York;
Mr. L. W. A. Zolldan, Jr., U. S. Naval Ordnance Laboratory, White Oak, Maryland.

It was reported that the following forty-six persons had been elected to membership on nomination of institutional members as indicated:

University of California, Los Angeles: Mr. R. J. Mercer, Mr. F. C. Reed, Mr. E. M. Scheuer.

University of Chicago: Mr. R. H. Szczarba.

Illinois Institute of Technology: Mr. R. C. Bzoch.

Johns Hopkins University: Mr. P. M. Anselone, Mr. Paul Berliner, Mr. J. H. De Boer, Mr. J. P. Falvey, Mrs. Nancy J. Hagelgans, Professor J. R. Hammond, Mr. J. T. Hundley, Mr. D. L. McQuillan, Mr. Richard Molloy, Professor N. O. Niles, Miss Marilyn J. Pettit, Mr. R. C. Sacksteder, Mr. B. C. Stebbings, Mr. C. E. Thompson, Mr. D. R. Tilley.

University of Miami: Mr. N. W. Hill, Jr., Miss Mabel A. Pauley.

Michigan State University: Mr. D. G. Johnson, Mr. J. R. Smart, Mr. W. W. Turner.

New York University: Mrs. Tilla S. Klotz.

Princeton University: Mr. A. M. Adelberg, Mr. M. L. Balinski, Mr. P. X. Gallagher, Mr. M. D. George, Mr. S. C. H. Gitler, Mr. R. A. Kurtz, Mr. James McKenna, Mr. B. C. Mazur, Mr. C. St. J. A. Nash-Williams, Mr. Kenichi Shiraiwa, Mr. R. M. Smullyan, Mr. J. R. Stallings, Mr. J. D. Stasheff, Mr. Nobuo Yoneda, Mr. R. O. Winder, Mr. Martin Zerner.

Stanford University: Mr. W. J. Buckingham, Mr. J. R. Hatcher, Dr. Eva M. Wirth.

Washington University: Mr. R. L. Pratt.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker-Vereinigung:

Professor W. M. Meyer-Koenig; London Mathematical Society: Professor E. T. Davies; University of Southampton, England, Professor G. B. Preston: Schweizerische Mathematische Gesellschaft: Dr. Walter Gautschi; Unione Mathematica Italiana: Mr. Tristano Manacarda.

Montana State University was elected to Institutional Membership and the Sun Oil Company and International Business Machines Corporation were elected to Corporate Membership in the Society.

The Secretary announced that by mail votes Professors W. S. Massey and A. H. Taub had been elected to the Executive Committee; a Summer Institute in 1958 on the topic, *Surface area and related topics* had been approved, the Invitations Committee for this Institute to be comprised of Tibor Rado, Chairman, L. C. Young, L. Cesari, Herbert Federer, and J. W. T. Youngs; the President was authorized to appoint two delegates of the Society to the constitutional convention of a proposed Institute of Mathematics. The delegates appointed were Professors A. E. Meder, Jr., and Saunders MacLane.

The following Presidential appointments were reported: As an arrangements Committee for the April Meeting in New York: S. C. Lowell, Chairman, J. W. John, P. D. Lax, R. D. Schafer, H. E. Wahlert, G. B. Whitham. As an Arrangements Committee for the Summer Meeting at Pennsylvania State University: Evan Johnson, Jr., Chairman, J. B. Bartoo, Orrin Frink, W. O. Gordon, H. L. Krall, H. M. Gehman, R. D. Schafer. As an Arrangements Committee for the Annual Meeting in Cincinnati: I. A. Barnett, Chairman, J. L. Baker, Arno Jaeger, H. T. LaBorde, H. D. Lipsich, C. I. Lubin, G. M. Merriman, W. E. Restemeyer, H. M. Gehman, J. W. T. Youngs. As a Committee to Nominate Candidates for the Executive Committee: E. G. Begle, J. W. Green, Saunders MacLane. To the Committee on Places of Meetings of the Association and Society: R. D. Schafer. To the Committee to Organize Summer Institutes: A. M. Gleason, Ralph Phillips. To the Committee on Applied Mathematics: R. S. Burington, R. C. Bellman. As Tellers for the 1956 election: David Gale, W. G. Lister, J. F. Randolph.

The President had appointed the following as delegates of the Society: Professor Mary V. Sunseri, at the Centennial Year Convocation of San Jose State College; Professor B. H. Bissinger, at the inauguration of F. de Wolfe Bolman, Jr. as President of Franklin and Marshall College; Professor J. H. Roberts at the inauguration of President William C. Friday of the University of North Carolina; Professor W. H. Gottschalk, at the Centennial Convention of the

National Educational Association; Professor K. C. Schraut, at the one-hundredth Anniversary of the founding of Wilberforce University; Professor W. L. Hart, at the inauguration of the Very Reverend James P. Shannon at St. Thomas College, St. Paul, Minnesota; Professor Harriet Griffin, at the inauguration of Sister Vincent Therese Tuohy as President of St. Joseph's College in Brooklyn; Professor W. C. McDaniel, at the installation of the president at Southeast Missouri State College.

The Secretary reported that the following persons have accepted invitations to deliver hour addresses: A. L. Whitman, Berkeley, California, April 20; René Thom, University of Chicago, April 19–20; Mahlon M. Day, State College of Washington, June 15; Harold Levine, University of California, Los Angeles, November 15–16; A. P. Calderon and M. Rosenlicht, at Pennsylvania State University, August 26–30.

The Council set meetings at the University of California, Los Angeles, November 15–16, 1957; at Stanford University April 18, 1958; at Chicago April 18–19, 1958; and the Summer Meeting at Massachusetts Institute of Technology August 25–30, 1958.

The Council discharged the following committees: Committee on the relationship between the Headquarters and the Mathematical Reviews; Committee to Look into the Relations between the Transactions and Proceedings; Committee to Survey the Society's Program of Periodical Publication. The President was authorized to appoint a committee to select the Gibbs Lecturer for 1958 and 1959 and a committee to award the Bôcher Prize for 1958.

The Council passed the following resolution:

Whereas, on January 1, 1951, Professor Edward G. Begle assumed the office of Secretary of the American Mathematical Society, and Whereas at that time the duties of the Secretary had attained an unprecedented importance due to the growth in Society membership and other factors, and Whereas Professor Begle performed the duties of Secretary faithfully and devotedly thereafter for a six-year period, during a time when many crises, financial, administrative, and other, were faced by the Society, and Whereas the Society is deeply indebted to Professor Begle for the unswerving loyalty and excellent judgment which he displayed during his term of office, Be it therefore resolved that the Council hereby express its profound gratitude to Professor Begle for his services to American mathematics through his work as Secretary of the American Mathematical Society, and Be it further resolved that copies of this resolution be sent to Professor Begle, spread upon the minutes of the Council, and appropriately published by the Society.

The Council approved an amendment to the by-laws setting the minimum annual dues of corporate members at One Thousand Dollars. The Council also approved a number of amendments to the by-laws having to do with the election of officers and members of the Council and the dates of Annual Meetings and meetings of the Board of Trustees. These amendments are made necessary by the fact that the next Annual Meeting will be in January, instead of at Christmas.

The recommendation of the Applied Mathematics Committee that a symposium on Applied Mathematics be held in the spring of 1958 in conjunction with a meeting of the Society was approved. The subject of this symposium will be *Combinatorial designs and analysis.*

The Council voted to approve the recommendation of the Colloquium Editorial Committee that Volume 8, *Non-Riemannian geometry* by L. P. Eisenhart be reprinted. It also voted to recommend to the Board of Trustees that in view of the large backlog in the Transactions, as many extra pages as they think feasible be published in the last volume of 1957.

On Saturday at 9:00 A.M. there was a meeting of representatives of the American Mathematical Society, Association for Symbolic Logic, Institute of Mathematical Statistics, Mathematical Association of America, National Council of Teachers of Mathematics and the Society for Industrial and Applied Mathematics.

The ladies of the Department of Mathematics entertained the Society at a tea on Friday afternoon.

In all there were six sessions for the presentation of contributed papers. Presiding Officers were Professors Ewing, MacLane, Roberts, Mickle, Herstein, and Randolph.

Abstracts of the contributed papers follow. Those with "*t*" after the abstract number were presented by title. In the case of joint papers, the name of the author who read the paper is followed by (p). Dr. Heinz Renggle was introduced by Professor B. J. Pettis.

ALGEBRA AND THEORY OF NUMBERS

500*t*. C. W. Curtis: *Modules whose annihilators are direct summands.*

Let $B=\Delta(G, \rho)$ be a ring which is a crossed product of a finite group G and a division ring Δ with factor set ρ. In particular B may be the group algebra of a finite group with coefficients in a field. Then B has a basis over Δ consisting of elements $b_1, b_2, \cdots$ in (1-1) correspondence with the elements of G such that $b_s b_t = b_{st}\rho_{s,t}$ and $\xi b_s = b_s \xi^s$ for all $s, t \in G$, $\xi \in \Delta$, where the mappings $\xi \rightarrow \xi^s$ are automorphisms of Δ. Let M be a right B-module, and let M' be the space of all Δ-linear functions on M. Let B_M be the set of all finite sums of the elements $\sum_{s\in G} \psi(xb_s)b_s^{-1}$ where $x \in M$, $\psi \in M'$. Then B_M is a two-sided ideal in B, and M is called a regular B-module if B_M contains a two sided identity element. Let $e_1B, \cdots, e_sB$ be the block ideals in B, where the e_i are central primitive idempotents. A reduced block component of B is the direct sum of a full set of nonisomorphic indecomposable right ideals eB, $e^2 = e$, belonging to a fixed block. Then the following statements are equivalent: (i) M is a regular module; (ii) $B = B_M \oplus (0:M)$, where $(0:M)$ is the annihilator of M; and (iii) $Me_i \neq 0$ implies that Me_i contains the ith reduced block component of B as a direct summand. This work is an extension of the author's previous investigation (Canadian J. Math. vol. 8 (1956) pp. 271–292.) (Received February 21, 1957.)

501. W. E. Deskins: *On the radical of a modular group algebra.*

Let G be a group of order g, p a prime divisor of g, F a field of characteristic p, and $F(G)$ the group algebra of G over F. If P is a Sylow p-subgroup of G, let L be the left ideal of $F(G)$ generated by the set of all elements of the form $P_i - P_j$. Define I to be the ideal $\bigcap GLG^{-1}$ for all G in G. Then I is nilpotent and Lombardo-Radici have proved (Rend. Sem. Mat. Roma vol. 3 (1939) pp. 239–256) that I is the radical of $F(G)$ if P is unique or if $g = pq$, q a prime. These results are extended here to more general classes of groups; for example: Theorem. *If G is a super-solvable group then I is the radical of* $F(G)$. If G contains an invariant p-subgroup then it is easy to show that $F(G)$ is bound to its radical, so the following question is raised: If $F(G)$ is a bound algebra, does G contain an invariant p-subgroup? A partial solution is provided by the following: Theorem. *If I is the radical of $F(G)$, if P is cyclic, and if $F(G)$ is a bound algebra, then G contains an invariant p-subgroup.* (Received February 20, 1957.)

502. Philip Dwinger: *Complete homomorphisms of complete lattices.*

A homomorphism α of a lattice L (all lattices in this paper are complete) is complete, if it preserves joins and meets of arbitrary subsets. Equivalent to this is the condition, that the congruence relation θ generated by α is complete, i.e. it enjoys the substitution property for infinite joins and meets. The complete congruence relations of L form a complete lattice $C^*[L]$ which is a $\cap\cup$-sublattice of the lattice $C[L]$ of all congruence relations of L. A necessary condition that θ is complete is the condition, that the residue classes of L modulo θ are closed intervals of L. Whether this condition is sufficient is answered affirmatively by the following theorem: A congruence relation θ is complete if and only if all residue classes of L modulo θ are closed intervals. If L is relatively complemented then this condition can be weakened to the condition that the ideal $\{x \mid x \equiv O(\theta)\}$ is a principal ideal of L. In this case $C^*[L]$ is a complete Boolean algebra and isomorphic to a $\cap\cup$-sublattice of L (see also Dilworth, Ann. of Math. vol. 51 (1950) pp. 350–359). If A is a complete Boolean algebra then $C^*[A]$ is a complete $\cap\cup$-sublattice of $C[A]$ and isomorphic to A. (Received February 20, 1957.)

503. C. C. Faith: *Normal extensions in which every element with nonzero trace is a normal basis element.* II.

If K/F is a normal extension, then $u \in K$ is a *completely basic element* (CBE for brevity) *of* K/F provided u generates a normal basis of K/Δ, for each inter-field Δ, $K \geq \Delta \geq F$. The author has shown previously that K/F always possesses a CBE when F is infinite, and that an extensive class $\mathfrak{C}$ exists consisting of normal extensions with the property that every normal basis element is a CBE (Bull. Amer. Math. Soc. Abstract 62-4-403). The existence of a CBE in an arbitrary normal extension is now reduced to that of Q/F, where F is finite of characteristic p, and Q/F is cyclic of deg q^e, q a prime $\neq p$. This is achieved with the aid of (1) a general lemma which asserts if

$K=K_1X\cdots XK_t$ over F, and if u_i is a CBE of K_i/F, $i=1,\cdots,t$, that $u=\prod_{i=1}^t u_i$ is a CBE of K/F, and (2) the knowledge that $P/F\in\mathfrak{C}$, when P/F has deg p^e, and F has characteristic p (Bull. Amer. Math. Soc. Abstract 63-2-162). Let N/F be an abelian extension, and let ζ be a primitive r root of unity in a field containing N, where r is the l.c.m. of the orders of the automorphisms of N/F. It is known (Abstract 62-4-403 loc. cit.) that $N/F\in\mathfrak{C}$ when $N\wedge F(\zeta)=F$. This result is now extended: If $K=P\times N$ over F (P/F defined as above), then $K/F\in\mathfrak{C}$. (Received February 20, 1957.)

504. J. W. Gaddum: *Quadratic forms and linear inequalities.*

Let the quadratic form $\sum_1^n a_{ij}x_ix_j$ be denoted by $z(x)$ and let A be the matrix of this form. A form is called conditionally definite (semi-definite) if $Z(x)>0$ (≥ 0) when $x\geq 0$. It is shown that $Z(x)$ is conditionally definite (semi-definite) if and only if for each principle minor B of A the system $By>0$ ($By\geq 0$) has a solution $y\geq 0$. The system $Ax\geq 0$ is related to the form $Z(x)$ in a slightly more complicated way. It can be stated that $Ax\geq 0$, $x\geq 0$ is solvable if and only if $Z(x)$ is conditionally semi-definite and $Z(x)>0$ if $x>0$. (Received February 20, 1957.)

505. D. Hawkins and W. E. Briggs (p): *The lucky number theorem.*

The lucky numbers of Ulam (Math. Mag. vol. 29 (1956) pp. 117–122) resemble prime numbers in their apparent distribution among the natural numbers and are defined by the following sieve. If S_n is the sequence $t_{n,m}$, $m=1, 2, 3, \cdots$, S_{n+1} is obtained for $n>1$ by removing from S_n every $t_{n,m}$ for which $t_{n,n}$ divides m. $S_2=(2, 3, 5, 7, 9, \cdots)$ and the lucky number sequence is $S=\lim_{n\to\infty} S_n=(2, 3, 7, 9, 13, 15, 21, \cdots)$. If $R(n, x)$ is the number of numbers not exceeding x in S_n, then the fundamental recurrence relation $R(n, x)=R(n-1, x)-[R(n-1, x)/s_{n-1}]$, s_k being the kth lucky number, yields the fact that $(1-1/s_1)(1-1/s_2)\cdots(1-1/s_n)\sim 1/\log n$ and that $s_n\sim n\log n$. S. Chowla has improved this last result to $s_n=n\log n+n(\log\log n)^2/2+o(n(\log\log n)^2)$. (Received February 14, 1957.)

506. R. H. Oehmke: *On simple flexible algebras of degree $t>3$.*

If A is an algebra over a base field F, the degree of A is the maximum number of pairwise orthogonal idempotents in any scalar extension of A over F. A flexible algebra is an algebra satisfying the identity $(xa)x=x(ax)$. If in addition the algebra also satisfies the identity $(x^2a)x=x^2(ax)$ it is called a noncommutative Jordan algebra. The results and methods of Albert [Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593] can be generalized to show that a simple flexible power-associative algebra of degree $t>2$ is a noncommutative Jordan algebra. R. D. Schafer [Proc. Amer. Math. Soc. vol. 6 (1955) p. 472] proved that a simple Jordan algebra of characteristic 0 is of degree 2, Jordan or quasi-associative. By use of a partial trace function it is possible to extend this result to include the characteristic $p\neq 2, 3$ and degree $t>1$ case. (Received February 21, 1957.)

507t. Ingram Olkin: *On a class of integral identities with matrix argument.* Preliminary report.

Bellman [Duke Math. J. vol. 23 (1956) pp. 571–577] obtained a generalization of the Ingham-Siegel integral identity (1) $\int_{X>0}|X|^a\prod_1^p|X^{(i)}|^{-a_i}\exp-\operatorname{tr}XA\{dX\}=\pi^{p(p-1)/4}\prod_1^p\Gamma(\sum_{j=i}^p a_j-(p-i)/2)|Y_i|^{-a_i}$, where $a=\sum_1^p a_i-(p+1)/2$, X: $p\times p$ is symmetric, $X^{(k)}=(x_{ij})$, $i, j=k, \cdots, p$, $Y_k=(y_{ij})$, $i, j=1, \cdots, k$. The proof is similar to that by Ingham. An alternative and more elementary proof based on certain matrix transformations is presented. The method permits the evaluation of a number of other integral identities of a form similar to (1). In particular, generalizations are given for (a) an integral indentity involving a correlation matrix in the argument, (b) the Beta function formula of Siegel, (c) a matric analogue of a certain Dirichlet integral. These integrals have a connection with distribution problems in multivariate analysis. (Received February 21, 1957.)

508t. D. S. Rim: *On relatively complete fields.*

It was proved by Kaplansky and Schilling (Bull. Amer. Math. Soc. vol. 48) that if K is relatively complete under rank one valuation, and if K has a unique separable extension of degree n for each integer n, then every subfield k with K/k finite algebraic is also relatively complete with respect to the induced valuation. One of the results of the paper is a generalization of this theorem to any field K which is not algebraically closed. In the case when K is algebraically closed, it is shown that the same statement is true if and only if the residue class field of k is real-closed. A simpler proof for Hensel's lemma, eliminating the usual approximation process is also given. (Received February 20, 1957.)

509. D. S. Rim (p) and George Whaples: *Axiomatic approach to cohomology theory of finite groups.*

Cohomology theory of finite groups is known to be essentially the theory of satellites of the 0-dimensional cohomology group. In this paper, without introducing satellites, the theory is axiomatized by demanding the decent behavior of induced homomorphisms and connecting homomorphisms, cohomological triviality of G-regular modules and specifying the 0-dimensional cohomology group. To accomplish a purely axiomatic approach, the cup product is constructed using only the connecting homomorphisms given in the axioms and one natural homomorphism, without introducing complexes at all. (Received February 20, 1957.)

510. M. P. Schützenberger: *D-representation of semigroups.*

A representation of a semigroup S by matrices over a subgroup of S is defined. This representation is an extension to the whole of S of a closely related representation given for a $\mathfrak{D}$-class, D, of S by D. D. Miller and A. H. Clifford (Trans. Amer. Math. Soc. vol. 82 (1956) pp. 270–280) whose results and notations are used below. Let L^i $(i \in I)$ and R_j $(j \in J)$ be the $\mathfrak{L}$- and $\mathfrak{R}$-classes of D; $H = L^1 \cap R_1$, a subgroup of S with idempotent e. For all $i \in I$ and $j \in J$, fixed arbitrary elements are chosen, satisfying: $a^1 = b_1 = e$; $a^i \in R_1 \cap L^i$; $b_j \in L^1 \cap R_j$; $a^{i*} \in L^1$; $b_j^* \in R_1$; $a^i a^{i*} = b_j^* b_j = e$. The $I \times J$ matrix P and, for any $s \in S$, the $I \times I$ matrix $M(s)$ and the $J \times J$ matrix $N(s)$ are defined by: $p_i^j = a^i b_j$, if $a^i b_j \in D$ and $= 0$, otherwise; $m_i^{i'}(s) = a^i s a^{i'*}$, if $a^i s \in L^{i'}$ and $= 0$, otherwise; $n_{j'}^j(s) = b_{j'}^* s b_j$, if $s b_j \in R_{j'}$ and $= 0$, otherwise. All the entries of the matrices P, $M(s)$ and $N(s)$ belong to $H \cup 0$ and, identically, $M(s)P = PN(s)$. The mapping; $s \rightarrow M(s) \otimes (N(s))^\dagger$ is shown to give a representation of S, independant (up to transformations by diagonal matrices) of the choice of $H \subset D$, of the a^{i}'s and so forth. The homomorphisms of S induced by the restrictions of this representation and by the homomorphisms of H are discussed. (Cf. Comptes Rendus vol. 242 (1956) pp. 2907–2908.) (Received February 20, 1957.)

511*t*. M. F. Smiley: *Jordan homomorphisms and right alternative rings.*

The methods of a previous paper (*Jordan homomorphisms onto prime rings*, Trans. Amer. Math. Soc. vol. 84 (1957) pp. 426–429) are used to give a brief and characteristic-free proof that certain strongly right-alternative rings are alternative. Our result includes those previously obtained by E. Kleinfeld (*Right alternative rings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 939–944) and by R. L. San Soucie (*Right alternative rings of characteristic two*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 716–719). (Received February 20, 1957.)

512. F. B. Wright: *On Hölder groups.*

Let G be a topological (additive) abelian group, and let M be a semigroup in G such that (1) M is open, (2) $0\notin M$, and (3) M is maximal with respect to (1), (2). Then the complement $b(M)$ of $M\cup -M$ is a closed subgroup of G. For each such semigroup, set $s(M)=M\cup b(M)$; then $s(M)$ is a semigroup. Let T be the intersection of all such groups $b(M)$; T is called the radical of G. (For details: a forthcoming paper in the American Journal of Mathematics.) The classical theorem of Holder on archimedean ordered abelian groups can be translated into this terminology: G is continuously isomorphic to a subgroup of the additive reals if and only if for every such M, it is true that $b(M)=0$ and the semigroup $s(M)$ is a maximal semigroup (in the strictest sense) in G. This suggests defining a Hölder group to be an abelian group whose radical is 0, and which has the property that for every M the semigroup $s(M)$ is a maximal semigroup in G. Then G is a Hölder group if and only if it is continuously isomorphic to a subgroup of a locally convex real linear topological space. (Received February 18, 1957.)

ANALYSIS

513. R. J. Blattner: *Lie algebras of measurable operators.* Preliminary report.

Let $\mathcal{A}$ be a ring of operators with countably decomposable center and $\mathcal{M}$ the algebra of measurable operators over $\mathcal{A}$. Then the family of all $*$-convergent sequences in $\mathcal{M}$ is the family of all convergent sequences for a metrizable topology on $\mathcal{M}$, called the $*$-topology. $\mathcal{M}$ is a nonlocally convex topological algebra in the $*$-topology. The exponential series $*$-converges for any normal operator, the resultant map (exp) being $*$-continuous. Let G be a Lie group, $\mathcal{L}$ its Lie algebra, ρ a representation of $\mathcal{L}$ as skew-adjoint operators in $\mathcal{M}$, and log the logarithm map from a suitable neighborhood of the identity in G to $\mathcal{L}$. Theorem: $\exp\circ\rho\circ\log$ is a local homomorphism of G into the unitary group of $\mathcal{A}$. This is proved by reducing to the two cases of abelian and compact semi-simple G. The first case is trivial, while the second can be reduced further to the case where $\rho(\mathcal{L})\subseteq\mathcal{A}$, which can easily be handled by the Campbell-Hausdorff formula. Corollary of the proof: there is no representation of the Heisenberg commutation relations as measurable operators. (Received February 22, 1957.)

514*t*. D. G. Bourgin: *Some fixed point theorems.*

The following is a typical result: Let X be a retract of the polytope $P=P_AI_a$ where the power of A is unrestricted. Let $\{Y_i\mid i=1,\cdots,n\}$ be n, $n\neq 1$, open subsets of X whose closures $\{\bar{Y}_i\}$ are retracts and are pairwise disjunct. Denote the boundary of Y_i by $\dot{Y}_i$. Let f map X into X and $\bar{Y}_i$ into $\bar{Y}_i$. Then f has a fixed point in X. The proof of the theorem depends on certain index notions. (Cf. D. G. Bourgin; Rend. Accad. Naz. dei Lincei vol. 22 (1956) pp. 43–48. A special finite dimensional case of the theorem, with a quite different proof which does not extend, has been found by F. Bagemihl; (Fundamenta vol. 40 (1953) pp. 3–12). (Received February 19, 1957.)

515. G. U. Brauer: *Remark on a result of Utz.*

The following theorem is proved: Let $c(t)$ be a positive function with a nonnegative derivative, for large t, and let $g(t)$ be a function such that $g(0)=0$, $g'(t)\leqq 0$ for all values of t. Then if $x(t)$ is a solution of the equation $x''(t)+g(x')+c(t)x(t)=0$, then $x(t)$ is oscillatory, $\lim_{t\to\infty} x(t)=\infty$, or $\lim_{t\to\infty} x(t)=-\infty$. Utz recently proved this theorem for the case where c is a positive constant. (Received February 11, 1957.)

516. R. H. Cameron: *Differential equations involving a parametric function.*

This paper deals with the differential system $ds/dt+f[t, y(t)+s(t)]=0$, $s(0)=0$ on the interval $[0, 1]$, where $y(t)$ is a parametric function continuous on $[0, 1]$ and vanishing at $t=0$. Conditions are given on $f(t, u)$ such that the system has solutions $s(t)$ on $[0, 1]$ for almost every choice (in the sense of Wiener measure) of the parametric function y. Nevertheless it is shown that there is a function $f(t, u)$ satisfying the conditions but such that the system does not have a solution s for *every* y. (Received February 20, 1957.)

517. A. M. Chak: *A generalisation of Lommel polynomials.*

This paper concerns the study of polynomials $R_{p,m,n}(x)$ obtained from the recurrence relation $J_{m+3,n+3}(x)-(3(n+n+3)/x)J_{m+2,n+2}(x)+(9(m+1)(n+1)/x^2)J_{m+1,n+1}(x)-J_{m,n}(x)=0$, where $J_{m,n}(x)$ is the generalized Bessel function studied by P. Humbert [Atti della Pont. Acc. delle Sci. Nouvi Lincei; 1930 (128–), 1934 (323–331), 1935 (154–158)]. These polynomials $R_{p,m,n}(x)$ are of degree p in $1/x$ and also of the same degree in m and n respectively. A number of recurrence relations, several term relations, etc. have been found out; also polynomials associated with $J_{1/3,2/3}(x)$ and $J_{-1/3,-2/3}(x)$ have been studied on the lines of the interesting relationship which Bessel functions of order half an odd integer have with sine and cosine functions. (Received February 18, 1957.)

518*t*. W. F. Darsow: *Ideals and states in C^*-algebras.*

With the appropriate modifications certain theorems of I. E. Segal in *Two-sided ideals in operator algebras* (Ann. of Math. vol. 50 (1949) pp. 856–865) extend readily to one-sided ideals. If K is a closed left ideal in a C^*-algebra A, then T is in K if and only if T^*T is in K. Let $s(K)$ be the set of all pure states of A that vanish on the self-adjoint elements in K; then T is in K if and only if $f(T^*T)=0$ for all f in $s(K)$. For a pure state f of A the *left-kernel* $k(f)$ of f is the closed left ideal of all T in A for which $f(AT)=0$; then a closed left ideal K is the intersection of the left-kernels $k(f)$ as f ranges over all of $s(K)$. (Received February 19, 1957.)

519. W. H. Fleming: *Functions with generalized gradient and generalized surfaces.*

Let $f(x)$ denote an integrable function of k real variables ($k\geqq 2$) vanishing outside a fixed cube K, for which the total gradient variation $I(f)$ defined by DeGiorgi [Annali di Mat. vol. 36 (1954) pp. 191–213] is finite. Each such f defines a vector measure $\Phi=(\Phi_1, \cdots, \Phi_k)$ called *exact*. A $k-1$ dimensional generalized surface L situated in K also defines a vector measure Φ, termed *closed* if L is closed [Fleming-Young, Trans. Amer. Math. Soc. vol. 76 (1954) pp. 457–484]. *Theorem*. Closed if and only if exact. The set F_N of all f for which $I(f)\leq N$ is convex and compact in the l_1 topology. *Theorem*. Every extreme point of F_N is a multiple of the characteristic function f_E of a set $E\subset K$. For $k=2$ the extreme points are completely characterized. *Mixture Theorem*. There exists a function $W(x, E)$, equal to f_E a.e. for each fixed E, such that to every f with $I(f)$ finite corresponds a signed measure ω on E-space for which $f(x)=\int[I(f_E)]^{-1}W(x, E)d\omega$ a.e. (Received February 18, 1957.)

520. W. R. Fuller: *Existence of cycles for weakly nonlinear real differential systems containing differences.*

Consider the system: (1) $\ddot{x}_1+\sigma_1^2x_1=\epsilon\{\sum_{l=0}^{r}\dot{x}_1(t-\lambda_l)f_l[x_1(t-\lambda_l)]+\sum_{l=1}^{r}x_1(t-\lambda_l)g_l[\dot{x}_1(t-\lambda_l)]\}+\epsilon q_1(y, \dot{y}; \epsilon)+\epsilon k(y, \dot{y}; \epsilon)$; $\ddot{x}_j+\alpha_j\dot{x}_j+\sigma_j^2x_j=\epsilon q_j(y, \dot{y}; \epsilon)$, $j=2, \cdots, n$; $\dot{x}_\mu+\beta_\mu x_\mu=\epsilon q_\mu(y, \dot{y}; \epsilon)$, $\mu=n+1, \cdots, N$; where λ_l, $l=0, \cdots, r$ are distinct, $\lambda_0=0$, $\epsilon>0$ is a small parameter, y, $\dot{y}$ denote the $N(r+1)$- and $n(r+1)$-vectors $[x(t-\lambda_l), l=0, \cdots, r]$, $[\dot{x}(t-\lambda_l), l=0, \cdots, r]$, $\sigma_j>0$, $\alpha_j>0$, $\beta_\mu>0$, $\gamma_j^2=4\sigma_j^2-\alpha_j^2>0$, $\alpha_j+i\gamma_j\neq 2im\sigma_1$ ($m=0, \pm 1, \cdots; j=1, \cdots, n; \mu=n+1, \cdots, N$), each function is analytic for $\epsilon<\epsilon_0$, $|x_h|, |\dot{x}_h|<K$; f_l, g_l even polynomials, $q_1(-y, -\dot{y}; \epsilon)=q_1(y, \dot{y}; \epsilon)$, $k(y_1, 0, \cdots, 0, \dot{y}_1, 0, \cdots, 0; 0)=0$. Then an algebraic equation $P(|a|^2)=0$ is given for each of whose simple roots $|a_0|^2=c_0^2>0$, a periodic solution of (1) exists: (2) $x_1=|a|\sigma_1^{-1}\sin(\tau t+\phi)+\epsilon W_1(\tau t+\phi, \lambda_1, \cdots, \lambda_r; \epsilon)$; $x_h=\epsilon W_h(\tau t+\phi, \lambda_1, \cdots, \lambda_r; \epsilon)$, $h=2, \cdots, N$; where W_1, W_h are periodic of period 2π in $\tau t+\phi$, ϕ is an arbitrary constant, and $\tau=\sigma_1+O(\epsilon)$, $|a|=c_0+O(\epsilon)$. The series for τ and $|a|$, and conditions assuring that $P=0$ has simple positive roots are given. For instance the Van der Pol like equation $\ddot{x}(t)+\epsilon[x^2(t)-1]\dot{x}(t)+\epsilon[x^2(t-\lambda)-1]\dot{x}(t-\lambda)+x=0$ has a cycle. The method of successive approximations successively developed by L. Cesari [Atti Acad. Italia (6) vol. 11 (1940) pp. 633–692], J. K. Hale [Bull. Amer. Math. Soc. Abstracts 60-1-118, 60-1-119, 62-6-668, 62-6-669] and others, is here modified and its convergence proved for systems with differences. (Received February 18, 1957.)

521. R. R. Goldberg: *Convolution transforms of almost periodic functions.*

Let $G(t)$ be any Pólya frequency function the reciprocal of whose bilateral Laplace transform has the form $E(s)=e^{bs}\prod_{k=1}^{\infty}(1-a_ks)e^{a_ks}$ where $\sum_{k=1}^{\infty}a_k^2<\infty$. Let $P_n(s)=e^{b_ns}\prod_{k=1}^{n}(1-a_ks)e^{a_ks}$ where $(b-b_n)^2=O(\sum_{k=n+1}^{\infty}a_k^2)$ as $n\to\infty$. The following theorem is proved: Necessary and sufficient conditions that a given function $f(x)\in C^\infty$ $(-\infty<x<\infty)$ be representable in the form $f(x)=\int_{-\infty}^{\infty}G(x-t)\phi(t)dt$ for some $\phi(t)$ almost periodic in the sense of Bohr are that the functions $\{P_n(D)f(x)\}_{n=1}^{\infty}$ (where D stands for differentiation) be uniformly bounded and be uniformly dominated by some almost periodic function. This last means that there should exist an almost periodic function $\beta(x)$ such that for any $\epsilon>0$ every translation number for $\beta(x)$ corresponding to ϵ is also a translation number for each $P_n(D)f(x)$ corresponding to ϵ. (Received February 15, 1957.)

522. J. H. B. Kemperman: *Continuous solutions of some functional equations on a topological group.*

Let X be a topological Hausdorff group satisfying the first axiom of countability, and let Y be a topological space. Let $f(s)$ be a function on an open subset U of X to Y, which is continuous on a Borel subset $D \subset U$. We suppose either that D is of second category or that X is locally compact and D is of positive right (regular) Haar measure. Finally, let $I_1 \subset X$, $I_2 \subset X$ with I_2 open and $I_1 I_2 \subset U$. (A) Suppose that $D \subset I_2 \subset U$ and that $f(st) = G(s, f(t))$ for $s \in I_1$, $t \in I_2$, where, for each $s \in I_1$, $G(s, y)$ is a continuous function from Y to Y. Then $f(s)$ is continuous at each point t_0 with the property that for each $d \in D$ there exists a finite sequence of points $s_i \in I_1$ with $s_i s_{i-1} \cdots s_1 d \in I_2$, $(i = 1, \cdots, k-1)$ and $s_k s_{k-1} \cdots s_1 d = t_0$. This implies some known results concerning measurable functions $f(s)$ on $s > 0$, cf. E. Hille, *Functional analysis and semi-groups*, Chapter VIII. (B) Suppose that $I_1 I_2$ is a connected set containing D, further, I_1 is open, while for each $t \in I_2$ there exists an open connected set V with $t \in V \subset I_2$. Finally, let Y be a topological vector space over the field $\mathfrak{F}$ and let, for $s \in I_1$, $t \in I_2$, $f(st) = \sum_1^q g_i(s) h_i(t)$, where $g_i(s) \in \mathfrak{F}$, $h_i(t) \in Y$, $(i = 1, \cdots, q < \infty)$. Then $f(s)$ is continuous on $I_1 I_2$. (Received February 20, 1957.)

523. M. Z. Krzywoblocki: *Mathematics and the first law of thermodynamics.*

Max Planck opposed using the differential notation for the heat input in the first law of thermodynamics, maintaining that the differential dQ cannot be regarded as the differential of a finite quantity Q. The author shows that this reasoning is correct only with reference to the exact differential. Using the elementary notion of the functions of bounded variation and that of the Riemann-Stieltjes integral, the author shows that the heat input function Q exists in almost all the cases of physical importance. (Received February 18, 1957.)

524. Karel deLeeuw: *Analytic almost periodic functions in a sector.*

Let α be a real number satisfying $0 \leq \alpha \leq \pi/2$ and S_α the subset $\{re^{i\theta}: -\alpha \leq \theta \leq +\alpha\}$ of the complex plane. Let $B(S_\alpha)$ be the space of bounded continuous functions on S_α that are analytic at interior points, supplied with the topology of uniform convergence on S_α. For each σ in S_α and f in $B(S_\alpha)$, let $T_\sigma f$ in $B(S_\alpha)$ be defined by $T_\sigma f(\tau) = f(\sigma + \tau)$. Define $A(S_\alpha)$ to be the closed linear subspace of $B(S_\alpha)$ consisting of those f which are such that the subset $\{T_\sigma f: \sigma \in S_\alpha\}$ has compact closure. Let $\hat{S}_\alpha$ be the subset $\{re^{i\theta}: \pi/2 + \alpha \leq \theta \leq 3\pi/2 - \alpha\}$ of the complex plane. *Theorem*: $A(S_\alpha)$ consists of precisely those functions that can be approximated uniformly on S_α by linear combinations of those exponential functions of the form $f(z) = e^{\lambda z}$ with λ in $\hat{S}_\alpha$. The case $\alpha = \pi/2$ of this theorem follows from results of Bohr. That for $0 < \alpha < \pi/2$ is established by obtaining a canonical direct sum decomposition of $A(S_\alpha)$ into four direct summands and identifying each of these direct summands. If $\alpha = 0$, there is a similar decomposition into two direct summands. (Received February 18, 1957.)

525t. M. J. Mansfield: *Some generalisations of full normality*, I.

A collection $\mathfrak{B}$ of subsets of a set X is said to be an $\mathfrak{m}$-*star refinement* ($\mathfrak{m}$ any cardinal number ≥ 2) of a collection $\mathfrak{A}$ if (i) $\mathfrak{B}$ is a refinement of $\mathfrak{A}$ and (ii) if $\mathfrak{C} \subset \mathfrak{B}$, $2 \leq |\mathfrak{C}| \leq \mathfrak{m}$, and $\bigcap \{C: C \in \mathfrak{C}\} \neq \phi$, then there is an $A \in \mathfrak{A}$ such that $\bigcup \{C: C \in \mathfrak{C}\} \subset A$. $\mathfrak{B}$ is an *almost*-$\mathfrak{m}$-*star* refinement of $\mathfrak{A}$ if (i) $\mathfrak{B}$ is a refinement of $\mathfrak{A}$ and (ii) if $M \subset St(x, \mathfrak{B})$ $(= \bigcup \{B \in \mathfrak{B}: x \in B\})$ for some $x \in X$ and $2 \leq |M| \leq \mathfrak{m}$, then there is an $A \in \mathfrak{A}$ such that $M \subset A$. A topological space X is said to be $\mathfrak{m}$-*fully normal* (resp. *almost*-$\mathfrak{m}$-*fully normal*) if each open covering of X admits an open $\mathfrak{m}$-star (resp. almost-$\mathfrak{m}$-star) refinement. Each fully normal space is $\mathfrak{m}$-fully normal for every $\mathfrak{m} \geq 2$, and each m-fully normal space is almost-m-fully normal. An almost-2-fully normal space is collectionwise normal (cf. H. J. Cohen, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 290–292). An example, due to R. H. Bing, is given of a collectionwise normal space which is not almost-2-fully normal. The class of almost-2-fully normal spaces is shown to coincide with the class of all spaces X for which the family of all neighborhoods of the diagonal in $X \times X$ is a uniformity for X (cf. Cohen, *op. cit.*). (Received February 18, 1957.)

526. N. G. Meyers: *Asymptotic behavior of solutions of elliptic equations.*

The paper is concerned principally with the behavior of solutions of linear elliptic equations of the form $\sum_{i,k=1}^{n} a_{ik}(x)\partial^2 u/\partial x_i \partial x_k + \sum_{i=1}^{n} b_i(x)\partial u/\partial x_i + c(x)u = f(x)$, $(n>2)$, defined in a neighborhood of infinity. The main result gives the leading term in the expansion of the solution about infinity. For simplicity it is stated here only for the case $b_i(x) = 0 = c(x)$. Theorem: If (1) $|a_{ik}(x) - \delta_{ik}| <$ constant $|x|^{-\alpha}$ where δ_{ik} is the Kronecker delta (2) $|x|^{n+\alpha}|f(x)| <$ constant (3) the Holder conditions $|a_{ik}(x) - a_{ik}(y)| <$ constant $d^{-\alpha}|x - y|^{\alpha}$, $|f(x) - f(y)| <$ constant $d^{-2-\alpha}|x - y|^{\alpha}$ $(d = \min |x|, |y|)$ hold (4) $u(x) \rightarrow u^0$ as $|x| \rightarrow \infty$ then $u = u^0 +$ constant$/|x|^{n-2}$ $+ O_k(1/|x|^{n-2+\alpha})$. Asymptotic estimates on derivatives of solutions are also obtained assuming conditions on the coefficients and their derivatives similar to those above. The method of proof exploits the Schauder estimates coupled with Green's integral representation of the solution. A similar method can also be applied to prove removability of singularities at finite points. (Received April 19, 1957.)

527. J. C. C. Nitsche: *A uniqueness theorem on minimal surfaces in cylindrical coordinates.*

Let S be a minimal surface which admits of a representation in cylindrical coordinates $x = r(s, \phi) \cdot \cos \phi$, $y = r(s, \phi) \cdot \sin \phi \cdot r(s, \phi)$ is assumed to be positive and periodic: $r(s, \phi + 2\pi) = r(s, \phi)$. The author proves the following *theorem*: Let the function $r(s, \phi)$ be of class C^3 for all values of s and ϕ and suppose that the curves of intersection of S with the planes $s =$ const. are convex. Then S must be a catenoid: $(*)$ $r(s, \phi) = [\gamma \cdot \sin (\phi + \delta) + (\cosh^2 (\alpha s + \beta) - \gamma^2 \cos (\phi + \delta)^{1/2}]/\alpha$. $\alpha, \beta, \gamma, \delta$ are real constants, $\alpha > 0$, $|\gamma| < 1$. The axis of revolution is parallel to the s-axis and passes through $x = \gamma \sin \delta$, $y = \gamma \cos \delta$. One may regard this as another example of a theorem of "Bernstein's type": The entire solutions of the corresponding differential equation under certain conditions depend on finitely many parameters only (cf. a remark by C. Loewner). The proof makes use of the following facts: The function $\zeta = \xi + i\eta$, where $\xi = s$, $\eta = \int_{0,0}^{s,\phi} W^{-1}[r_s r_\phi ds + (r^2 + r_\phi^2)d\phi]$, $W = (r^2 + r_\phi^2 + r^2 r_s^2)^{1/2}$, yields a one-to-one and isothermic mapping of the s, ϕ-plane onto the ζ-plane. $F(\zeta) = (r + ir_\phi)(W + rr_s)^{-1}e^{-i\phi}$ is an analytic function of ζ and a schlicht function of $\zeta^* = e^{\zeta}$ in $0 < |\zeta^*| < 1$ without zeros. Hence $F(\zeta) =$ const $\cdot e^{-\zeta}$. Thus a differential system is obtained whose integration leads to $(*)$. (Received February 19, 1957.)

528. Heinz Renggli: *On quasiconformal mappings.*

By $\lambda\{\gamma\}$ we denote the extremal length of a family $\{\gamma\}$ of curves γ [L. Ahlfors and A. Beurling, Acta Math. 83 (1950) p. 101]. Let f be a homeomorphism of a region D of the complex plane onto another region. Then f is said to be quasiconformal and denoted by F, if $\lambda\{\gamma'\} = 0$ if and only if $\lambda\{\gamma\} = 0$ where $\{\gamma'\}$ is the image of $\{\gamma\}$ under F. We then have: (i) the class $\{N(SB)\}$ of sets $N(SB)$ [l.c.] is quasiconformally invariant, i.e. let F be a mapping of the complement in the extended plane of a set $N(SB)$; then the complement of the image is also a set $N(SB)$, (ii) let F map a Jordan region onto another Jordan region; then F induces a homeomorphism of the boundary, (iii) let F map a Jordan region D onto the unit-circle and let $N(SB)$ be situated on the Jordan curve bordering D; then the image of $N(SB)$ in the boundary of the unit-circle is also a set $N(SB)$. The statements are proved by a special study of families $\{\gamma\}$ with $\lambda\{\gamma\}=0$. For (iii) reflection in the boundary of the unit-circle is used. (Received February 18, 1957.)

529t. Diran Sarafyan: *Singular points of exact differential equations.*

Concerning the singular points of $y'=f(x, y)$ the following theorem has been given in a previous paper (Bull. Amer. Math. Soc. Abstract 63-2-256): The necessary and sufficient condition under which through any point $P_0(x_0, y_0)$ of a closed simply connected region R, there exists a unique integral curve is that $\exp \int_a^{x_0}(\partial f/\partial y)(x, y_0)dx=0$. This theorem will be applied to the exact differential equation $M(x, y)dx+N(x, y)dy=0$ which can also be written $y'=-M/N=f(x, y)$. It will be assumed that the functions M, N and $f(x, y)$ are continuous in the region R. Now $\partial f/\partial y=(M/N^2)(\partial N/\partial y)-(1/N)(\partial N/\partial X)$. But $\partial N/\partial x=dN/dx+(M/N)(\partial N/\partial y)$. Thus by substitution one gets $\partial f/\partial y=-(1/N)(dN/dx)$. Then $\int_a^{x_0}(\partial f/\partial y)dx=-\int_a^{x_0}dN/N=[\log(1/|N|)]_a^{x_0}$. The application of the above stated theorem indicates that, there are no singular points in the region R. Outside of R they are determined from $1/N(x_0, y_0)=0$ and/or $1/M(x_0, y_0)=0$. (Received February 20, 1957.)

530. A. B. Simon: *Subalgebras of group algebras.* Preliminary report.

Let S be a measurable subset of a locally compact Abelian group G and $L=L(G)$ the group algebra of G. We denote by L_S the set of all functions of L whose support lies in S. In the case L_S forms a subalgebra of L, we call it a vanishing subalgebra. It is proved that L_S is a vanishing subalgebra if and only if there is a subset T whose closure is a subsemigroup of G and $L_T=L_S$. We say that a subset A of L is separating if the set of Fourier transforms of elements of A separate points of the character group. It is shown that L_S is separating if and only if S generates G. (Received February 18, 1957.)

531. Hidehiko Yamabe: *On a problem of Bellman.*

In the Bulletin of American Mathematical Society, January 1957, R. Bellman proposed a problem. Let $f(x)$ and $g(x)$ be two strictly positive functions over $[0, 1]$. Then are the eigenvalues λ of $u''+\lambda(f+\epsilon g)u=0$ analytic in ϵ when $R(\epsilon)\neq 0$. An affirmative answer is given for real. This is shown by showing a solution $\beta(\lambda, \epsilon; x)$ of $u''+\lambda(f+\epsilon g)u=0$, $u(0)=1$, $u(1)=0$, which is analytic in λ and ϵ and is 0 at $x=1$ and $\beta_\lambda\neq 0$ at $x=1$. From this the answer follows by the implicit function theory. β is the (1.2) member of the product integral of $\left(\begin{smallmatrix}0 & 1\\ -\lambda(f+\epsilon g) & 0\end{smallmatrix}\right)$ which gives the solution for any initial data. (Received April 8, 1957.)

APPLIED MATHEMATICS

532. R. P. Kanwal: *On curved shock waves in three-dimensional steady rotational gas flows.*

In this paper we discuss the problem of shock waves in three-dimensional steady rotational gas flows of fluids devoid of viscosity and heat conductivity. The main object is to obtain, by the help of the Rankind-Hugoniot relations, formulas for the determination of the derivatives of velocity components, pressure, density and entropy behind the shock surface when the flow in front is known. The detailed treatment is given to the case when the flow in front is uniform. Gauss-Weingarten formulas and various other results of the geometry of surfaces concerning principal normal curvature are found useful in the analysis. As anticipated, the flow is found in general to be rotational behind the shock surface. The explicit determination of the components of vorticity has been carried out. This leads to the formulation of a general theorem regarding the characterization of surfaces behind which the flow remains irrotational. It is found that the plane, the right circular cone, the cylinder and the developable helicoid are the only such surfaces. (Received April 15, 1957.)

533. W. R. Wasow: *On the accuracy of implicit difference approximations to the equation of heat flow.*

Let $u(x, t)$ be the isolution of the boundary value problem: $\partial u/\partial t=\partial^2 u/\partial x^2$, for $0<t$, $0\leqq x\leqq\pi$; $u(x, 0)=f(x)$, $u(0, t)=u(\pi, t)=0$. Let $U(x, t)$ satisfy the difference equation $[U(x, t+k)-U(x, t)]k^{-1}=[U(x+h, t+k)-2U(x, t)+U(x-h, t+k)]\cdot(1-s)h^{-2}+[U(x+h, t)-2U(x, t)+U(x-h, t)]sh^{-2}$ and the same subsidiary conditions as $u(x, t)$. It is proved that the truncation error $U-u$ is of the order $O(t^{-1}h^{\alpha})$, where $\alpha=1$ for $1/2<s\leqq 1$, $\alpha=2$ for $s=1/2$, provided $f(x)\in C^3$, $f(0)=f(\pi)=0$, and $0<p\leqq h/k$ (p independent of h and k). For the convergence of U to u as $h\rightarrow 0$, $k\rightarrow 0$, without any restriction on h and k it is sufficient that $f'(x)$ is piecewise continuous, that $f(0)=f(\pi)=0$, and $s\geqq 1/2$. For any $f(x)$ that is bounded, $U=O(\max|f| \log(1/h))$. This last fact implies that the difference equation is always reasonably stable. (Received March 18, 1957.)

GEOMETRY

534. W. R. Abel (p) and L. M. Blumenthal: *Metric arcs with a tangent property.*

This paper continues the systematic study of metric properties of metric arcs, begun in a previous article (*Metric arcs with Menger curvature*, I., Bull. Amer. Math. Soc. Abstract 63-1-52). A metric arc has property T_A at a point p, provided that $r, s, t\in A$, $r<s<t$ implies $\lim \sphericalangle rst=\pi$ as $r, s, t\rightarrow p$. A metric continuum K has property T_K at a point p, provided $r, s, t\in K$, $\lim \max \sphericalangle$ (triple r, s, t) $=\pi$ as $r, s, t\rightarrow p$. It is proved that (1) if K has property T_K at each of its points, then K is a rectifiable arc or a rectifiable simple closed curve, (2) properties T_A, T_K are equivalent for $K=A$, and hence a metric arc with property T_A at each point is rectifiable. The latter statement remains valid if $\lim \sphericalangle rst<\pi/2+\arcsin(1/\pi)$, as $r, s, t\rightarrow p$. These results are intimately related to theorems announced without proof by C. V. Pauc (Rend. R. Accad. Naz. Lincei. vol. 24, ser. 6, 1936; Thése, Fac. Sci. l'Univ. Paris, Hermann et Cie., Paris, 1941). (Received February 27, 1957.)

LOGIC AND FOUNDATIONS

535*t*. Hugo Ribeiro: *Universal completeness*. I.

Terminology and some of the methods are as in Tarski, *Contributions to the theory of models*, Proc. Acad. Soc. Amsterdam, A, no. 5 (1954) pp. 572–588 and no. 1 (1955) pp. 56–64. Equational completeness for abstract algebras has been defined by Kalicki and Scott, see Indag. Math. vol. 17 (1955) pp. 650–659 and bibliography therein. The following is a slightly different version of the definition given by the author in Bull. Amer. Math. Soc. vol. 62 (1956) p. 176, where the result on the class of all linearly ordered sets should have been attributed to Langford, Proc. London Math. Soc. vol. 25 (1926) pp. 115–142. A consistent set, Σ, of sentences of the formalized theory, $T(K)$, constructed for given similarity class, K, of relational systems is said to be universally complete if and only if for any universal sentence, ϕ, of $T(K)$, either ϕ follows from Σ or there is, for some n, a (universal) sentence, α (involving only the identity symbol and the other logical constants) saying that there are at most n elements and such that $\phi \leftrightarrow \alpha$ follows from Σ. Langford's results on universal completeness for several classes of linearly ordered sets follow from that for the class of all linearly ordered sets by the remark that if Σ is consistent and has a universally complete subset then Σ is universally complete. Every set of equalities which is universally complete is also equational complete. Several of the sets of equalities considered by Kalicki and Scott are universally complete. (Received February 21, 1957.)

536. Hugo Ribeiro: *Universal completeness* II. Preliminary report.

For terminology see abstract I. In the following Σ is a consistent set of sentences and K is the correspondent class of systems. (I) If Σ is universally complete, $\mathfrak{A}, \mathfrak{B} \in K$, $\mathfrak{A}$ finite and card $\mathfrak{A} \leq$ card $\mathfrak{B}$ then $\mathfrak{A}$ is isomorphically embeddable in $\mathfrak{B}$. (II) Let K satisfy the condition of (I) and the following one: if $\mathfrak{S} \in K$ and S' is finite subset of the set of $\mathfrak{S}$ then there is $\mathfrak{R} \in K$ such that $\mathfrak{R}$ is finite and the restriction of $\mathfrak{S}$ to S' is isomorphically embeddable in $\mathfrak{R}$. Then Σ is universally complete. (III) Σ is universally complete if and only if for any $\mathfrak{W} \in K$ and for any finite subset W' of the set of $\mathfrak{W}$ the restriction of $\mathfrak{W}$ to W' is isomorphically embeddable in all elements of K with the eventual exception of finite elements $\mathfrak{W}$ such that card $\mathfrak{W} <$ card $\mathfrak{W}$ and which, up to isomorphism, are finitely many. The following are examples of classes defined by universally complete sets of sentences: the class of all Boolean algebras, the class of all cyclic groups whose order is power of a fixed prime and smaller than a fixed number, the class of all finite projective geometries over the same field. Other classes, K, are discussed in connection with I, II and III. (Received February 21, 1957.)

STATISTICS AND PROBABILITY

537t. J. H. B. Kemperman: *An asymptotic expansion concerning the Smirnov test.*

Let $2^n p_n(a, b, d)$ denote the number of sets $(\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i = \pm 1$, $-a < \epsilon_1 + \cdots + \epsilon_i < b$, $(i = 1, \cdots, n)$, $\epsilon_1 + \cdots + \epsilon_n = d$; here, a, b, n denote positive integers, d is an integer with $-a < d < b$, $n - d$ even. Then (*) $p_n(a, b, d) = 4/c \sum_1^K \sin k\pi a/c \sin k\pi(a+d)/c \cos^n k\pi/c$, where $c = a+b$, $K = [(c-1)/2]$. From (*), we obtain the following asymptotic expansion: (1) Let $H_{2r}(x) = D^{2r} \exp(-x^2)$ and $g_r = 2^{-r} \sum_{k=-\infty}^{\infty} [H_{2r}((d+2kc)/(2n)^{1/2}) - H_{2r}((2a+d+2kc)/(2n)^{1/2})]$; (2) Let the constants $A_{ij} \geq 0$ be defined by the expansion $\exp(u/2 + u/s \log \cos s^{1/2}) = \sum_0^\infty \sum_0^\infty A_{ij} s^i (-u)^j$, (holding for $|s| < \pi^2/4$); (3) Let $c_i = (-1)^i (2/\pi)^{1/2} \sum_{j=0}^{i} A_{ij} g_{i+j}$; (4) Let R_m be defined by $p_n(a, b, d) = \sum_{i=0}^{m-1} c_i n^{-i-1/2} + R_m n^{-m-1/2}$; (5) Assertion: for each integer $m \geq 0$, both c_m and R_m are bounded functions of n, in fact, uniformly bounded with respect to a, b, d. Note that $2^{2n} p_{2n}(a, b, 0)/C_{2n}^{n} = \Pr(-a/n < F_2(x) - F_1(x) < b/n$ for all $x)$, where $F_1(x)$, $F_2(x)$ denote the empirical distributions of two independent samples of size n from a continuous population. (Received February 20, 1957.)

TOPOLOGY

538*t*. D. G. Bourgin: *Real functions on spaces with homeomorphisms of even period.*

Let P be a unicoherent locally connected compact Hausdorff space with a finite Borel measure. Let R_1 and R_2 be two measure preserving homeomorphisms of periods $2m_1$ and $2m_2$ and suppose R_1^{m} and R_2^{m} are fixed point free. Let f be a continuous real valued function on P which satisfies $f(rR_i^{2l})=f(r)$, $l=1, 2, \cdots$. Then, for some $\bar{r}$, $f(\bar{r})=f(\bar{r}R_1)=f(\bar{r}R_2)$. A special case of this theorem with $m_1=m_2=2$ and P the projective 3 space has been given by the writer previously (Rend. di Mat. e delle sue applicazioni 15, pp. 177–189, 1956; Theorem 3). The proof is similar to that of the special case and pivots on introducing an identification space Y, admitting a fixed point free involution R' for which P is a covering space, and observing that the inverse projection of an R_1' (or R_2') symmetric closed connected separating set in Y is an R_1 (or R_2) symmetric connected separating set in P. (Received February 14, 1957.)

539*t*. A. H. Clifford: *Connected ordered topological semigroups with idempotent endpoints.* I.

By a *thread* we shall mean a system $S(\circ, <)$ such that: (1) $\circ$ is an associative binary operation in S; (2) $<$ is a total (i.e. linear) order relation in S; (3) the mapping $(x, y)\rightarrow x\circ y$ of $S\times S$ into S is continuous in the order topology; (4) S is connected; (5) S has a least element f and a greatest element e, and these are idempotent. The systematic study of threads was initiated by W. M. Faucett (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 741–747 and 748–756). The present paper continues the study to the point of obtaining a complete determination of all threads with zero. (All threads without zero will be described in Part II.) A thread S is called *standard* if its endpoints are the zero and identity elements of S. P. S. Mostert and A. L. Shields (Ann. of Math., to appear) determined all these when S is a real interval; the general case is similar. If $S=[f, e]$ is any thread, the intervals $[f, 0]$ and $[0, e]$ are standard subthreads. If e is the identity element, S is completely determined by the homomorphism $x\rightarrow xf$ of $[0, e]$ onto $[f, 0]$, a result found independently by H. Cohen and L. I. Wade (Trans. Amer. Math. Soc., to appear). Every commutative thread with zero is a trivial extension of a thread of this type. Every noncommutative thread is a trivial extension of a thread like Faucett's Example 3, p. 747, with the real unit interval replaced by any standard thread. (Redeived February 25, 1957.)

540. H. H. Corson: *On the relatively compact subsets of a space.*

A subset A of a topological space X is called relatively countably compact if every sequence in A has a cluster point in X. It is proved that if a completely regular topological space has property (1) every open covering has a point finite refinement, or (2) its product with one of its compactifications is normal then (3) every relatively countably compact subset is relatively compact. It also is shown that if a space X has (3) then every closed continuous image of X has (3), and that X has (3), if it is the union of a countable number of closed subsets having (3). An example is given of a normal topological group which does not have (3), and is therefore not paracompact. (Received February 18, 1957.)

541. Edward Halpern: *A primitive hyperalgebra.*

A *hyperalgebra* over A (commutative ring with unit) consists of an A-module H, associative product $\phi: H\otimes H\rightarrow H$ with unit 1, and associative coproduct $\psi: H\rightarrow H\otimes H$

with $\psi(1)=1\otimes 1$ which is multiplicative. An *augmentation* is an algebra augmentation $\alpha: H\rightarrow A$ satisfying $(i\otimes\alpha+\alpha\otimes i)\psi(x)=x\otimes 1+1\otimes x$, where i is the identity and $x\in H^+$, the kernel of α. An element $x\in H^+$ is *primitive* if $\Delta(x)=\psi(x)-x\otimes 1-1\otimes x=0$; H is primitive if generated (under ϕ) by primitive elements (and 1). Define subalgebras $\pi^{(k)}$, $k\geqq 0$, inductively as follows: $\pi^{(0)}$ is generated by the primitive elements (and 1), $\pi^{(k+1)}$ is generated by elements such that $\Delta(x)\in\pi^{(k)}\otimes\pi^{(k)}$. The hyperalgebra is of *class* k if $H=\pi^{(k)}$, $H\neq\pi^{(k-1)}$. In a graded hyperalgebra, $H=\sum_{p\geq 0}H^p$ (weak) and ϕ and ψ are homogeneous of degree zero; it is assumed $H^0=A\ 1$ and the augmentation is "standard." The following theorem is proved: If H is a graded hyperalgebra of class 2 over a field of characteristic zero and ϕ is anticommutative then the hyperalgebra $H/\pi^{(0)}$ (with naturally induced product and coproduct) is primitive. (Received February 18, 1957.)

542*t*. Edward Halpern: *An exact sequence.*

A spectral sequence of A-modules (E_r) is *canonical* if E_r is defined and bigraded for $r\geqq 2$ with $E_r^{p,q}$ zero if p or q is negative and $E_r^{0,0}\approx A$ (naturally induced for $r>2$) and the differential d_r has bidegree $(r, 1-r)$. If $E_\infty^{p,q}=0$ for $p+q>0$ and $E_\infty^{0,0}\approx A$ (naturally induced) it is *acyclic*. The following theorem is proved: If (E_r) is acyclic and canonical and $E_2^{p,q}=0$ for $p\neq 0, m, n$ where $2\leqq m\leqq n$, $n-m\neq 1$, then $0\rightarrow E_2^{0,1}\rightarrow^{\alpha}E_2^{m,2-m}\rightarrow^{\beta}E_2^{n,3-n}\rightarrow^{\gamma}E_2^{0,2}\rightarrow^{\alpha}\cdots\rightarrow^{\gamma}E_2^{0,q}\rightarrow^{\alpha}E_2^{m,q-m+1}\rightarrow^{\beta}E_2^{n,q-n+2}\rightarrow^{\gamma}E_2^{0,q+1}\rightarrow^{\alpha}\cdots$ is exact, where α, β, γ are given essentially by d_m, d_{n-m}, d_n respectively. The cohomology $H^*(\Omega)$ of the loop spaces over the (a) complex projective plane, (b) quaternionic projective plane, (c) Cayley plane, is computed using this sequence. The results when A is a principal ideal domain of characteristic zero are: $H^*(\Omega)$ is isomorphic to a tensor product of an exterior algebra with one generator of degree $m-1$ and a twisted polynomial algebra in "one variable" of degree $3m-2$ where $m=2, 4, 8$ in cases (a), (b), (c) respectively. Corresponding results are obtained if the characteristic is not zero. (Received February 18, 1957.)

543. L. F. McAuley: *On complete collectionwise normality and paracompactness.*

The concept of complete collectionwise normality is defined. It bears a relation to Bing's notion of collectionwise normality [Canadian Journal of Mathematics (1951)] like that of complete normality to normality. Some results are as follows. (1) In a regular semi-metric topological space, paracompactness is equivalent to complete collectionwise normality. (2) There is a separable normal Hausdorff space which is collectionwise normal and which satisfies the first axiom of countability but which is not completely collectionwise normal. (3) In a Moore space, complete collectionwise normality is equivalent to collectionwise normality. (4) A normal, pointwise paracompact and separable semi-metric topological space is hereditarily separable, completely collectionwise normal, and paracompact but may fail to have a countable basis. As a consequence of (4), a normal, separable and pointwise paracompact Moore space is metrizable. (Received February 20, 1957.)

544*t*. M. J. Mansfield: *Some generalisations of full normality*, II.

It is shown that every linearly ordered space is $\aleph_0$-fully normal and that, for any ordinal α, the linearly ordered space $W(\omega_{\alpha+1})$ is $\aleph_\alpha$-fully normal but not almost-$\aleph_{\alpha+1}$-fully normal. It is an open question as to whether an almost-$\mathfrak{m}$-fully normal space is $\mathfrak{m}$-fully normal, even for $\mathfrak{m}$ finite. It is known that if a space X is k-fully normal for some finite k, then X is n-fully normal for every finite $n=2, 3, \cdots$. It is not known if the analogous statement for almost-k-fully normal spaces is true or not. An example is given of a 2-fully normal space which is not almost-$\aleph_0$-fully normal; this space, however, is not a Hausdorff space. It is an open question as to whether every 2-fully normal Hausdorff space is almost-$\aleph_0$-fully normal. The questions of topological completeness and separation of arbitrary subsets by means of open sets are investigated for $\mathfrak{m}$- and almost-$\mathfrak{m}$-fully normal spaces. In particular, it is shown that every almost-$\aleph_0$-fully normal space is countably paracompact. (Received February 18, 1957.)

545. P. S. Mostert: *On one-parameter transformation groups in the plane.*

THEOREM. Let E be the plane and R the real line acting on E as a group of transformations without fixed points. If the space E/R of orbits is Hausdorff, then E is fibred as a direct product of R and a cross sectioning line. That is, the action of R is equivalent to a group of translations. The main tools used in the proof are the local cross section theorem for local orbits and the Jordan curve theorem. (Received February 18, 1957.)

546. D. E. Sanderson: *$H(E^3)$ is uniformly locally arcwise connected.*

By specializing previous results by the author (Bull. Amer. Math. Soc. Abstract 59-6-717) to the case of Euclidean 3-space, E^3, the affirmative answer is given to the following question raised by R. H. Bing at the Summer Institute on Topology at Madison, Wisconsin in 1955: given $\epsilon>0$, does there exist a $\delta>0$ such that if f is a homeomorphism of E^3 onto itself moving no point as much as δ, then there is an isotopy of f to the identity moving no point as much as ϵ? Considering the author's theorem (Bull. Amer. Math. Soc. Abstract 60-4-545) that for any $\epsilon>0$ an arbitrary member of the space $H(E^3)$ of autohomeomorphisms of E^3 is ϵ-isotopic to a piecewise linear homeomorphism, it suffices to prove Bing's conjecture for piecewise linear homeomorphisms. This is accomplished by deforming f onto the identity on an expanding infinite sequence of polyhedral 2-spheres, noting that the requirement of compactness for such a fitting process can be replaced by the requirement of a uniform structure for the 2-spheres and certain neighborhoods involved. The required deformation between the 2-spheres is carried out by an application of a theorem of Alexander (Proc. Nat. Acad. Sci. vol. 9 (1923) pp. 406–407). (Received February 21, 1957.)

547. Patrick Shanahan: *An axiomatic characterisation of the reduced homology theory.*

A homology theory on an admissible category $\mathfrak{A}$ is a covariant ∂-functor defined on $\mathfrak{A}$ satisfying the seven axioms of Eilenberg and Steenrod. (For terminology see Eilenberg and Steenrod, *Foundations of algebraic topology*.) Let H be a homology theory on the admissible category $\mathfrak{A}$ consisting of all pairs of spaces and all maps of such pairs. It is shown that if $\tilde{H}$ is the reduced homology theory associated with H, then $\tilde{H}$ is a covariance ∂-functor on $\mathfrak{A}$ having seven properties analogous to the axioms of Eilenberg and Steenrod. Conversely, if H is a covariant ∂-functor defined on $\mathfrak{A}$ and enjoys the properties alluded to above, then $\tilde{H}$ is isomorphic to the reduced theory of some homology theory on $\mathfrak{A}$. The properties thus serve to axiomatize the reduced theory. They are closely allied to the seven axioms of Eilenberg and Steenrod; in fact the basic difference is in the dimension axiom. (Received February 20, 1957.)

J. W. T. YOUNGS
Associate Secretary

THE APRIL MEETING IN BERKELEY

The five hundred thirty-fifth meeting of the American Mathematical Society was held on April 20, 1957, at the University of California in Berkeley. Registrants numbered 144, including 124 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor A. L. Whiteman of the University of Southern California delivered the main address, on *Recent developments in the theory of cyclotomy*. He was introduced by Professor D. H. Lehmer.

Presiding over the sessions for contributed papers were Professors C. B. Allendoerfer, Leon Henkin, A. F. Moursund, and Bertram Yood. The abstracts of these papers are appended. Those having the abstract number followed by the letter "*t*" were read by title, the others in person. For papers having more than one author and presented in person, the letter (p) follows the name of the author who presented it. Dr. Cordes and Dr. Helmberg were introduced by Professor Klee.

ALGEBRA AND THEORY OF NUMBERS

548*t*. S. Chowla and E. G. Straus: *On the lower bound in the Cauchy-Davenport theorem.*

Let A, B be sets of residues (mod p) where p is prime. Let $A+B=\{a+b \mid a\in A, b\in B\}$ and let $n(S)$ denote the number of elements in S. The Cauchy-Davenport theorem states $n(A+B)\geq \min\{p, n(A)+n(B)-1\}$. In the present note it is shown that if $n(A+B)<p-1$ then the lower bound $n(A)+n(B)-1$ is attained only if $n(A)=1$ or $n(B)=1$ or if A, B are in arithmetic progression with the same difference between consecutive terms. This theorem is applied to several number theoretic problems. For example, it is proved that if $(p-1, k)<(p-1)/2$ then every residue (mod p) can be expressed as the sum of no more than $[k/2]+1$ kth powers. (Received April 12, 1957.)

549. W. J. Coles: *On a theorem of van der Corput on uniform distribution.*

Van der Corput has shown (Acta Math. vol. 56 (1931) pp. 373–456) that the two-dimensional sequence $\{P_n\}$, $P_n=(\alpha_n, \beta_n)$, is uniformly distributed mod 1 if and only if for all integer pairs (u, v) other than $(0, 0)$ the one-dimensional sequence $(u\alpha_n+v\beta_n)$ is uniformly distributed mod 1. Let $F^{(N)}(x_0, x_1; y_0, y_1)$ be N^{-1} times the number of points of the type $(\alpha_n+a_n, \beta_n+b_n)$ $(n=1, \cdots, N)$, a_n, b_n integral, contained in the rectangle $x_0\leq x\leq x_1$; $y_0\leq y\leq y_1$. Let $D^{(N)}$ be the l.u.b. of $F^{(N)}$ over $0\leq x_0<x_1\leq 1$, $0\leq y_0<y_1\leq 1$. $D^{(N)}$ is the *discrepancy* of the set $P_1, \cdots, P_N$. Similarly one defines the discrepancy of one-dimensional sets. Let $D_{uv}^{(N)}$ be the discrepancy of the set of points $(u\alpha_n+v\beta_n)$ $(n=1, \cdots, N)$. *Theorem*: There is an absolute constant $\mathfrak{C}$ such that $D^{(N)}\leq \epsilon+\mathfrak{C}(D_{10}^{(N)}+D_{01}^{(N)}+\sum_{(u,v)=1, v>0} f_{uv}(\epsilon)D_{uv}^{(N)})$ for any $\epsilon>0$, where $f_{uv}(\epsilon)$

$= \min(1/|uv|, 1/|\epsilon uv|^2)$. Since the series on the right converges uniformly in N (for fixed ϵ), van der Corput's sufficiency criterion follows from this quantitative criterion. The theorem is deduced from a similar theorem about an integrable function $r(x, y)$ periodic mod 1 in x, y. Also obtained is the following: there is an absolute constant $\mathfrak{C}$ such that $|D| \leq \mathfrak{C}(\epsilon + \sum_{(u,v)=1, v>0} \sum_{1 \leq u \leq 1/\epsilon, 1 \leq v \leq 1/\epsilon} D_{uv}/|kuv|)$ for any $\epsilon > 0$. (Received January 14, 1957.)

550. Lawrence Goldman and A. P. Hillman (p): *On statements of Poincaré and Schlesinger on linear differential equations.*

Let $\mathfrak{K}$ be the field of rational functions of x with complex coefficients. Let $A(y)=0$ be a noncomposite homogeneous linear differential equation (h.l.d.e.) of order n with coefficients in a field $\mathfrak{F}$ normal over $\mathfrak{K}$. Let $A_1(y)=0, \cdots, A_s(y)=0$ be the maximum set of distinct equations obtained by applying automorphisms of $\mathfrak{F}$ over $\mathfrak{K}$ to $A(y)=0$. Poincaré (Acta Math. vol. 4 (1884) p. 205) stated without proof that every solution of $A(y)=0$ satisfies an h.l.d.e. $B(y)=0$, with coefficients in $\mathfrak{K}$, of order ns. Schlesinger (*Handbuch der Linearen Differentialgleichungen*, vol. 1, pp. 217–219) gave a proof assuming that there cannot exist $y_1, \cdots, y_s$, not all zero, in an extension of $\mathfrak{F}$, such that $A_i(y_i)=0$ and $y_1+\cdots+y_s=0$. Although Poincaré's statement is correct, the following shows that such y_i may exist: Let $\alpha^3+x\alpha^2+(x-3)\alpha-1=0$, $\mathfrak{F}=\mathfrak{K}(\alpha)$, and $A(y) = y'+y[\alpha'(2\alpha+1)-\alpha^2-\alpha]/3\alpha(\alpha+1)$. One may choose $y_1=[e^x/\alpha(\alpha+1)]^{1/3}$, $y_2=\alpha y_1$, $y_3=-(\alpha+1)y_1$. $B(y)$ exists of order 2. Sets of h.l.d.e. with dependent solutions are investigated in another paper. (Received February 18, 1957.)

551. C. V. Holmes: *Automorphisms of monomial groups.*

The automorphisms of monomial groups $\sum_n(H)$ consisting of finite substitutions were determined by Professor Oystein Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. vol. 51 (1942). If the restriction that the given set be finite be removed, monomial groups with infinite substitutions result. Denote such monomial groups by $\sum(H; B, C, D)$, where $d \leq C$, $D \leq B^+$, $d = \aleph_0$, B denoting the order of the given set which together with the group H generate the monomial group, C a cardinal such that all substitutions of the group have fewer than C factors different from the identity of H, D a cardinal such that each monomial substitution of the group maps fewer than D elements of the given set onto elements distinct from themselves, B^+ the successor of B. It is shown that any automorphism of the group $\sum(H; B, d, d)$ is the product of two automorphisms, an inner automorphism of the complete monomial group $\sum(H; B, B^+, B^+)$, and an automorphism T^+ which is generated by an automorphism T of H, T^+ maps any substitution into the same substitution with factors h_α replaced by $h_\alpha T$. A similar result is obtained for monomial groups $\sum(H; B, d, C)$, where $d < C < B^+$. In addition the automorphism groups of these monomial groups are determined. (Received February 20, 1957.)

552. C. T. Long: *Note on normal numbers.*

This note contains an easy proof of the fact that a real number α is normal to base r if and only if there exist positive integers $m_1 < m_2 < m_3 < \cdots$ such that α is simply normal to base r^{m_i} for each $i \geq 1$. It is also proved that if $m_1, m_2, \cdots, m_s$ is an arbitrary collection of distinct positive integers, then there exists at least one real number simply normal to each of the bases $r^{m_1}, r^{m_2}, \cdots, r^{m_s}$ but not normal to base r. (Received February 20, 1957.)

553t. R. C. Lyndon: *On positive formulas in the predicate logic.*

Let L be the first-order predicate logic with both predicates and operation symbols, and possibly the predicate $=$, without special identity axioms. Consider L-formulas $A, B, C, \cdots$ in prenex normal form. A predicate ϕ is said to occur positively, or negatively, in A if some occurrence of ϕ in A is not, or is, preceded by the negation symbol. A is called positive in ϕ if ϕ does not occur negatively in A. Using Gentzen's extended Hauptsatz one obtains: *Theorem* 1. If $A \rightarrow B$ is L-provable, then, for some C, both $A \rightarrow C$ and $C \rightarrow B$ are L-provable, and no predicate occurs positively, or negatively, in C unless it so occurs in both A and B. This improves a result of W. Craig (Harvard thesis). Let ϕ_i, ϕ_i', for $i = 1, 2, \cdots, p$, be n_i-place predicates, and $I(\phi_i, \phi_i')$ the formula stating that $\phi_i x_1 \cdots x_{n_i}$ implies $\phi_i' x_1 \cdots x_{n_i}$ for all $x_1, \cdots, x_{n_i}$. *Theorem* 2. If $A \wedge I(\phi_1, \phi_1') \wedge \cdots \wedge I(\phi_p, \phi_p') \rightarrow B$ is L-provable, where A does not contain the ϕ_i' nor B the ϕ_i, then, for some C, both $A \rightarrow C$ and $C \wedge I(\phi_1, \phi_1') \wedge \cdots \wedge I(\phi_p, \phi_p') \rightarrow B$ are L-provable, and C is positive in $\phi_1, \cdots, \phi_p$ and contains only predicates occurring in A. If, moreover, B results from A by replacing each ϕ_i by ϕ_i', then $A \leftrightarrow C$ is L-provable. (Received February 21, 1957.)

554. R. C. Lyndon: *On positive classes of relational systems and algebras.*

For terminology see Tarski, Indigationes Mathematicae 16, p. 572, and the preceding abstract. A relational system $\langle B, S_1, \cdots, S_n \rangle$ is called an enlargement of a system $\langle A, R_1, \cdots, R_n \rangle$ if $A = B, R_1 \subseteq S_1, \cdots, R_n \subseteq S_n$. Given a class K of (similar) relational systems, $E(K)$ denotes the class of all enlargements of systems in K; $R(K)$ denotes the class of all systems isomorphic to residue-class systems of systems in K; $A(K)$ denotes the intersection of all arithmetical classes that include K. An arithmetical class K (in the wider sense) is called (i) relationally positive, (ii) equationally positive, or, simply, (iii) positive if it can be characterized by a set of L-formulas which are positive (i) in all predicates different from $=$, (ii) in $=$, or (iii) in all predicates. Theorem 2 of the preceding abstract implies: *Theorem* 1. For every arithmetical class K, $AE(K)$ is relationally positive, $AR(K)$ equationally positive, and $ARE(K)$ positive. *Corollary* 2. An arithmetical class K is (i) relationally positive iff $E(K) \subseteq K$, (ii) equationally positive iff $R(K) \subseteq K$, (iii) positive if $E(K) \cup R(K) \subseteq K$. Applied to a class K of algebras, 2(ii) yields a result stated without proof by Łoś (*Mathematical interpretations of formal systems*, Amsterdam, 1955, p. 107). *Theorem* 3. There are arithmetical classes K such that $E(K)$, $R(K)$, $RE(K)$ are not arithmetical. (Received February 21, 1957.)

555t. R. L. San Soucie: *Generalized weakly standard rings.*

If R is an arbitrary ring, not necessarily associative, define $K = [k \in R \mid (k, x, y) = (x, y, k) = 0$, all $x, y \in R]$. We call R *nuclear* if and only if $(x, y) \in K$, $(x, y, x) \in K$, all $x, y \in R$. If R is either commutative or associative, then R is nuclear. The main theorem of this paper is a partial converse: (*) A prime ring is nuclear if and only if it is either associative or commutative. Since a primitive ring is prime, and since a simple ring R with $R^2 \neq 0$ is also prime, we have: (**) Primitive rings and simple rings are nuclear if and only if they are either associative or commutative; a semi-simple nuclear ring is a subdirect sum of primitive nuclear rings. The class of nuclear rings includes the weakly standard rings of the author [Amer. J. Math. vol. 79 (1957) pp. 80–86], the accessible rings of Kleinfeld [Canadian J. Math. vol. 8 (1956) pp. 335–340], and the standard algebras of Albert [Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593]. Thus, Theorems (*) and (**) are generalizations of Albert's Theorem 14, Chapter V, and contain Kleinfeld's Theorems 1 and 3, as well as the author's Theorems 1–4, all as special cases. The proof of Theorem (*) does not depend upon the previous work, so the author is giving new proofs of these results in a more general framework. (Received February 6, 1957.)

ANALYSIS

556. W. G. Bade: *Multiplicity of spectral measures.*

Let T be a representation of the algebra of bounded Borel functions on a compact space in the algebra of bounded operators on a separable Banach space X. Recently Dieudonné (J. Math. Pures Appl. (1955) p. 155) has obtained a multiplicity theory for such representations. A new approach is obtained which achieved Dieudonné's results without the assumption of separability. This method also yields considerable new information about the invariant subspaces in X. (Received February 21, 1957.)

557*t*. W. G. Bade: *On unbounded finitely additive measures.*

The following theorem is proved: On every infinite field of sets there may be defined a real valued finitely additive measure whose range is unbounded. This settles in the negative a question raised by Hewitt (Mat. Tidsskr. B. (1951) p. 81) on whether every finitely additive measure whose domain is a σ-field is necessarily bounded. (Received February 21, 1957.)

558. A. V. Balakrishnan: *A generalization of the semigroups of stable distributions of P. Levy.*

Let $\phi_\alpha(it) = (-\cos \pi\alpha/2)|\cos \pi\alpha/2|^{-1}|t|^\alpha \exp -i\pi\alpha t/|t|$ for $-\infty < t < \infty$, $0 < \alpha$, $\cos \pi\alpha/2 \neq 0$. Then it is shown that there is a one parameter semigroup of functions $S_\alpha(t)$, $0 < t < \infty$, for each α, in $L_1(-\infty, \infty)$, such that $\phi_\alpha(it)$ is the logarithm of the Fourier transform of $S_\alpha(1)$. Moreover, for each $f \in L_1(-\infty, \infty)$, $\|S_\alpha(t)*f - f\| \to 0$ as $t \to 0$. For $0 < \alpha \leqq 2$, the $S_\alpha(t)$ are the stable distributions of P. Levy; for $\alpha > 2$, they are no longer distributions. Again, if $T(\xi)$ is a strongly continuous one-parameter group of endomorphisms over a B-space X, with infinitesimal generator A, and $\|T(\xi)\| \leqq M$, then setting for each $x \in X$, $T_\alpha(t)x = \int_{-\infty}^{\infty} T(\xi)xS_\alpha(t, \xi)d\xi$, $0 < t < \infty$, $T_\alpha(t)$ is a strongly continuous semigroup over X. For $x \in D(A^{n+1})$, $n = [\alpha]$, the infinitesimal generator C_α of $S_\alpha(t)$ can be represented $C_\alpha(x) = -((\cos \pi\alpha/2)/\Gamma(-\alpha))|\cos \pi\alpha/2|^{-1} \cdot \int_0^\infty (T(\xi)x - \sum_0^n (A^k x \xi^k/k!))\xi^{-\alpha-1}d\xi$. Among the additional properties proved is the spectral mapping theorem: $\sigma[C_\alpha] = \phi_\alpha(\sigma A)$. (Received February 18, 1957.)

559. F. H. Brownell: *Asymptotic distribution of eigenvalues for the lower part of the Schrödinger operator spectrum.* II.

Consider the eigenvalue problem $-\nabla^2 u(x) + V(x)u(x) = \lambda u(x)$ over $x \in R_n$ and $u \in L_2(R_n)$ with eigenvalues $\lambda_j \leqq \lambda_{j+1}$ repeated according to multiplicity starting with the least and with real eigenfunctions u_j. $N(\lambda) = \sum_{\lambda_j \leqq \lambda} 1$, the number of eigenvalues $\leqq \lambda$ satisfies $N(\lambda) \sim [(2\pi^{1/2})^n \Gamma(n/2+1)]^{-1} \int_{A(\lambda)} [\lambda - V(x)]^{n/2} d\mu_n(x)$, with $A(\lambda) = \{x \in R_n | V(x) \leqq \lambda\}$ and μ_n n-dimensional Lebesgue measure, as the right side $\to +\infty$ with increasing λ. This formula has been proved by Titchmarsh, Ray, and others under a variety of smoothness and growth conditions on $V(x)$, but all assuming $V(x) \to +\infty$ as $|x| \to +\infty$, which implies $\lambda_j \to +\infty$ as $j \to +\infty$. In an earlier abstract (Bull. Amer. Math. Soc. 62-4-566) we reported that this formula held under weaker conditions on $V(x)$ not given in detail. To clarify the record we specify that if $V(x) \to k$

finite real as $|x| \to +\infty$ in such a way that the right side integral $\to +\infty$ as $\lambda \to k$, then, under mild additional conditions on $V(x)$, the formula still holds as $\lambda \to k$, despite the fact that every $\lambda \geqq k$ will be contained in the spectrum and generally the continuous spectrum. (Received February 18, 1957.)

560*t*. F. H. Brownell: *Improved error estimates for the asymptotic eigenvalue distribution of the membrane problem for polygonal boundaries.*

Consider D an open, bounded, connected subset of the euclidean plane R_2, let B the boundary of D be the disjoint union of $p+1$ polygons, D being simply connected if $p=0$. Consider the eigenvalue problem $-\nabla^2 u = \lambda u$ over D, u having continuous second partials there, and u satisfying the Dirichlet or Neumann boundary condition on B. Then $N(\lambda)$, the dimension of the eigenspace corresponding to eigenvalues $\leqq \lambda$, satisfies $N(\lambda) = (4\pi)^{-1}\mu_2(D)\lambda + O(\lambda^{1/2})$ as $\lambda \to +\infty$, where $\mu_2(D)$ is the two-dimensional Lebesgue measure of D. Previously only $O(\lambda^{1/2} \ln \lambda)$ in this relation was known for such D. Similar improvements appear clear for dimension $n > 2$, also. The result comes from applying a trivial modification of the Tauberian theorem of Ganelius (Kungl. Fys. Salls. Lund Forhand. vol. 24, no. 20 (1954)) to the evaluation $\int_0^\infty e^{-t\lambda} dN_\nu(\lambda) = (4\pi)^{-1}\mu_2(D)t^{-1} - (-1)^\nu(8\pi^{1/2})^{-1}l(B)t^{-1/2} + [(1-P)/6 + \sum_k E_\nu(\alpha_k)] + O(\exp(-\delta^2/t))$ as $t \to 0^+$, where $\nu = 0$ or 1 designates the Dirichlet or Neumann boundary conditions respectively, $\delta > 0$, and $E_\nu(\alpha_k)$ depends only on the jump angle α_k of the tangent vector to B at the kth corner. This evaluation is obtained in exact analogue to the evaluation of $\int_\rho^\infty (1/\lambda(\lambda+\omega^2))dN_\nu(\lambda)$ as $\omega \to +\infty$ found previously by the author (Journal Math. & Mech. vol. 6 (1957) pp. 119–166, particularly p. 152.) (Received March 4, 1957.)

561. J. B. Butler, Jr.: *A proof of Rellich's theorem for normal operators.*

Let $A(s)$ be a bounded operator on $\mathfrak{H}$, analytic in the complex parameter s at $s=0$. Wolf proved: If (i) μ_0 is an isolated eiqenvalue of $A(O)$ of multiplicity m, and (ii) $A(s_n)$ is normal on a sequence of points s_n with $\lim_{n\to\infty} s_n = O$, then there exists δ and an open set Δ containing μ_0 such that, for $|s| < \delta$, the spectrum of $A(s)$ inside Δ consists of m analytic functions $\mu_i(s)$, $i = 1, \cdots, m$, and the projections $P_i(s)$ corresponding to the eigenvalues $\mu_i(s)$ are analytic [Math. Ann. vol. 124 (1952)]. A short proof is derived from a theorem of Kato [J. Math. Soc. Japan vol. 4 (1952)] by establishing (I) $\mu_i(s)$ has a branch point at $s=O$ implies $P_i(s)$ has a pole. Further results: (II) (ii) is equivalent to: $A(s)$ is normal on an analytic arc through $s=O$. (III) If $A(s)$ is a closed operator having domain $\mathfrak{D}$ common with $A^*(s)$ and independent of s if $A^{-1}(O)$ is compact, and if $A(s)f$ and $A^*(\bar{s})f$ are analytic for $f \in \mathfrak{D}$, then (ii) implies that the spectrum of $A(s)$ consists of a denumerable sequence of functions $\{\mu_i(s)\}$ defined and analytic on an analytic arc through $s=O$. (Received February 18, 1957.)

562. Paul Civin: *A maximum modulus property of maximal subalgebras.*

Let B be a regular Banach algebra with a unit and with space of maximal ideals $\mathfrak{M}(B)$. Let N be a subalgebra of B which is not a maximal ideal of B and whose Gelfand representation separates the points of $\mathfrak{M}(B)$. Let π denote the mapping which represents B as a collection of continuous functions on $\mathfrak{M}(B)$. Suppose $\pi(N)$ is not dense in $\pi(B)$ and that N is maximal with respect to this property. The space $\mathfrak{M}(B)$

can then be embedded topologically in the maximal ideal space $\mathfrak{M}(M)$ of N. As so embedded $\mathfrak{M}(B)$ is the Šilov boundary of $\mathfrak{M}(N)$ with respect to $\pi(N)$, i.e. $\mathfrak{M}(B)$ is the smallest closed set of $\mathfrak{M}(N)$ on which each function in $\pi(N)$ attains its maximum modulus. (Received December 17, 1956.)

563. H. O. Cordes: *On a theorem of Hurwitz-Radon and certain integral estimates for partial differential operators.*

According to a theorem of Hurwitz and Radon it is always possible to find quadratic forms $q_i(x, x)=\sum_{j,k=1} a^i_{jk}x_jx_k$, $i=1, \cdots, 4$, such that $\sum_{i=1}^4 q_i^2(x, x)=(\sum_{i=1}^4 x_i^2)^2$ for all quadruples (x_1, x_2, x_3, x_4) and the analogous statement holds for some vector-spaces of higher dimensions. It will be shown that this property can be used to give strong estimates of the form $\int_{r\leq r_0}(\Delta u)^2r^\alpha dx\leq c\int_{r\leq r_0}(Lu)^2r^\alpha dx$ where Lu is an elliptic differential operator. These estimates can be used to obtain an existence theorem for the Dirichlet problem of certain quasilinear elliptic differential equations. (Received February 22, 1957.)

564. C. J. A. Halberg, Jr., and H. G. Tucker (p): *Conditions for the Toeplitz summability of all powers of a bounded real sequence.*

Let $A=(a_{nj})$ be a regular (Toeplitz) matrix method of summation and let $x^{(k)}=\{x_j^k\}$ denote the "kth power" of a bounded real sequence. The principal result of this paper is embodied in the following theorem. A necessary and sufficient condition that $Ax^{(k)}$ converge for all k is that there exist a function $F(t)$ which is the limit almost everywhere of a sequence of functions $F_n(t)$, $F_n(t)$ being the sum of all a_{nj} over those j for which $x_j\leq t$. An application of this theorem is given. The sufficiency of the condition is applied to give an extremely simple proof of the strong law of large numbers in the case of bounded, independent, identically distributed random variables. (Received February 4, 1957.)

565t. Erwin Kreyszig: *On a class of systems of partial differential equations.*

Each solution (regular at the origin) of a system (1) $U_{x_\alpha x_\alpha}+U_{y_\alpha y_\alpha}+\phi_\alpha(x_\alpha, y_\alpha)\,U=u_{z_\alpha z_\alpha^*}+F_\alpha(z_\alpha, z_\alpha^*)u=0$, $z_\alpha=x_\alpha+iy_\alpha$, $z_\alpha^*=x_\alpha-iy_\alpha$, $\alpha=1, 2$, can be represented in the form (2) $u=\int_{-1}^1\int_{-1}^1E_1(z_1, z_1^*, t_1)\times E_2(z_2, z_2^*, t_2)[f_1\{z_1(1-t_1^2), z_2(1-t_2^2)\}+f_2\{z_1(1-t_1^2), z_2^*(1-t_2^2)\}](1-t_1^2)^{-1/2}(1-t_2^2)^{-1/2}dt_1dt_2$ where f_1, f_2 are analytic functions of 2 complex variables regular at the origin. Cf. Bergman and Schiffer (Studies, pres. to R. v. Mises, 1954). A system (1) is said to be of class $\mathfrak{E}$ if in the representation (2) of its solutions generating functions of the form (3) $\mathfrak{E}_\alpha=\exp(\sum_{\mu=1}^{m_\alpha} q_{\alpha\mu}(z_\alpha, z_\alpha^*)t_\alpha^{2\mu})$, $\alpha=1, 2$, can be chosen. Necessary and sufficient conditions were obtained in order that (1) be of class $\mathfrak{E}$. If (4) $f_1=z_1^\beta z_2^\gamma$, $f_2=z_1^{*\sigma} z_2^{*\tau}$, $\beta, \gamma, \sigma, \tau=0, 1, \cdots$, the corresponding solution of a system (1) of class $\mathfrak{E}$ satisfies 4 ordinary linear differential equations in each of the 4 variables; the coefficients are rational functions of the functions $q_{\alpha\mu}$; the order is independent of β, γ, σ, τ and depends on F_α only. The infinite set of these solutions forms a base of approximation; that is, each solution of a system of class $\mathfrak{E}$, regular in a convex domain $B_{z_1z_2}$ can be approximated by a linear combination of these solutions, uniformly in each closed subdomain of $B_{z_1z_2}$. (Received February 28, 1957.)

566t. Erwin Kreyszig: *On solutions of systems of partial differential equations.*

In the previous abstract an infinite set of independent particular solutions of systems of class $\mathfrak{E}$ was obtained which satisfy ordinary differential equations. There exists another infinite set of solutions having the same property. These solutions correspond to the meromorphic associated functions $f_1(z_1, z_2) = (z_1 - a_k)^{-k}(z_2 - b_n)^{-n}$, $f_2(z_1, z_2^*) = (z_1 - c_p)^{-p}(z_2^* - d_q)^{-q}$, $k, n, p, q = 1, 2, \cdots$, where a_k, b_n, c_p, d_q are arbitrary constants, different from zero (cf. formula (2) of the previous abstract). Resulting theorems: I. Solutions of systems of class $\mathfrak{E}$ with rational associated functions f_1, f_2 satisfy ordinary linear differential equations in each of the 4 variables. II. If a system of class $\mathfrak{E}$ has rational coefficients the singularities of its solutions with rational associated functions lie on algebraic manifolds of the $z_1z_1^*z_2z_2^*$-space. In this way the theory of ordinary differential equations can be used for characterizing properties of solutions of those systems, e.g. with regard to the behavior of the solutions in the neighborhood of singularities and outside of the domain of validity of the representation (2) (cf. the previous abstract). The system $U_{x_\alpha x_\alpha} + U_{y_\alpha y_\alpha} + U = 0$, $\alpha = 1, 2$, is of class E. Choosing $E_\alpha = i(z_\alpha z_\alpha^*)^{1/2}/4_\alpha$ and associated functions (4) (cf. the previous abstract) the corresponding solutions are products of Bessel functions which satisfy ordinary differential equations of the fourth order. (Received February 28, 1957.)

567. T. J. McMinn: *Linear measures of sets of the Cantor type.*

From the closed unit interval remove the concentric open interval of length λ, $0 < \lambda < 1$. From each of the resulting closed intervals of length $(1-\lambda)/2$ remove the concentric open interval of length $\lambda(1-\lambda)/2$, etc. Let $T_{\lambda r}$ be the Cartesian product of the resulting set (the Cantor set for $\lambda = 1/3$) with the uniform magnification of it by a factor $r > 0$. Letting L and G be Carathéodory and Gillespie linear measures respectively, Randolph (J. London Math. Soc., 1941) has shown that $L(T_{\lambda 1}) = G(T_{\lambda 1}) = 0$ or ∞ according to whether $\lambda > 1/2$ or $\lambda < 1/2$ and $3/5^{1/2} \leqq L(T_{1/2\,1}) \leqq 2^{1/2} < 3(2/5)^{1/2} \leqq G(T_{1/2\,1}) \leqq 2$. We obtain $L(T_{1/2\,r}) = (1+r^2)^{1/2}$ and $G(T_{1/2\,r}) = 1 + r$ for $1/2 \leqq r \leqq 2$. We also determine $L(T_{1/2\,r})$ and $G(T_{1/2\,r})$ for $2 < r$ as well as obtain certain other results concerning L and G linear measures of homogeneous linear transformations of $T_{1/2\,1}$. (Received February 4, 1957.)

568. J. S. Maybee: *A family of hyperbolic equations and the Cauchy problem for the wave equation.*

Let $\Box$ represent the wave operator in n-dimensions and let τ^m represent the mth partial derivative with respect to t. The paper studies equations of the form $\tau^m \Box u = f(x, t)$. By introducing two complex parameters μ and ν a Riesz kernel $W(t, x, \mu, \nu)$ is constructed for this equation with the following properties, (i) $\tau^m \Box W(t, x, \mu, \nu) = W(t, x, \mu-1, \nu-1)$, (ii) $\tau^m W(t, x, \mu, \nu) = W(t, x, \mu-1, \nu)$. (iii) $\Box W(t, x, \mu, \nu) = W(t, x, \mu, \nu-1)$ and (iv) W vanishes on the forward light cone with vertex at the origin. A two parameter semi-group of operators $J^{\mu,\nu}f$ is then introduced using the kernel W. It is shown that $J^{0,0}f = f$ and that, if m is chosen sufficiently large, $J^{1,1}f$ is well defined. The operators $J^{\mu,\nu}$ correspond to the Riesz operators introduced by Gårding. The Cauchy problem for the wave equation (which corresponds to the case $m = 0$) with the Cauchy data given on the plane $t = 0$ is solved by using the operator $J^{\mu,\nu}$. The method reproduces the solution of the wave equation given by Volterra for the case $n = 3$. (Received February 18, 1957.)

569. Werner Meyer-Koenig (p) and Karl Zeller: *The Tauberian gap theorem for Taylor's method.*

The series (1) $\sum u_k$ is called summable (T_α), i.e. summable by Taylor's method of order α, if $v_m = (1-\alpha)^m \sum_{k=m}^{\infty} \binom{k}{m} \alpha^{k-m} u_k$ $(m = 0, 1, \cdots)$ and $\sum v_m$ exist, and is called regularly summable (T_α), if it is summable (T_α) and if $\sum u_k z^k$ is regular at the point

$s=\alpha$. Let $0<\alpha\leq 1/2$ and (1) a gap series with $u_k=0$ for $k\neq k_l$ $(l=0, 1, \cdots)$, where the $k_l\geq 0$ are integers with $k_{l+1}-k_l>\theta(k_l)^{1/2}$ for a positive θ. Then if (1) is regularly summable (T_α) it is convergent. The proof uses a functional analytic method recently given by the authors in Math. Zeit. vol. 66 (1956) pp. 203–224. (Received February 20, 1957.)

570. R. S. Phillips: *Dissipative operators and dissipative hyperbolic systems.*

A linear operator L with domain $\mathfrak{D}(L)$ in a hilbert space H is said to be dissipative if $(Ly, y)+(y, Ly)\leq 0$, $y\in\mathfrak{D}(L)$, and to be maximal dissipative (m.d.) if it is not the proper restriction of any other dissipative operator. An operator L is the infinitesimal generator of a strongly continuous semi-group of contraction operators on H if and only if L is a m.d. operator with dense domain. Further if L is m.d. with dense domain then so is L^*. Let Δ be a domain in real E_m and let $y(\chi, t)$ be a function of (χ, t) with values in a complex E_k. Consider the initial value problem: $y_t=E^{-1}[\sum(A^iy)_i+By]$, $\chi\in\Delta$, $t>0$; $y(\chi, 0)=f(\chi)$. Here E, A^i, and B are matrix-valued functions of χ, E is positive definite, the A^i are hermetian, and $B+B^*+\sum A^i_i\leq\Theta$ for each $\chi\in\Delta$. Defining a minimal operator L_0 and a maximal operator L_1 corresponding to the spatial part of the above differential operator, one obtains semi-group generators either as m.d. extensions of L_0 or as m.d. restrictions of L_1. The generators of solutions for which energy neither enters from the interior of Δ nor through the boundary are the m.d. operators L such that $L_0\subset L\subset L_1$. A characterization of such operators is obtained by means of certain "maximal negative" subspaces of boundary data. (Received January 31, 1957.)

571. E. B. Tolsted: *Nontangential limits of subharmonic functions.*

Let $u(r, \theta)$ be a subharmonic function in the unit circle $r<1$, and let $F(\rho, \phi)$, the generalized mass distribution function associated with $u(r, \theta)$, satisfy the condition $\int\int(1-\rho)dF(\rho, \phi)<+\infty$, where the integral is Stieltjes-Radon and is extended over the unit circle $\rho<1$. Littlewood showed that such a subharmonic function has a radial limit at almost all points of the circumference of the unit circle. Priwaloff published a generalization of Littlewood's theorem, allowing nontangential approaches to the circumference, but Tamarkin discovered an error in Priwaloff's proof and Zygmund constructed a counter-example to his theorem. In this paper certain averages of the subharmonic function over small circles are shown to have non-tangential limits, and it is shown that the subharmonic function itself has a non-tangential limit at almost all points on the circumference *provided* that the condition on the mass distribution function is strengthened to $\int\int(1-\rho)f(\rho, \phi)\log^+f(\rho, \phi)d(\rho, \phi)<+\infty$, where $f(Q)$ is a certain non-negative, Lebesgue-integrable function, called the density function associated with $u(r, \theta)$. (Received January 23, 1957.)

APPLIED MATHEMATICS

572. S. D. Conte: *An implicit finite difference approximation to vibration problems in two space variables.*

The basic equation applying to the vibrations of a plate can be written in the form (1) $\Delta\Delta w+w_{tt}=0$ where Δ is the biharmonic operator. This equation is approximated by the finite difference system (2) $2^{-1}\Delta_x^4[w^*_{i,j,n+1}+w_{i,j,n-1}]+2\Delta_x^2\Delta_y^2w_{ijn}+\Delta_y^4w_{ijn}+(w^*_{ij,n+1}-2w_{ijn}+w_{i,j,n-1})/\Delta t^2=0$, (3) $2^{-1}\Delta_y^4[w_{i,j,n+1}+w_{i,j,n-1}]=\Delta_y^4w_{ijn}-(w_{ij,n+1}-w^*_{i,j,n+1})/\Delta t^2$ where for example, $w_{ijn}=w(i\Delta x, j\Delta y, n\Delta t)$, $\Delta_t^2w_{ijn}=(w_{i,j,n+1}-2w_{ijn}+w_{i,j,n-1})/\Delta t^2$, $\Delta_x^4w_{ijn}=(w_{i+2,j,n}-4w_{i+1,j,n}+6w_{ijn}-4w_{i-1,j,n}+w_{i-2,j,n})/\Delta x^4$. The stability and convergence of (2) and (3) is investigated under appropriate boundary and initial conditions. (Received February 15, 1957.)

573. D. H. Hyers (p) and J. A. Ferling: *On the local uniqueness problem for water waves of permanent type*, I.

The question of the existence of periodic surface waves of permanent type in a channel of infinite depth was settled independently by T. Levi-Cività and N. Nekrasov around 1922–1924, for the case of small, but finite, amplitudes. However, little mention was made at that time of the *uniqueness* problem. The purpose of the present article is to initiate a study of the latter, first in the same case of small finite amplitudes. After the initial set-up of the problem, the present methods differ considerably in details from those used in the classical work mentioned above. They spring from some ideas of Lichtenstein, Liapounoff and others, as developed in Friedrich's Lectures on Functional Analysis (N.Y.U., 1950) in connection with bifurcation theory. In addition to the local uniqueness theorem, we also prove, as a sort of by-product, that a conjecture of Levi-Civita (Math. Ann. vol. 93 (1925) p. 284) is true at least locally, where the latter term is defined in a somewhat special sense. (Received February 28, 1957.)

574. H. M. Lieberstein: *On the generalised radiation problem of A. Weinstein.*

The following uniqueness theorem has been proved. Let $u^{(k)}$ be a solution of the two dimensional Euler-Poisson-Darboux (EPD) equation, $u_{xx}^{(k)} = u_{yy}^{(k)} + (k/y)u_y^{(k)}$, such that $u^{(k)}(x, 0) = f(x)$ and $u^{(k)}(x, x) = 0$, $f(x)$ given in the interval $[0, a]$ and consider the triangle T with vertices $(0, 0)$, $(0, a)$, $(a/2, a/2)$. Then there exists at most one $u^{(k)}(x, y)$, $-\infty < k < 1$, having two continuous derivatives on a triangle G whose interior contains T and its sides except for the base line. Solutions of this boundary value problem with certain conditions on $f(x)$ have been given by A. Weinstein [Summa Brasiliensis Mathematical, vol. 3, Fasc. 7, September, 1955]. In another paper the author will construct solutions of a more general mixed problem for which the present uniqueness theorem applies. General solutions of the EPD equation as given by E. K. Blum [Duke Math. J., vol. 21 (1954) pp. 257–274] and Darboux [*Leçons sur la théorie générale des surfaces*, Book IV, vol. 11, Paris, 1914–1915], together with an inversion of a certain integral equation yield the uniqueness for $-2 < k < 1$. It is shown that the uniqueness for all negative k follows through an induction using the Weinstein recursion, $u_y^{(k)} = yu^{(k+2)}$. Certain properties of the general solutions for negative k which are established by using propagation on the characteristic $y = x$ play an important role. Uniqueness for the classical radiation problem [Courant and Hilbert, Mathematische Physik, vol. II], which requires a singular solution of the EPD equation, is included as a special case. (Received February 18, 1957.)

GEOMETRY

575. Albert Nijenhuis: *Differential-geometric operations on rings.*

From a commutative algebra R with unit over a field K the R-module Φ_p $(p = 1, 2, \cdots)$ is defined as the free module with generators $(r, s_1, \cdots, s_p)$; $r, s_1, \cdots, s_p \in R$, modulo relations: (a) "linearity" in each of the $p+1$ entries; (b) "triviality" of elements of K: $(r, s_1, \cdots, s_i, \cdots, s_p)$ for $s_i \in K$; (c) "product rule": $(r, s_1, \cdots, s_i t_i, \cdots, s_p) - (rs_i, \cdots, t_i, \cdots, s_p) - (rt_i, \cdots, s_i, \cdots, s_p)$; (d) "skew-symmetry": $(r, s_1, \cdots, s_i, \cdots, s_j, \cdots, s_p) + (r, s_1, \cdots, s_j, \cdots, s_i, \cdots, s_p)$.

Multiplication by $r(s, t_1, \cdots, t_p) \rightarrow (rs, t_1, \cdots, t_p)$. There is a mapping $\wedge: \Phi_p \times \Phi_q \rightarrow \Phi_{p+q}$ by $(r, s_1, \cdots, s_p) \wedge (t, u_1, \cdots, u_q) \rightarrow (rt, s_1, \cdots, s_p, u_1, \cdots, u_q)$, and a mapping $d: \Phi_p \rightarrow \Phi_{p+1}$ by $d(r, s_1, \cdots, s_p) \rightarrow (1, r, s_1, \cdots, s_p)$; satisfying $d^2 = 0$. The element of Φ_p represented by $(r, s_1, \cdots, s_p)$ is thus $rds_1 \wedge \cdots \wedge ds_p$.—$R$ generalizes the C^∞ functions on a manifold; K the constant functions; Φ_p the p-forms.—The existence of the de Rham cohomology groups for R is now obvious. The derivations of R (relative to K) are in (1, 1) correspondence with the elements of Φ_1^*; vector p-forms are elements of $\mathrm{Hom}_R(\Phi_1, \Phi_p)$; cf. [A. Frölicher-A. Nijenhuis-Indag. Math. vol. 18 (1956) pp. 338–359]; and the theory of derivations on $\Phi = \oplus_{p=0}^{\infty} \Phi_p$ $(\Phi_0 = R)$ and the existence of the bracket operator $[L, M]$ for vector forms are established as in the paper quoted. The de Rham groups, and concepts like almost product rings and almost complex rings need further study. (Received February 18, 1957.)

576. C. E. Watts: *On Riemannian connections.*

Let X be a C^∞-manifold, $C^\infty(X)$ the ring of real-valued C^∞ functions on X. Let T_q^p be the $C^\infty(X)$-module of cross-sections of the vector-bundle over X whose fiber at x is $T_x^* \otimes \cdots \otimes T_x^* \otimes T_x \otimes \cdots \otimes T_x$, where T_x is the tangent space at x, T_x^* its dual, and where T_x^*, T_x appear as factors p times and q times respectively. A Riemannian metric tensorfield induces symmetric $C^\infty(X)$-module isomorphisms $\gamma: T_1^p \rightarrow T_0^{p+1}$. An affine connection is considered as a $C^\infty(X)$-derivation $\delta: T_1^0 \rightarrow T_1^1$, and induces a derivation $D: T_q^1 \rightarrow T_q^p$. Let A^p be the module of exterior p-forms on X, and $a: T^p \rightarrow A_0^p$ the antisymmetrization operator. Then δ is a Riemannian connection iff $\gamma\delta = D\gamma$ and $da = aD$. A coordinate-free existence and uniqueness proof for the Riemannian connection is obtained, and explicit formulas for it are derived, involving only γ and d, or alternatively only γ and the bracket operation in T_1^0. (Received February 20, 1957.)

LOGIC AND FOUNDATIONS

577t. S. Feferman and R. L. Vaught: *Model-theoretic and decidability theorems concerning generalized products.*

For notation see preceding abstract. (I) and (II) include previous work of Mostowski (J. Symbolic Logic vol. 17 (1952) pp. 1–32), Vaught (Proc. Int. Cong. Math. 1954 vol. 2, pp. 409–410), and Feferman (Bull. Amer. Math. Soc. Abstracts 61-2-342 and 61-2-343), and with their aid (1)–(4) and (6), below, have been obtained. (1) *The generalised product and, hence, arbitrary cardinal and ordinal products and sums* (with $\mathfrak{I}$ fixed) *preserve* arithmetical equivalence (cf. Feferman, loc. cit.; for infinite ordinal sums this was found by Fraïssé (Zeit. f. math. Logik u. Grund. der Math. vol. 2 (1956) pp. 76–92)). (2) *If a sentence σ holds in the cardinal sum of $\langle \mathfrak{A}_i | i \in I \rangle$, then for some finite $J \subset I$, σ holds in the cardinal sum of $\langle \mathfrak{A}_i | i \in J' \rangle$ provided $J \subset J' \subseteq I$.* (3) *The decision problem for (the theory of) the generalised power [generalised weak power] of a system $\mathfrak{A}'$ to the exponent $\mathfrak{I}$ $[\mathfrak{I}']$ reduces to the decision problems for $\mathfrak{A}'$ and $\mathfrak{I}$ $[\mathfrak{A}'$ and $\mathfrak{I}'' = \langle P''(I), \subseteq, S_0', \cdots, S_{n-1}' \rangle]$.* Consider any ordinal $\xi = \{\alpha | \alpha < \xi\} \cdot X <' Y$ means for some α, β, $X = \{\alpha\}$, $Y = \{\beta\}$, and $\alpha < \beta$; $+$ and $\cdot$ denote ordinal addition and multiplication. (4) *The decision problems for $\langle \omega^\xi, + \rangle$ and $\langle P''(\xi), \subseteq, <' \rangle$ are inter-reducible.* Write $X \sim Y$ if X and Y have the same power; $j \approx k$ if j and k have the same number of prime divisors. A reduction to decision methods of Presburger (also used for (4)) and Mostowski-Tarski for addition of cardinals yields: (5) *$\langle P''(I), \subseteq, \sim \rangle$ and $\langle P(I), \subseteq, \sim \rangle$ have decidable theories.* (6) *The theory of $\langle \omega, \cdot, \approx \rangle$ is decidable.* (Received February 15, 1957.)

578t. S. Feferman and R. L. Vaught: *Elimination of quantifiers in the theory of generalized products.*

$P(I)$, $P'(I)$, $P''(I)$ denote, respectively, the sets of all (i) subsets (of a set I), (ii) finite subsets or their complements, (iii) finite subsets. Given an indexed family $\langle \mathfrak{A}_i | i \in I \rangle$ of relational systems $\mathfrak{A}_i = \langle A^{(i)}, R_0^{(i)}, \cdots, R_{m-1}^{(i)} \rangle$ and finitary relations $S_0, \cdots, S_{n-1}$ among the members of $P(I)$, the ***generalized product*** $\mathfrak{B}$ of $\langle \mathfrak{A}_i | i \in I \rangle$ relative to $\mathfrak{J} = \langle P(I), \subseteq, S_0, \cdots, S_{n-1} \rangle$ is a system formed by the set B, consisting of all functions f on I such that $f(i) \in A_i$ for each $i \in I$, and by (infinitely many) relations $Q^{\theta}_{\phi_0, \ldots, \phi_{p-1}}$. Each $Q^{\theta}_{\phi_0, \ldots, \phi_{p-1}}$ is determined by a finite list $\phi_0, \cdots, \phi_{p-1}$ of formulas of the theory (= first order theory) of the $\mathfrak{A}_i$, having q free variables, and a formula θ of the theory of $\mathfrak{J}$ having p free variables. Members $f_0, \cdots, f_{q-1}$ of B are in the relation $Q^{\theta}_{\phi_0, \ldots, \phi_{p-1}}$, provided $U_0, \cdots, U_{p-1}$ satisfy θ in $\mathfrak{J}$, where $U_r = \{ i | i \in I$ and $f_0(i), \cdots, f_{q-1}(i)$ satisfy ϕ_r in $\mathfrak{A}_i \}$, $r = 0, \cdots, p-1$. Similarly, the ***generalized weak power*** $\mathfrak{B}' = \langle B', \cdots \rangle$ of a single system $\mathfrak{A}' = \langle A', e, R'_0, \cdots, R'_{m-1} \rangle$ (having a distinguished element e) to the exponent $\mathfrak{J}' = \langle P'(I), \subseteq, P''(I), S'_0, \cdots, S'_{n-1} \rangle$ is defined, where $S'_0, \cdots, S'_{n-1}$ are finitary relations among the members of $P''(I)$, and B' is the set of all functions f on I to A' such that $\{ i | f(i) \neq e \}$ is finite. Two theorems, (I) and (II) are derived. (I) shows effectively that the set of relations of $\mathfrak{B}$ is closed under the Boolean and projective operations, (II) the same for $\mathfrak{B}'$. (Received February 15, 1957.)

STATISTICS AND PROBABILITY

579. R. K. Getoor: *Additive functionals of a Markov process.*

Functionals of the form $\int_0^t v[x(\tau)]d\tau$ are considered where $x(t)$ is a temporally homogeneous Markov process in E^n and v is a non-negative measurable function on E^n. In investigating the distribution of such a functional one considers $r(t, x, y) = E\{\exp(-u\int_0^t v[x(\tau)]d\tau) | x(0) = x; x(t) = y\} p(t, x, y)$ where $p(t, x, y)$ is the transition density of $x(t)$ with respect to some Radon measure, m, which we assume exists. It is "intuitively clear" that if p satisfies a diffusion equation $\partial p/\partial t = \Omega p$ then r satisfies the equation $\partial r/\partial t = (\Omega - uv)r$. The purpose of this paper is to give an exact meaning to above statement under certain regularity assumptions on p. The first step in such a procedure is to give an unequivocable definition of the conditional expectation appearing in the definition of r. In accomplishing this we prove some theorems concerning conditional distributions of Markov processes which may be of some independent interest. (Received February 11, 1957.)

580. Herman Rubin: *Estimation of the location of a change of parameter.* Preliminary report.

Suppose for $t > 0$ the random variable $X(a+t) - X(a)$ is a Poisson process with mean βt and $X(a) - X(a-t)$ is an independent Poisson process with mean γt, β, γ known, and $X(t) - X(u)$ observable for all t, u. This problem is treated as a translation parameter problem, and optimal invariant estimates obtained. This can be applied to the problem of estimating the location of a discontinuity in density (H. Chernoff and H. Rubin, *Proc. Third Berkeley Symposium in Probability and Statistics*). It is seen that, apart from a homogeneity factor, the distribution of the estimates depends only on β/γ, and hence the estimates are superefficient, as already noted in the above reference. Also the assumption that β, γ are known is unnecessary. (Research performed under OOR contract.) (Received February 20, 1957.)

TOPOLOGY

581*t*. L. W. Anderson: *Complements in topological lattices.*

If L is a distributive lattice with 0 and 1 and if a and b are elements of L such that $a \vee b = 1$ and $a \wedge b = 0$ then the function $x \rightarrow (a \wedge x, a \vee x)$ is a lattice isomorphism of L onto $(a \wedge L) \times (a \vee L)$. Moreover, if L is a topological lattice then this function is a homeomorphism. *Corollary*: If L is a locally compact connected subset of the Euclidean plane and if L is a topological lattice with 0 and 1 and if a and b are elements of L distinct from 0 and 1 such that $a \vee b = 1$ and $a \wedge b = 0$ then L is topologically and lattice theoretically the same as the closed unit square endowed with the usual lattice operations. (Received February 11, 1957.)

582*t*. Anatole Beck: *On invariant sets.*

The solution of the following problem is announced: Given a one (real)-parameter group of homeomorphisms of the Euclidean plane onto itself, then the set of all points which are fixed under all these homeomorphisms is a closed set. It is wished to know precisely which sets can be obtained in this way. The answer is: All closed sets. It is found, in fact, that this result is true in a very broad collection of metric spaces, among which are all the metric real linear spaces. It is shown that in any metric space, every closed set (including the empty set) can be represented as the set of fixed points of some one-parameter group of homeomorphisms if and only if the empty set itself can be so represented and also that for some metric spaces (including the unit disc, the two-sphere, and the projective plane), the only closed set failing such a representation is exactly the empty set. (Received February 6, 1957.)

583. Gilbert Helmberg: *A theorem on equidistribution in compact groups.*

The paper gives a generalization of results contained in a paper by Beno Eckmann, *Über monothetische Gruppen*, Comm. Math. Helv. vol. 16 (1944). The following theorem is proved and discussed with respect to its consequences for the generation of a group and its subgroups by a set of elements: "Let G be a compact group with a countable base and $\{R^{(\lambda)}\}$ $(\lambda \in \omega)$ a complete system of inequivalent irreducible unitary representations of G, where $R^{(\lambda_0)}$ is the identity representation; furthermore let g_k $(k = 1, 2, \cdots, n)$ be n elements of G such that for each $\lambda \neq \lambda_0$ there is at least one element $g_{\bar{k}}$ for which the determinant $|R^{(\lambda)}(g_{\bar{k}}) - E^{(\lambda)}| \neq 0$ $(E^{(\lambda)} =$ identity matrix in $R^{(\lambda)})$. Then the set $G' = \{g : g = g_1^{i_1} g_2^{i_2} \cdots g_n^{i_n}, 0 \leq i_k < +\infty, k = 1, 2, \cdots, n)\}$ is equidistributed in G." (Received February 20, 1957.)

584*t*. L. E. Ward, Jr.: *Completeness in semi-lattices.*

A *semi-lattice* is a pair $(X, \leq)$ where X is a set, $\leq$ is a partial order on X and glb (x, y) exists for each $(x, y) \in X \times X$. The semi-lattice $(X, \leq)$ is *complete* if glb A exists for each $A \subset X$. Lattice-theoretical results of Frink, Tarski, and Davis are generalized as follows: I. For the semi-lattice $(X, \leq)$ to be complete, it is necessary and sufficient that, for each $x \in X$, the set $\{a : a \leq x\}$ be compact in the interval topology. II. Let $(X, \leq)$ be a semi-lattice, compact in its interval topology. If $f: X \rightarrow X$ is an isotone function, then the set P of fixed points of f is nonempty; further, $(P, \leq)$ is a complete semi-lattice. III. For the semi-lattice $(X, \leq)$ to be compact in its interval topology it is necessary and sufficient that every isotone $f: X \rightarrow X$ have a fixed point. (Received February 18, 1957.)

585. R. L. Wilder: *Montone mappings of manifolds*. II.

In an earlier paper [see Summary of Lectures and Seminars, Summer Inst. on Set Theoretic Topology, Madison, 1955, pp. 30–32] it was shown that an $(n-1)$-monotone image of an orientable n-gcm is (if nondegenerate and finite dimensional) an orientable n-gcm of the same homology type. It is the purpose of the present paper to extend this result, both to the orientable noncompact case and the locally orientable case, using proper mappings (i.e., such that counter-images of compact sets are compact). The orientable noncompact case does not offer great difficulty. For the locally orientable case the additional assumption must be made that for each point x of the image space, the counter-image of x lies in an orientable portion of the original space; the image-space is then a locally orientable n-gm of the same homology type as the original space. To show that the additional assumption is necessary, even in the compact case, an example is given of a classical 3-manifold S and a mapping f of S of the desired monotone type onto a space S' where S' is not even a generalized manifold. (Received February 19, 1957.)

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LOCALLY TAME CURVES AND SURFACES IN THREE-DIMENSIONAL MANIFOLDS[1]

O. G. HARROLD, JR.

1. Introduction. The program to be described here is concerned primarily with an imbedding problem in the topology of 3-manifolds. A preliminary remark or two will relate this problem with those of a more general nature.

The compact 1-manifold without boundary, i.e. the simple closed curve, has long been characterized both as a subset of the plane and as an abstract space. The Jordan curve theorem and its converse, due to Schoenflies, accomplish the former. The latter is accomplished, for instance, by the theorem of Wilder [29][2] that among the locally compact, locally connected continua, the simple closed curve is distinguished by the fact that for any pair of points A and B, the continuum is a union of two irreducibly connected sets from A to B having in common only these points.

The compact orientable 2-manifolds without boundary likewise admit characterizations both as subsets of three-space, say, and as abstract spaces. The first step in the direction of the former was taken by Brouwer around 1912 and completed by Wilder in 1930. For citations to the rather extensive literature relating to the latter we refer to van Kampen's article [21]. Specifically, however, it may be pointed out that among the Peano spaces, the topological 2-sphere is characterized elegantly as the set satisfying the Jordan curve theorem nonvacuously. Zippin established the result in this form and gave an analogous characterization of the closed 2-cell [31]. Had the first characterization of the 2-sphere as an abstract space followed its characterization as a subset of three-space, one would be tempted to feel that perhaps a characterization of the 3-cell or 3-sphere along the lines of Zippin's work for dimension 2 would await the characterization of the 3-cell, say, as a subset of three-space. There are two characterizations of the 3-sphere or 3-cell as abstract spaces published to date that we are aware of. One is due to Bing [5], the other due to Woodard [30]. Bing has used his characterization in the solution of several problems.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

2. The Schoenflies extension problem. It will be recalled that Schoenflies not only contributed a converse to the Jordan curve theorem in the plane but also gave a proof of the theorem itself. The basic idea used in the proof of the Jordan theorem was the ultimate in simplicity. If a homeomorphism h is given from a standard circle to a curve J in the same plane π in such a way as to preserve orientation, then there is a homeomorphism of π onto itself, call it H, which is an extension of h. Since a standard circle separates the plane and is the common boundary of each of its residual domains, the same properties are established for J via the homeomorphism H. The problem, then, reduces to showing that h has an extension H. The solution is referred to as the Schoenflies extension theorem.

In his thesis Antoine [4] considered many phases of the corresponding extension problem in three-space. Among the numerous examples in his paper one of the most remarkable is that of a compact, totally disconnected perfect set, i.e. a Cantor set, call it P, such that if S is any topological 2-sphere in three-space whose interior and exterior both meet P, then S meets P. Since every Cantor set in three-space is known to lie on some arc, this example implies there are arcs in three-space that cannot be thrown onto a linear interval by a homeomorphism acting on the whole space. (The explicit construction of such Cantor sets in n-space, $n \geq 3$, and proofs of their properties have been carried out by Blankenship [8]. Thus "wild" arcs exist in every euclidean n-space, E^n, $n \geq 3$.)[3]

Generally, the problem Antoine sets for himself is this: Given a homeomorphism h between two sets A and B, does an extension of h exist to some pair of neighborhoods U and V of A and B, respectively? An interesting situation is that in which U and V are all of 3-space. In this fortuitous circumstance A and B are equivalent under the group of homeomorphisms of three-space. In general, if the required neighborhoods U and V exist, it will not be possible to extend the homeomorphism to all of space. The case in which A is the boundary of a plane triangle and B is a polyhedral trefoil knot suggests that in some cases h may be extended to neighborhoods U and V of A and B, but neither U nor V may be taken to be all of 3-space. Considerations of the knot-group may be used to establish this latter fact. By means of the set P referred to above, we see there are cases in which not only may we fail to take U and V as all of three-space but U and V fail to exist at all as neighborhoods of A and B, respectively. In a sense one is not surprised about the failure of extension possibilities for polyhedral *closed* curves A and B due to global differences in the manner in which they are imbedded. However, the failure of the extension to exist for any neighborhoods U and V of the nonpolyhedral arc A and interval B showed a type of phenomenon entirely unexpected at that time. Antione also includes in his examples a topological 2-sphere A such that any homeomorphism of A onto a standard 2-sphere cannot be extended to all space. Thus the Schoenflies method of proof of the Jordan curve theorem does not admit generalization to higher dimensions.

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[3] The symbol E^n stands for euclidean n-space with a fixed rectangular coordinate system.

3. **The extension theorem for polyhedra.** In 1924 J. W. Alexander published three short notes in the Proceedings of the National Academy dealing with these matters [1; 2; 3]. In the second of these notes a "horned" sphere is constructed failing to have the extension property. In the third note further properties of one of Antoine's sets are described.

In the first note Alexander proves that if a polyhedron in 3-space is a topological 2-sphere then both the interior and the exterior (in a compactified space) have closures that are closed 3-cells. (As a corollary to the proof, one finds, almost immediately, that a polyhedral torus always has at least one unknotted complementary domain.) Progress on certain problems of 3-dimensional topology such as triangulation theorems, limited analogues of the Schoenflies theorem, all of recent date, depend at some stage on the result for polyhedral 2-spheres. In the thirty-three years that have elapsed since this was attained no generalization of this mild sounding result to higher dimensions has been forthcoming. It might be noted in passing that Bing's characterization of 3-space among Peano spaces by means of sequences of partitionings does *not* use this result.

4. **Homotopy in the complement and consequences.** In their paper on *Uniform local connectedness and contractibility*, Eilenberg and Wilder [9] consider the consequences of postulating homotopy uniform local connectedness of the complement of a topological $n-1$ sphere in n-space. They show that each component of the complement is in this case simply connected. (Of course uniform local connectedness in the homology sense over a proper coefficient group is always present in the appropriate dimensions.) Thus the domains complementary to Alexander's horned sphere are not both uniformly locally connected in the homotopy sense and simple alterations of the construction can destroy the contractibility of either domain.

The above mentioned results gave some hope that suitable homotopy properties of a domain complementary to a topological $n-1$ sphere might lead to an extension theorem. That this hope was futile was shown by the paper by Fox and Artin on *Some wild cells and spheres in 3-dimensional space* [10]. In this paper an example is given of a topological 2-sphere whose complement is an open 3-cell, the closure of the complement is not a closed 3-cell. This shows, rather pointedly, that the homotopy properties of the complement, at least "in the large," cannot be expected to yield an analogue of the Schoenflies extension theorem. The wildness of the examples in the Fox and Artin paper is established by destroying the usual homotopy properties of the complement either in the large, in the small, or both. To establish the quoted properties for their sets the authors bring into play the knot group of the set in question. Since all of the complements studied in that paper require an infinite set of generators, their techniques were essentially new in this field. (The first application of such group techniques to topology wherein the group requires an infinite set of generators and the presentation is given explicitly is due to Newman and Whitehead [26] as far as the author is aware.)

Let us recall the terminology introduced in [10].

Let K be a set in a geometric complex C. Then K is said to be *tamely imbedded* in C if and only if there is a homeomorphism of C onto itself throwing K onto a polyhedron. In the contrary case K is called *wildly* imbedded in C. The set K is called *locally tamely* imbedded at p provided there is a neighborhood N of p and a homeomorphism h_p of the closure of $N(=\overline{N})$ onto a polyhedron in C such that $h_p(\overline{N}\cap K)$ is a polyhedron. If K is locally tamely imbedded at every point, call K *locally tame*. Clearly, if K is tamely imbedded in C, it is locally tamely imbedded. The converse proposition is established in [6] and [25] for $n=3$.

The aim of the present program, in essence, is to discover when a particular topological type is locally tamely imbedded.

In 1949 the author considered complements of cells in n-space having uniformly abelian local fundamental groups [15]. Roughly speaking, this property means that small paths determining a product in the sense of the path group commute on a set of small diameter and uniformly so.

The principal result states: Let C be a closed, topological i-cell, $i=1, 2, \cdots, n$ in the euclidean n-sphere, S^n, such that $S^n\backslash C$ has uniformly abelian local fundamental groups then $\pi_1(S^n\backslash C)$ is trivial. An inspection of the method of proof for $i=n-2$ is instructive. For $n=3$ a simple arc such that each point of the arc has neighborhoods satisfying the stipulated conditions is shown to have a simply connected complement. A glance at the proof, however, shows that the local condition may be suppressed for *one* end-point. Consequently, an arc has a simply connected complement provided that each point, save possibly one end-point, has a local basis consisting of spherical neighborhoods, each having a closure that meets the arc in a connected polygon. Example 1.2 of [10] shows the arc may be wild.

A set is called *locally polyhedral* at p if there are arbitrarily small neighborhoods of p whose closure meets the set in a finite polyhedron. The above result may then be re-stated: An arc that is locally polyhedral save for one end-point has a simply connected complement. It is natural to ask if "almost" locally polyhedral *spheres* have complementary domains that are simply connected.

Moise and the author [20] showed that if a topological 2-sphere is locally polyhedral at all except possibly 3 points, at least one complementary domain is simply connected. If at most two points are exceptional, either both domains are simply connected or one domain has a closure that is a closed 3-cell. If only one point is exceptional, both domains are simply connected and at least one of them has a closure that is a closed 3-cell. The 3-cell, of course, need not be tame.

5. **Covering an arc by almost polyhedral spheres.** Inasmuch as a topological 2-sphere that is locally polyhedral save for 1 or 2 points has complementary domains much like those of a polyhedral 2-sphere (even if it is wildly imbedded) it seemed that if an arc would possess coverings by topological spheres of arbitrarily small diameters that are almost polyhedral one might hope to enclose the arc by a strictly polyhedral sphere lying near the arc. That is, if the covering 2-sphere associated with a point is required to be locally polyhedral save where it meets the arc and, if, at such common points, other such spheres exist enclosing these points of entry and exit along the arc, then the union of a finite collection of such spheres ought to have an unbounded complementary domain whose frontier is *strictly* polyhedral, and, if the spheres are properly related, this frontier should be a polyhedral 2-sphere. In view of the examples in [10] some restriction on the *cardinal* of the set wherein the almost polyhedral sphere meets the arc is necessary. The condition on the cardinality turns out to be the essential condition. By the approximation theorem of Bing [7], the almost polyhedral nature of the enclosing spheres may be proved in E^3.

6. **Locally peripherally unknotted curves and surfaces.** Let C be a topological k-manifold with or without boundary in a euclidean space E^n. For $k=0$ we consider C to be locally peripherally unknotted and

proceed inductively: *C is said to be locally peripherally unknotted at x provided that for each positive number ϵ there exists a closed n-cell L of diameter less than ϵ whose interior contains x and such that the common part of C with the boundary of L is a locally peripherally unknotted $k-1$ cell or $k-1$ sphere according as x lies on the boundary of C or not.* This property is evidently invariant under homeomorphisms of E^n on itself. It implies, for $k=1$, $n=3$, what was called property $\mathcal{P}$ in [17]. A strictly polyhedral C in E^3 has the property.

7. **The enclosure property.** If C is a set in E^n and ϵ positive it is convenient to say C has the *enclosure property* provided that there is a polyhedral $n-1$ sphere in the ϵ-neighborhood of C whose interior contains C. In the compactified 3-space any set with an open 3-cell complement has this property and conversely for compact C. For arcs we may summarize: If C is a tame arc, C is locally peripherally unknotted; if C is a locally peripherally unknotted arc, C has the enclosure property. Neither implication may be reversed. (See Examples 1.4 and 1.2 of [10].)

8. **Locally unknotted curves and surfaces.** Let M denote a closed k-cell and ∂M its combinatorial boundary. Suppose N is a closed $k-1$ cell contained in M such that $(\partial M)\cap N$ is a $k-2$ sphere and under some parameterization of M, N is a parameter $k-1$ cell in M. We say shortly that N *spans* the boundary ∂M of M. For $k=2$ every arc N in M whose intersection with the boundary is a pair of points, contains an arc that spans ∂M.

If C is a topological k-manifold in E^n with or without boundary, call *C locally unknotted at x if and only if there is some $k+1$ cell D such that $C\cap D$ is the closure of a neighborhood of x in C that lies on a spanning k-cell of ∂D.* If $n=3$, $k=1$, this becomes what was called property $\mathcal{Q}$ in the local form [18].

If C is a simple closed curve in E^3 that bounds any 2-cell, it is locally unknotted at every point. Suppose F is a 2-cell and $\partial F=C$. First we move F slightly by a homeomorphism on F, keeping the points of C fixed so that the image F' of F is locally polyhedral at points[4] of $F'\backslash C$. Let F' become inflated to obtain a 3-cell G, again leaving the points of C fixed, and G locally polyhedral at points of $G\backslash C$. (The first step is justified by Bing's approximation theorem, the second step elementary on account of the polyhedral character of F'). For any point x of C we may draw the desired 2-cell D on ∂G.

Now, if it is also known that C is locally peripherally unknotted, then by Theorem VII of [18], C is tame and in fact bounds a tame 2-cell. Thus C is unknotted in the classical sense. Since locally tame arcs and simple closed curves are evidently locally unknotted and locally peripherally unknotted, we may say *a 1-manifold C is locally tame if and only if it is both locally unknotted and locally peripherally unknotted at every point.*

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[4] The complement of C in F' is denoted by $F'\backslash C$.

In the first proof of the theorem to the effect that a curve that is locally unknotted and locally peripherally unknotted is locally tame, a 2-cell D that is locally polyhedral modulo[5] its boundary is postulated and used for a complicated auxiliary construction. The embedding of D itself was not studied. In a later work (unpublished) the authors showed that D is tame, and, as a corollary that $C=\partial D$ is tame [19]. (This result may also be gotten from Theorem VII of [18] and a theorem of Moise on "smoothing an annulus" [25].)

At this point we have attained a characterization of the 1-manifold and a fragmentary result for surfaces. The next sequence of steps seems clear. One might seek a more general result for surfaces or one might hope to tame more general 1-dimensional sets. Both steps will be necessary before attacking the general 2-dimensional complex.

In connection with the Example 1.4 referred to we have a sample of the difficulties to be expected when the imbedding of more general 1-dimensional sets than 1-manifolds is undertaken. This arc is a union of two tame arcs having in common only an end-point of each arc. At the end-point the union is locally knotted, hence the theorem about locally tame sets being tame cannot be applied. It is natural to expect that conditions specifying a tamely imbedded finite graph should be closely related to the conditions obtained for an arc or simple closed curve. Since an arc or simple closed curve is tame if and only if every sub-arc is tame, one might wonder if this condition would be necessary and sufficient for the tame imbedding of a finite graph. P. Doyle[6] has recently exhibited an example of a triod such that every sub-arc is tame, the complement is an open 3-cell, but the triod itself is wildly imbedded. A form of local peripheral unknottedness (or property $\mathcal{P}$) applicable to 1-dimensional regular curves has been studied by C. Masaitis [23]. For finite dendrites he obtains an enclosure property analogous to that obtained for arcs. Since every dendrite, finite or not, has a homeomorphic image in the plane, it might be permissible to call a dendrite tame if some homeomorphism on space maps the dendrite into a subset of a plane.

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[5] The set D is locally polyhedral modulo C if it is locally polyhedral at the points of $D\backslash C$.

[6] University of Tennessee thesis, 1957.

Other known characterizations of tame curves will now be described before discussing surfaces explicitly.

9. **Tame curve types have finite crookedness.** If P is a simple closed polygon, let $k(P)$ be the sum of suitably defined "external" angles measured along P according to some selected orientation. For an arbitrary simple closed curve define $k(C) = \text{l.u.b. } k(P)$, P inscribed in C. If $\mathfrak{C}$ is the class of curves in E^3 whose members are equivalent to C under some space isotopy, define $k(\mathfrak{C}) = \text{g.l.b. } k(C)$, $C \in \mathfrak{C}$. This number may be referred to as the total curvature of the class $\mathfrak{C}$.

If $\bar{r} = \bar{r}(t)$ is a representation of the curve C, let $\mu(C, \bar{b})$ equal the number of maxima of the function $\bar{r} \cdot \bar{b}$ in a fundamental period $0 \leqq t \leqq 2\pi$, where $\bar{b}$ is a fixed unit vector. J. W. Milnor sets $\mu(C) = \min_b \mu(C, \bar{b})$ and refers to $\mu(C)$ as the crookedness of C. For a class $\mathfrak{C}$, put $\mu(\mathfrak{C}) = \min \mu(C)$, $C \in \mathfrak{C}$. The fundamental relation between $k(\mathfrak{C})$ and $\mu(\mathfrak{C})$ is given by Milnor,

$$2\pi\mu(\mathfrak{C}) = k(\mathfrak{C}).$$

The class of tame unknotted curves is then characterized by $k(\mathfrak{C}) = 2\pi$, and, for a class of tame curves generally, $k(\mathfrak{C}) < +\infty$ [24]. Whether this device could be used to study the imbedding of other configurations seems not to have been investigated.

10. **Tame curves and regular curve families.** In 1933 H. Whitney defined regular curve families and later showed that a regular curve family filling an open set in E^2 or E^3 may be cross-sectioned [27; 28]. It is easily seen that this implies each trajectory of the family is locally tame. For if a point p lies on a nondegenerate trajectory J, then nearby points p', p'' of the same trajectory such that $p' < p < p''$ will determine cross sections A', A'' from which we can pick a pair of 1-spheres B', B'', enclosing p', p'' on A', A'', respectively, that correspond under the parametrization. Then disks of A', A'' bounded by B', B'' and the ring swept out by following B' along until it falls on B'' give a topological 2-sphere enclosing p that meets J only at p' and p''. Evidently J is locally peripherally unknotted at p. If we join B' to p' by an appropriate arc α in A' and consider the locus swept out by α as p' moves to p'', we see J is locally unknotted[7] at p. Hence J is locally tame. It is obvious that a tame curve is (locally) a member of a regular curve family.

11. **The enclosure properties for surfaces.** In his thesis [12] H. C. Griffith generalized property $\mathcal{P}$ to apply to k-cells in n-space. If C is

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[7] To be exact, we see J has local property $\mathfrak{Q}$ at p, then argue as in paragraph 8.

a closed k-cell in n-space, Griffith requires that each parameter $k-1$ cell T be enclosed in an arbitrary neighborhood of T by a topological $n-1$ sphere that is locally polyhedral save where it meets C and the common part of C with the enclosing $n-1$ sphere must be a pair of spanning $k-1$ cells (or a single $k-1$ cell if T lies on the boundary of C). The hypothesis that the intersection consists of *spanning* $k-1$ cells may be eliminated for $k=2$, $n=3$. The relation between this property and local peripheral unknottedness for $k=2$, $n=3$ is not clear at present. It is not difficult to see, however, that the above defined property $\mathcal{P}$ does imply that the boundary of C be tame. As we remarked earlier, the enclosure property implies C has an open 3-cell complement. But, whereas in the 1-dimensional case property $\mathcal{P}$ implies the enclosure property, this has not been proved true for surfaces without further hypotheses.

Griffith defines a *strong enclosure property* as follows: a set C has the strong enclosure property provided each pair U, V of disjoint open subsets of ∂C contain, respectively, the initial point of a pair of disjoint rays, each of which is otherwise disjoint from C, and which are such that every neighborhood of C contains a polyhedral sphere enclosing C and meeting each ray of the pair in a single point. There are examples to show that for a k-cell, $k \geqq 2$, the enclosure property does not imply the strong enclosure property.

12. **The disk and uniform disk properties.** Let T be a spanning cell of the boundary ∂C of C or a subset of ∂C. In the former case let M', N' be the components of $C \backslash T$, in the latter case let M' be the component of $C \backslash T$ and N' null. Putting $M = M' \cup T$, $N = N' \cup T$, T determines a triple (T, M, N) of closed non-null sets. If σ is the usual Hausdorff metric, put

$$\rho(T_1, T_2) = \min \{\sigma(M_1, M_2) + \sigma(N_1, N_2), \sigma(M_1, N_2) + \sigma(N_1, M_2)\}.$$

This is a metric over the collection of spanning cells of the boundary of C.

The k-cell C is said to have the disk property provided that to each parameter $k-1$ cell T of C and positive ϵ there exists a set D such that (i) D is a disk; (ii) $\partial D \cap C = 0$; (iii) $D \cap C$ is a parameter cell (not necessarily under the same parameterization that corresponds to T); (iv) If $C \backslash D$ has two components, then D separates them in every sufficiently small neighborhood of C, and (v) $\rho(D \cap C, T) < \epsilon$.

If, as $\epsilon \to 0$, it is possible to choose D so that the distance from C to the boundary of D does not approach 0, D is said to have the *uniform disk property*.

If in the choice of T, we restrict ourselves to cells T corresponding to a particular homeomorphism h of I^k onto C^k, $I=[0, 1]$, the other conditions remaining unaltered, C^k is said to have the disk property relative to h, or the uniform disk property relative to h, respectively.

Griffith proves these weaker forms of property $\mathcal{P}$ and the uniform disk property together imply the enclosure property. Then, using the stated forms of property $\mathcal{P}$ and the uniform disk property, the strong enclosure property follows. Finally, any two of the three properties imply the third. Recently, Griffith [13] has shown that among the 2-cells in 3-space having the disk property and strong enclosure property a tame 2-cell C is characterized by the fact that if D_1 and D_2 are disks satisfying (i), (ii), (iii), and (iv) above and T is a parameter 1-cell such that $T\cap D_i$ is a point, $i=1, 2$, and $\epsilon>0$, then there exists a disk D satisfying (i), (ii), (iii), (iv), and (v) for this T and ϵ and such that $D\cap D_i\cap C$ is a point, for $i=1, 2$.

13. **Relations between the concepts of local unknottedness and local peripheral unknottedness.** Examples 1.2 and 1.4 of [10] show that local unknottedness and local peripheral unknottedness are independent conditions for $k=1$ and $n=3$. For $k=2$ and $n=3$ local unknottedness implies local peripheral unknottedness as follows. If p is in C, there is a 3-cell D such that $C\cap D$ is (i) the closure of a neighborhood M of p in C, (ii) $\overline{M}$ lies on a spanning cell K of ∂D. By definition of D there is a homeomorphism h of D onto the locus σ: $\{(x, y, z); x^2+y^2+z^2\leq 1\}$ and $h(K)$ may be taken to be a common part of σ with the x-y plane, since K spans ∂D. If $\overline{M}=K$, we are through. If $\overline{M}\neq K$, then $\overline{M}$ is a closed 2-cell with p on its boundary. The Schoenflies extension theorem may be applied to map $h(\overline{M})$ onto the locus σ': $\{(x, y, z); x^2+y^2\leq 1, z=0, y\geq 0\}$. This generates a homeomorphism of σ on itself in an obvious manner so that the final image of D is a sphere and the image of $\overline{M}$ is a semi-circular planar disk. Since σ could have been taken as a standard simplex, this shows C is actually *locally tame*, hence, as a corollary, locally peripherally unknotted.

The condition local peripheral unknottedness is weaker than local unknottedness for $k=2$ as may be seen by the following example, communicated by H. C. Griffith and R. L. Plunkett. Let C be obtained from a planar rectangle by removing the interiors of a sequence of disjoint circles converging to the center of the rectangle. Fill in each open circular hole by a disk of diameter not more than twice the diameter of the hole which has the imbedding of the example 1.2 of [10] when that arc is replaced by a disk. The set C so formed is a disk that is locally peripherally unknotted at the center. It is not locally tame at this point, hence locally knotted. The possibility remains that if a disk C is locally peripherally unknotted at every point that C would be locally unknotted at each point.

14. **Extension problems in euclidean 4-space.** The definition in §6 above shows that if C is a locally peripherally unknotted manifold in E^n and E^n is regarded as a hyperplane of E^{n+m}, then C is locally peripherally unknotted in E^{n+m}. Similarly, if C is locally unknotted in E^n, it is automatically locally unknotted in E^{n+m}. The extent to which these properties may characterize tame curves and surfaces in E^n, $n>3$, is open.

It will be noticed that the definition of "tameness" for $n>3$ is less restrictive than for $n=3$ in the sense that a k-cell that is tame in E^4 may or may not be "flat," i.e. there may exist no semi-linear homeomorphism on E^4 carrying a polyhedral 2-cell into a subset of some E^3 in E^4. For instance, if C is the join of a polyhedral knot in E^3 and a point p of $E^4\backslash E^3$, then C is polyhedral, hence tame, but not flat in the sense of Gugenheim [14]. It is easy to see that C is locally peripherally unknotted in E^4. The decision as to whether C is locally knotted at p or not is resolved less easily. (It is evident that no "flat" 3-cell D exists in E^4 of which C is a spanning 2-cell for ∂D.)

The construction of Antoine may be used to show that there are 1-cells in E^n that are locally peripherally knotted. Presumably there might be a 1-cell in E^4 that lies on no 2-cell in E^4 and perhaps a 2-cell in E^4 that lies on no 3-cell. Fulfillment of both of these conditions would be necessary for a 1-cell in E^4 to be equivalent to a flat 1-cell. Thus, even a locally unknotted *and* locally peripherally unknotted arc in E^4 might fail to be equivalent to a segment because of the lack of existence of the appropriate 3-cell.

V. L. Klee has an extension theorem for subsets of linear normed spaces that may be used to show that every arc in E^3 is tame in E^4 (even "flat") [22, Theorem 3.3]. Hence every simple closed curve in E^3 is locally tame in E^4. If J is a simple closed curve in E^3 it may be shown that it bounds a disk in E^4 that is locally polyhedral modulo J. It is not clear such a disk would be locally tame at the points of J.

The immediate prospect of obtaining extension theorems in 4-space analogous to those quoted above in paragraphs 8, 12 seems remote until a suitable generalization or substitute is found for the Alexander theorem. Since a polyhedral 2-sphere may be knotted in E^4, it would appear a classification of the ways in which such a sphere may be imbedded in E^4 would be an essential step. R. H. Fox and J. W. Milnor have extended results of Gugenheim in this direction [11; 14].

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THE JUNE MEETING IN PULLMAN

The five hundred thirty-sixth meeting of the American Mathematical Society was held at the State College of Washington in Pullman, Washington, on Saturday, June 15, 1957, preceded by a meeting of the Mathematical Association of America on Friday. There were 97 registrants, including 70 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor M. M. Day delivered an address on *Rotundity and smoothness.* He was introduced by Professor Klee, and the sessions for contributed papers presided over by Professors Paul Civin and J. M. Kingston.

Following are abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been given by title. On joint papers, the presenter's name is followed by "(p)". Dr. Schmidt was introduced by Professor Ostrom.

ALGEBRA AND THEORY OF NUMBERS

586. W. E. Barnes and H. Schneider (p): *The group-membership of a polynomial in an element algebraic over a field.*

Let F be a field, R an arbitrary extension ring of F, and let a be an element of R algebraic over F, with minimum polynomial $m(x)$. If $q(x) \in F[x]$, the polynomial $q(a)$ is called a *group-element* in R if there exists a subgroup of the multiplicative semigroup of R to which $q(a)$ belongs. It is proved that $q(a)$ is a group-element in R if and only if the greatest common divisors in $F[x]$ of $q(x)$ and $m(x)$ and of $q(x)^2$ and $m(x)$ are equal, in which case $q(a)$ is a group-element even in $F[a]$. It follows that $q(a)$ is a group-element in R for every polynomial $q(x)$ in $F[x]$ if and only if the irreducible factors of $m(x)$ are simple. This extends a result due to Farahat and Mirsky on the group membership of a matrix polynomial (American Math. Monthly vol. 63 (1956) pp. 410–412). (Received May 1, 1957.)

587. R. A. Beaumont (p) and R. S. Pierce: *Partly transitive modules with proper isomorphic submodules.*

The problem of classifying those R-modules M over a principal ideal domain R which have proper isomorphic submodules is considered. Such modules are called I-modules. For modules of finite rank, a generalization of a theorem of Kaplansky is obtained: Let M be a module of finite rank which splits. Then M is an I-module if and only if M/T, where T is the torsion submodule of M, is not divisible. Using standard techniques from the theory of abelian groups, it is proved that divisible modules and reduced modules may be considered separately, and the divisible case is disposed of easily. Reduced torsion-free modules are I-modules. It is shown that if the cardinality of a minimal set of generators of a primary reduced module M exceeds $\|R\|^{\aleph_0}$, where $\|R\|$ is the cardinality of R, then M is an I-module. Let x and y be elements of a primary module M and let $U(x) = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ and $U(y) = (\beta_1, \beta_2, \cdots, \beta_n, \cdots)$ be their Ulm sequences. We write $U(x) \prec U(y)$ if $\alpha_n \leq \beta_n$ for all i and $\alpha_i = \beta_i$ whenever $\beta_i \geq \mu$, where μ is the least ordinal such that M_μ, the submodule of elements of height $\geq \mu$, has finite rank. Then M is called partly transitive if whenever x and y in M satisfy $U(x) < U(y)$, then there exists an isomorphism θ of M into itself such that $\theta(x) = y$. It is proved that a countably generated primary module is partly transitive. It follows as a corollary from this result that a primary reduced module with a minimal set of generators of cardinality $\aleph_0$ is an I-module. (Received May 3, 1957.)

588. R. A. Beaumont and R. S. Pierce (p): *Partly invariant submodules of a torsion module.*

Let M be a module over a principal ideal domain R. A submodule N of M is called partly invariant if every isomorphism of M into (not necessarily onto) itself carries N into itself. Of course, any partly invariant submodule is characteristic and any fully invariant submodule is partly invariant. This paper surveys the partly invariant submodules of an arbitrary p-primary torsion module M which is partly transitive (in the sense of the preceding abstract). Let $H = M_\mu$ be the submodule of elements of height $\geq \mu$ in M, where μ is the least ordinal such that M_μ has finite rank. Let G be a characteristic submodule of H and $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ a finite U-sequence of ordinals $<\mu$ (see Kaplansky's *Infinite Abelian groups*). Then the set of all x in M such that $p^k x \in G$ and $h(p^i x) \geq \alpha_{i+1}$ for $0 \leq i < k$ is a partly invariant submodule of M, which is fully invariant if and, assuming M is fully transitive, only if G is fully invariant in H. Moreover, every partly invariant submodule of M, which is not fully invariant, is obtained in this way. These results are of interest only if $R/(p)$ has two elements, since otherwise every characteristic submodule is fully invariant (see Kaplansky, loc. cit.). An example is given of a 2-primary group having characteristic subgroups which are not partly invariant and partly invariant subgroups which are not fully invariant. (Received May 3, 1957.)

589. A. A. Goldstein and Ward Cheney (p): *On overdetermined systems of linear equations*, I.

Let (A_{ij}) be an $n \times (n+1)$ matrix of rank n. Let x minimize $\sum_i (b_i + \sum_j A_{ij}x_j)^2$ where $b \in E_{n+1}$, $x \in E_n$, $i = 1 \cdots n+1$ and $j = 1 \cdots n$. Put $r_i = b_i + \sum_j A_{ij}x_j$. If r has zero components, let them be, say, the first k. If y minimizes $\sum_i (b_i + \sum_j A_{ij}y_j)^2 |r_i|$ subject to the conditions $b_i + \sum_j A_{ij}y_j = 0$ for $i = 1 \cdots k$, then $y = u$ also minimizes $M = \sup_{i \leq n+1} |b_i + \sum_j A_{ij}u_j|$. Put $v_i = (r, r) \operatorname{sgn} r_i + \sum_s |r_s|$ and let s be chosen so that $b_i + \sum_j A_{ij}s_j = v_i$. Then $s = u$ minimizes M. A vector u which minimizes M is unique if and only if $k = 0$. (Received May 6, 1957.)

590. A. P. Hillman: *A syzygy system in differential algebra.*

Let $y_1, \cdots, y_n$ be indeterminates over a partial differential field $\mathfrak{F}$ of characteristic zero and let $\mathfrak{R} = \mathfrak{F}\{y_1, \cdots, y_n\}$. Let A be in $\mathfrak{R}$ but not in $\mathfrak{F}$. Let S be the separant of A. Let $A_1, \cdots, A_r$ be distinct partial derivatives of A. If $M_1, \cdots, M_r \in \mathfrak{R}$, a necessary and sufficient condition for the syzygy $M_1A_1 + \cdots + M_rA_r = 0$ is that there exist N_{ij} in $\mathfrak{R}$ and a nonnegative integer t such that $S^tM_i = \sum_{j=1}^{r} N_{ij}A_j$ for $i = 1, \cdots, r$ and $N_{ij} + N_{ji} = 0$ for $i, j = 1, \cdots, r$. This is generalized in the following: Let $B_0, \cdots, B_r \in \mathfrak{R}$ with B_0 not in $\mathfrak{F}$. Let S be the separant of B_0. Let $\delta_0, \cdots, \delta_s$ be distinct power products in the fundamental differentiations of $\mathfrak{F}$. Then $\sum_{i=0}^{g}\sum_{j=0}^{s} M_{ij}(\delta_j B_{g-i}) = 0$ for $g = 0, 1, \cdots, r$ with M_{ij} in $\mathfrak{R}$ if and only if there exist

N_{ijk} in $\mathcal{R}$ and a nonnegative integer t such that $S^tM_{ij}=\sum_{k=0}^{i}\sum_{l=0}^{j} N_{kjl}(\delta_l B_{t-k})$ and $N_{ijk}+N_{ikj}=0$ for $i=0, \cdots, r$ and $j, k=0, \cdots, s$. (Received May 1, 1957.)

591*t*. J. H. Hodges: *Some matrix equations over a finite field.*

Let $GF(q)$ denote the finite field of $q=p^n$ elements, p odd. In this paper we consider several problems of the following type. Let B be a symmetric matrix of order t and A be an arbitrary $m\times t$ matrix, both with elements in $GF(q)$. Determine the number of $t\times m$ matrices X over $GF(q)$ such that $X'A+A'X=B$. If the equation has any solutions, it is shown that their number is q^e, where $e=t(m-r)+r(r-1)/2$ and r is the rank of A. A necessary and sufficient condition is given for the existence of solutions of the equation. (Received April 30, 1957.)

592*t*. J. H. Hodges: *On orthogonal matrices of polynomials over a finite field.*

J. L. Brenner (*Polynomial parametrizations*, Amer. Math. Monthly vol. 58 (1951) pp. 327–329) has proved that the only orthogonal matrices of order 2 with elements in $GF[p, x]$ are constant matrices if $p>2$ and if $p=2$ are of the form $A=I+p(x)C$, where C is the matrix of order two all of whose elements are $+1$ and $p(x)$ is an arbitrary element of $GF[p, x]$. L. Carlitz (*A note on orthogonal matrices*, Amer. Math. Monthly vol. 60 (1953) pp. 253–255) has given examples of nonconstant orthogonal matrices of order p over $GF[p, x]$ for $p>2$. In (*Orthogonal matrices of modular polynomials*, Duke Math. J. vol. 21 (1954) pp. 225–232) Brenner gives a construction for producing nonconstant orthogonal matrices of order 3, therefore of arbitrary order $m>2$, over $GF[p, x]$ for $p>2$. In the present paper it is shown that nonconstant orthogonal matrices of order $m>2$ with elements in $GF[q, x]$, $q=p^n$ odd, may be constructed using results by Eckford Cohen concerning sums of squares in $GF[q, x]$ (*Sums of an even number of squares in* $GF[p^n, x]$, II, Duke Math. J. vol. 14 (1947) pp. 543–557 and *Sums of an odd number of squares in* $GF[p^n, x]$, Duke Math. J. vol. 15 (1948) pp. 501–511). (Received April 30, 1957.)

593*t*. Joseph Landin and Irving Reiner: *Automorphisms of the two-dimensional general linear group over a euclidean ring.*

Let R be a commutative principal ideal domain which is integrally closed in its quotient field, and assume (i) R is euclidean, (ii) the group U of units of R contains more than two elements, (iii) every element of R can be expressed as a finite linear combination of units, with coefficients which are rational integers (when R has characteristic 0) or elements of $GF(p)$ (when R has characteristic $p>0$). It is shown that every automorphism of $GL_2(R)$ can be expressed as a product of automorphisms of the following types: (i) inner, (ii) $u\rightarrow u^\sigma$, where σ is an automorphism of the ring R, (iii) $u\rightarrow\lambda$ (det u)u, where λ is a homomorphism of U into itself satisfying the condition: $\lambda(\alpha^2)=\alpha^{-1}$ if and only if $\alpha=1$. (Received March 14, 1957.)

594*t*. Howard Osborn: *A class of bilinear forms*. I.

Let E be a finite-dimensional vector space over an arbitrary field, A an endomorphism of E, and $\mathfrak{A}$ the ring generated by A and scalar multiplications. As an $\mathfrak{A}$-module E is the direct sum $E_1\oplus\cdots\oplus E_t$ of unique cyclic $\mathfrak{A}$-submodules such that the order ideal of E_i contains that of E_j when $i\leq j$. For any bilinear form f on $E\times E$ let fA represent that form such that $fA(u, v)=f(u, Av)$. If g is a given form on $E\times E$ let $\mathfrak{S}(A; g)$ be that class of symmetric forms f such that $fA+g$ is also symmetric, and let $\mathfrak{A}(A;g)$ be that class of alternating forms f such that $fA+g$ is also alternating. *THEOREM: Let* x_j' *be an arbitrary linear functional on* $E_1\oplus\cdots\oplus E_j$, *and let* z_j *generate* E_j. *Then there exists a unique* $f\in\mathfrak{S}(A;g)$ *such that* $x_j'(x_j)=f(x_j, z_j)$ *for all* $x_j\in E_1\oplus\cdots\oplus E_j$, $j=1,\cdots,t$, *and if* $t>1$ *there exists a unique* $f\in\mathfrak{A}(A;g)$ *such that* $x_{j-1}'(x_{j-1})=f(x_{j-1}, z_{j-1})$ *for all* $x_{j-1}\in E_1\oplus\cdots\oplus E_{j-1}$, $j=2,\cdots,t$. (Received May 1, 1957.)

595*t*. Howard Osborn: *A class of bilinear forms*. II.

Let $\mathfrak{C}(A)$ represent the linear space of bilinear forms f such that A is self-adjoint with respect to f, $f(Au, v)=f(u, Av)$, and let $\mathfrak{C}(A)$ represent the centralizer of A in the ring of all endomorphisms of E. Then $\mathfrak{C}(A)$ and $\mathfrak{C}(A)$ are isomorphic right $\mathfrak{C}(A)$-modules, where $f\cdot B=fB$ for $B\in\mathfrak{C}(A)$, and a fortiori $\mathfrak{C}(A)$ and $\mathfrak{C}(A)$ are isomorphic $\mathfrak{A}$-modules. Furthermore, for fields of characteristic $\neq 2$, $\mathfrak{S}(A;0)$ and $\mathfrak{A}(A;0)$ are $\mathfrak{A}$-modules such that $\mathfrak{C}(A)=\mathfrak{S}(A;0)\oplus\mathfrak{A}(A;0)$. This leads to a development of $\mathfrak{C}(A)$ as an $\mathfrak{A}$-module, from which one obtains an easy characterization of $\mathfrak{C}(A)$ as an algebra over $\mathfrak{A}$. In particular $\mathfrak{C}(A)$ is the direct sum over $\mathfrak{A}$ of a special Jordan algebra and a Lie algebra, these two algebras corresponding to $\mathfrak{S}(A;0)$ and $\mathfrak{A}(A;0)$, respectively. (Received May 1, 1957.)

596*t*. R. M. Robinson: *The converse of Fermat's theorem*.

It is shown that a theorem of Lehmer concerning the form of the prime factors of a composite number satisfying Fermat's congruence is in a certain sense the strongest possible result. Particular attention is given to the use of Euler's criterion as a test for primeness. The discussion here is based on the following theorem: *Let* $N=k\cdot 2^n+1$, *where* $k>0$, $n>0$, *and* k *is odd. Then there exists* a *such that* $a^{(N-1)/2}\equiv -1 \pmod{N}$ *if and only if every prime factor* p *of* N *satisfies the congruence* $p\equiv 1 \pmod{2^n}$. As a by-product of this study, the following rather unexpected result is obtained: *If* $N>1$ *is odd, and* $a^{(N-1)/2}\equiv -1 \pmod{N}$, *then* $(a/N)=-1$. (Received April 30, 1957.)

597*t*. H. J. Ryser: *The term rank of a matrix*.

This paper continues the author's study of the combinatorial properties of a matrix A of m rows and n columns, all of whose entries are 0's and 1's (Bull. Amer. Math. Soc. Abstract 62-6-636). Let $\mathfrak{A}$ be the class of 0, 1 matrices A of m rows and n columns, with row sum vector $R=(r_1,\cdots,r_m)$ and column sum vector $S=(s_1,\cdots,s_n)$. Let the components of R and S be positive, and let $R^1=(r_1-1,\cdots,r_m-1)$. Let $\bar{A}'$ be the 0, 1 matrix of size m by n with row sum vector R' and such that the 1's in each row of $\bar{A}'$ are shifted to the left as far as possible. Let the column sum vector of $\bar{A}'$ equal $\bar{S}'=(\bar{s}_1',\cdots,\bar{s}_n')$. Renumber the subscripts of S so that $s_1\geqq\cdots\geqq s_n$, and define $s_i'=s_i-1$, $i=1,\cdots,n$. Let $\bar{s}_0'=s_0'=0$. Let M be the largest integer in the set $\sum_{i=0}^{k}(s_i'-\bar{s}_i')$, $k=0,\cdots,n$. Then if $\bar{\rho}$ is the maximal term rank for the matrices in $\mathfrak{A}$, $\quad =m-M$. This result is applied to a variety of combinatorial problems dealing with term ranks of 0, 1 matrices in the class $\mathfrak{A}$. (Received March 8, 1957.)

598. Wolfgang Schmidt: *The measure of the set of admissible lattices*.

If S is a point set in R_n, then, as usual, we say a point lattice is admissible relative to S, if it has no point in S, except possibly the origin. Let $A(S)$ be the set of admissible lattices of determinant 1. We prove the theorem: Suppose S is a Borel set with measure $V=V(S)\leq c_1 n$. Then $m(A(S))=\int_{A(S)}d\mu=e^{-V}(1-R)$, where $|R|<\exp(2V-2c_2 n)$.

Here $d\mu$ is the invariant measure over the space of lattices of determinant 1, and $c_1 > c_2 > 0$ are constants. The proof is based on results of C. A. Rogers on mean values over lattices. A consequence of the above theorem is an improvement of the Minkowski-Hlawka theorem: If $V(S) \leqq c_2 n$, then there exist admissible lattices of determinant 1. (Received April 29, 1957.)

599*t*. S. K. Stein: *The union of arithmetic sequences.*

Theorem 1. If A_i, $1 \leqq i \leqq n$, and B_j, $1 \leqq j \leqq m$ are each collections of intersecting incomparable arithmetic sequences and $\bigcup A_i = \bigcup B_j$ then $m = n$ and the B_j coincide (in some order) with the A_i. *Theorem* 2. If A_i, $1 \leqq i \leqq n$ and B_j, $1 \leqq j \leqq m$ are each collections of disjoint incongruent arithmetic sequences and $\bigcup A_i = \bigcup B_j$ then $m = n$ and the B_j coincide (in some order) with the A_i. The first theorem is easily proved by the Chinese Remainder Theorem, the second, by translation of the problem into one on the factorization of polynomials with complex coefficients. Theorem 2 yields a new proof of Mirsky-Newman's theorem (given in P. Erdös *On a problem concerning congruence systems* (In Hungarian)) which asserts that the integers J are not the finite union of disjoint incongruent arithmetic sequences (unless the collection is simply J itself). *Theorem* 3. The complement of a finite union of disjoint incongruent arithmetic sequences of differences $d_1 < d_2 < \cdots < d_n$ is again an arithmetic sequence if and only if $d_k = 2^{k-1}$. The proof is similar to that of Theorem 2. (Received April 18, 1957.)

600*t*. Robert Steinberg: *Groups generated by reflections.*

H. S. M. Coxeter [see *Regular polytopes*, London, 1948, and New York, 1949, pp. 212, 227] has proved by verification two theorems ((1) and (2) below) concerning finite orthogonal groups generated by reflections in real finite-dimensional Euclidean spaces. In this paper, general proofs are given for these theorems and some related results. (1) Let G be a finite irreducible g.g.r., let $W_1, W_2, \cdots, W_n$ be the walls of a fundamental region and $R_1, R_2, \cdots, R_n$ the reflections in these walls. Then, if $R_1R_2 \cdots R_n$ has order h, the number of reflecting hyperplanes is $nh/2$. The method of proof leads to an explicit enumeration of the reflecting hyperplanes in terms of the W's and R's. In the case that G is associated with a simple Lie group [see H. Weyl, *Continuous groups*, I.A.S. notes], an enumeration of the positive roots in terms of the fundamental set results, and this is used to prove (2): if $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a fundamental positive set of roots and if $\sum r^i\alpha_i$ is the dominant root, then $\sum r^i = h-1$. (Received May 4, 1957.)

ANALYSIS

601*t*. G. L. Krabbe: *Spectral invariance of convolution operators on $L^p(-\infty, \infty)$.*

Suppose that g is a function of bounded variation on $(-\infty, \infty)$ satisfying the condition $g(-\infty) = 0$, and let $\mathfrak{M}^1$ be the set of all such functions. Denote by g_{*p} the mapping $f \rightarrow g^*f$ defined for all f in L^p (as usual, $(g^*f)(\theta) = \int f(\theta-\alpha)dg(\alpha)$). This note concerns the spectrum $\sigma(g_{*p})$ of the operator g_{*p}; let $\mathfrak{F}$ be the set of all the members g of $\mathfrak{M}^1$ such that $\sigma(g^*_p) = \sigma(g_{*1})$ for all $p > 1$. Theorem 1: $g \in \mathfrak{F}$ if the Fourier transform Fg of g is of the form $\sum a_n \exp(2\pi i\theta\alpha_n)$ with $\sum |a_n| < \infty$ $(n = -\infty, \cdots, \infty)$; note that in this case $g_{*p} = \sum a_n t(\alpha_n)$, where $t(\alpha_n)$ is the translation operator defined by $(t(\alpha_n)f)(\theta) = f(\theta + \alpha_n)$. Theorem 2: $\mathfrak{F}$ contains all absolutely continuous functions.

Theorem 3: If $g \in \mathfrak{F}$, then $\sigma(g_{*p})$ is the closure $r(Fg)$ of the range of $\mathfrak{F}g$ (for any $p \geq 1$). The theorems 2 and 3 combine to form a generalization of a Hilbert space result due to H. Pollard [*Integral transforms*, Duke Math. J. vol. 13 (1946) pp. 307–330] (see also N. Dunford's extension in the Proc. Symposium Spectral Theory, Oklahoma, 1955, pp. 57–62). Since $\mathfrak{F}$ is also the set of all g in $\mathfrak{M}^1$ with $\sigma(g) \subset r(\mathfrak{F}g)$, it follows immediately from results of N. Wiener and Pitt that $\mathfrak{F} \neq \mathfrak{M}^1$. We characterize a subset $\mathfrak{F}_1$ of $\mathfrak{F}$ which includes the sets involved in the theorems 1 and 2; it seems probable that $\mathfrak{F}_1 = \mathfrak{F}$. (Received April 17, 1957.)

602*t*. Marvin Rosenblum: *On the Hilbert matrix*, I.

1. For fixed $k < 1$ the generalized Hilbert matrix is $H_k = ((m+n+1-k)^{-1})$, $m, n = 0, 1, 2, \cdots$. By an *eigenvalue* of H_k we mean a complex number λ such that there exists a non-null sequence of complex number $\{x_n\}_0^\infty$ with the property that $\sum_{n=0}^\infty (n+m+1-k)^{-1}x_n$ converges to λx_m for all non-negative integers m. Theorem. Every complex number with positive real part is an eigenvalue of H_k. (Received April 11, 1957.)

603. R. G. Selfridge: *Summability of Walsh transforms.*

The Walsh functions form a complete system over the unit interval, and one can generate Walsh-Fourier series. A transform can also be generated for functions in $L_p(0, \infty)$ which has an inverse which converges to the initial function under reasonable conditions. The inverse transform is also C, 1 summable to the initial function under certain conditions. Previously C, 1 summability had not been proven almost everywhere. This paper, based primarily on a paper by N. J. Fine (Proc. N. A. S. vol. 41 (1955) pp. 588–591) shows that the inverse transform of the transform of a function in $L(0, \infty)$ is summable almost everywhere to the initial function. The same result also holds for functions in $L_p(0, \infty)$ for $1 < p \leq 2$. (Received March 18, 1957.)

604*t*. V. L. Shapiro: *Uniqueness of surface spherical harmonic series.*

Let $f(x)$ be in L^1 on S, the surface of the unit sphere in three dimensional Euclidean space. If $f(x)$ has for its Laplace series $\sum_{n=0}^\infty Y_n(x)$, then $\|Y_n\| = o(n^{1/2})$ where $Y_n(x)$ is a surface spherical harmonic of order n and $\|Y_n\|^2 = \int_S |Y_n(x)|^2 dS(x)$, $dS(x)$ being the natural two dimensional area element on S. In this paper we prove three theorems concerning arbitrary series of surface spherical harmonics which satisfy the condition $\|Y_n\| = o(n^{1/2})$. *Theorem* 1. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let $f^*(x)$ and $f_*(x)$ be respectively the upper and lower Abel limits of the series. Suppose (i) $\|Y_n\| = o(n^{1/2})$, (ii) $f_*(x)$ and $f^*(x)$ are finite for x on S, (iii) $f_*(x) \geq A(x)$ where $A(x)$ is in L^1 on S. Then $f_*(x)$ is in L^1 on S and the given series is its Laplace Series. *Theorem* 2. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let p be a fixed point on S. Suppose (i) $\|Y_n\| = o(n^{1/2})$, (ii) $f_*(x)$ and $f^*(x)$ are finite on $S-p$, (iii) $f_*(x)$ and $f^*(x)$ are in L^1 on S. Then the given series is the Laplace series of $f_*(x)$. *Theorem* 3. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let E be a denumerable point set on S. Suppose (i) $\|Y_n\| = o(n^{1/2})$ (ii) $f_*(x) = f^*(x) = 0$ for x in $S-E$. Then $Y_n(x) = 0$ for all n. It is easy to see that Theorems 2 and 3 are best possible results. (Received February 27, 1957.)

605. Abraham Spitzbart and Nathaniel Macon (p): *Numerical differentiation formulas.*

In a recent paper (Amer. Math. Monthly vol. 64 (1957) pp. 79–82) R. T. Gregory described an algorithm for finding the coefficients in certain formulas which express the derivatives of an interpolation polynomial at a set of equally spaced points as linear combinations of the given functional values. The method requires the solution of systems of linear equations or, equivalently, the inversion of certain Vandermonde matrices. In this paper, explicit representations for these same coefficients are obtained, and in such a form that their use is in no way restricted to the tabular points. These representations are in terms of Stirling numbers, which are well known and extensively tabulated. This in turn enables one to determine directly the matrix inverses mentioned above. Applications to more general matrices and general interpolation problems will be made later. (Received April 18, 1957.)

APPLIED MATHEMATICS

606. V. W. Bolie: *Electromagnetic propagation in an almost homogeneous medium.*

This paper concerns the developnent from Maxwell's electromagnetic equations an equation of propagation in an almost homogeneous medium. The equation is applied to the problem of determining the secondary wave produced by an isolated gaussian-shaped perturbation in the refractive index. An exact solution is obtained for points located on the axis of symmetry parallel to the direction of propagation of the incident wave. An approximate solution for points remote from the anomaly is obtained and its validity is compared with the more restricted exact solution. An interesting limit process is encountered in the derivation of the formula for the scattering cross section of the refractive index perturbation. (Received April 22, 1957.)

607t. Philip Wolfe: *Simplex method for quadratic programming.*

Given matrices: A, m by n; b, m by 1, $b \geqq 0$; C, m by n, positive semidefinite. To solve: Minimize $x'Cx$ subject to $x \geqq 0$, $Ax = b$. Let $'$ denote transposition, $x = (x_1, \cdots, x_n)'$, $v = (v_1, \cdots, v_n)$, $u = (u_1, \cdots, u_m)$, $s = (s_1, \cdots, s_n)'$. *Initiation*: Let $Ax^0 = b$. Choose $x^0 \geqq 0$ and the diagonal matrix E so that $|E_{ii}| = 1$ and $Cx^0 + Es^0 = 0$. Let $v^0 = 0$, $u^0 = 0$. *Recursion*: Given the basic feasible vector $[x^k, v^k, u^k, s^k]$ satisfying the constraints below, take one step in the simplex method minimization of $\sum_i s_i$ subject to $x \geqq 0$, $v \geqq 0$, $s \geqq 0$, $Ax = b$, $x'C + uA - v + Es = 0$ with the added condition: Do not allow $x_j^{k+1} > 0$ if $v_j^k > 0$, or $v_j^{k+1} > 0$ if $x_j^k > 0$. *Theorem*: The procedure will terminate in at most $C_{m+2n,m+n}$ steps with $s = 0$; the final x solves the minimization problem. (Received April 15, 1957.)

GEOMETRY

608. T. G. Ostrom: *Dual transitivity in finite projective planes.*

The author defines a plane π to be dually transitive if, for every pair of points P and P_1, and every pair of lines l and l_1 (where P is not on l and P_1 is not on l_1) there is a collineation of π which maps P into P_1 and l into l_1. Dual transitivity will follow if, for every such pairs of points and lines, there is a duality which maps P into l_1 and l into P_1. The main result is that if the plane is finite of order n, where n is not a square, then the plane is Desarguesian. See the author's paper *Double transitivity in finite projective planes*, Canadian J. Math. vol. 8 (1956) pp. 563–567. (Received April 29, 1957.)

609. F. A. Valentine: *The intersection of two convex surfaces and property* P_3.

A set S in a linear space L has property P_3 if for each triple of points x, y, z in S at least one of the segments xy, yz, xz is in S. This property enables one to establish the following: *Let S_1 and S_2 be two compact convex bodies in a finite dimensional normed linear space L_n, and suppose their interiors have a nonempty intersection. Let $B(S_i)$ denote the boundary of S_i $(i=1, 2)$. If $B(S_1)\cdot B(S_2)$ is in the interior of the convex hull of $S_1 \cup S_2$, then $B(S_1)\cdot B(S_2)$ is the union of a finite number of disjoint compact $(n-2)$-dimensional manifolds.* Furthermore: *Let S be a closed set in a topological linear space L, where dimension $L>2$. Assume S has property P_3, and that S is not contained in any two-dimensional variety of L. If the set Q of points of local nonconvexity of S has an isolated point, then Q has at most two points* (a point p is a point of local convexity of S if there exists a neighborhood N of p such that for each pair of points x and y in $S\cdot N$, it is true that $xy \subset S$). (Received April 22, 1957.)

LOGIC AND FOUNDATIONS

610*t*. J. C. E. Dekker: *On certain equations in isols*, II.

For notations and terminology see the author's two abstracts, J. Symbolic Logic vol. 20 (1955) pp. 204, 205. The relation $(\exists U)[A+U=B]$ is denoted by $A \leqq B$ or $B \geqq A$. An abstract of the first part of the present paper appears in the March, 1957 issue of the J. Symbolic Logic. In the following C and k stand for arbitrary constants in Λ, C finite or infinite, but k finite. The letters X, Y, U, V are used as variables ranging over Λ. The function $X(X+1)\cdots(X+k)$ is denoted by $\Pi(X, k)$. *Theorems.* (1) $XC=YC$ and $C\geqq 1$ imply $X=Y$, (2) $X^C=Y^C$ and $C\geqq 1$ imply $X=Y$, (3) $C^X=C^Y$ and $C\geqq 2$ imply $X=Y$, (4) $\Pi(X, k)=\Pi(Y, k)$ implies $X=Y$, (5) $X^X=Y^Y$ and X, $Y\geqq 1$ imply $X=Y$, (6) $X^2+Y^2=2XY$ implies $X=Y$, (7) $j(X, Y)=j(U, V)$ implies $X=U$ and $Y=V$. It follows from (6) that the ring Λ^* has no nilpotents. While the function $j(X, Y)$ maps $\varepsilon\times\varepsilon$ onto ε, it maps $\Lambda\times\Lambda$ into Λ, but not onto Λ. (Received April 25, 1957.)

611*t*. J. C. E. Dekker: *Congruences in isols with a finite modulus.*

For notations and terminology see the preceding abstract. Let m denote a positive integer. We say that *m divides X* (written: $m|X$) if $(\exists U)[mU=X]$. If m and X have no common divisor besides 1, they are called *relatively prime.* $X\equiv Y \pmod{m}$ means: $X\geqq Y$ and $m|X-Y$. It is easily seen that: if $X_1\equiv Y_1 \pmod{m}$ and $X_2\equiv Y_2 \pmod{m}$, then $X_1+X_2\equiv Y_1+Y_2 \pmod{m}$ and $X_1X_2\equiv Y_1Y_2 \pmod{m}$. On the other hand, the three conditions $ZX\equiv ZY \pmod{m}$, $Z\neq 0$, Z and m relatively prime do not imply $X\equiv Y \pmod{m}$. *Theorem.* Let p be any prime. Then (1) $(\forall X)[X\not\equiv 0 \pmod{p}\rightarrow X^{p-1}\equiv 1 \pmod{p}]$ is false, but (2) $(\forall X)[X^p\equiv X \pmod{p}]$ is true. *Corollary.* Let p be any prime. Then (1) $p|XY$ does not imply $p|X$ or $p|Y$, but (2) for $n\geqq 1$, $p|X\leftrightarrow p|X^n$. *Indication of proof.* The isol X is called *indecomposable,* if it is not the sum of two infinite isols. The first part of the theorem follows from the fact (due to Myhill) that there exist infinite, indecomposable isols. The second part can be established directly by a modification of the proof of $(\forall x)[x^p\equiv x \pmod{p}]$ due to M. J. Perott [Bull. des Sc. Math. vol. 24 (1900) pp. 175, 176] or indirectly as a consequence of $\Pi(X, p-1)\equiv 0 \pmod{p!}$. (Received April 25, 1957.)

STATISTICS AND PROBABILITY

612. D. G. Chapman: *A comparative study of several one-sided goodness-of-fit tests.*

Let X be a real random variable with d.f. $F \in \Omega_c$, the class of continuous distribution functions on R. Several strongly distribution-free tests of the hypothesis H_0: $F = F_0$ (where F_0 is completely specified) against the alternative $F < F_0$ have been proposed. These tests are reviewed in the light of criteria that have been suggested as desirable for tests in general and in the light of two new criteria: partial ordering and monotonicity. Further let $\rho^-(F_0, F) = \sup_{-\infty<x<\infty} [F_0(x) - F(x)]$. Sharp upper and lower bounds are found for those tests of H_0 which meet the specified criteria, against alternatives F such that $\rho^-(F_0, F) = \Delta$. (Received April 26, 1957.)

613. R. K. Getoor: *Conditional expectations of L^* random variables.*

Let $(\Omega, \mathfrak{F}, P)$ be a probability space then Mourier (Proc. 3rd Berk. Symp. on Math. Stat. and Prob. vol. II) has defined an L^* random variable to be a mapping, X^*, from Ω into the dual, $\mathfrak{X}^*$, of a separable Banach space, $\mathfrak{X}$, which is weak* measurable, i.e., $\langle x, X^*(\omega)\rangle$ is $\mathfrak{F}$-measurable for every $x \in \mathfrak{X}$. Let $\mathfrak{F}'$ be a σ-subalgebra of $\mathfrak{F}$ and suppose that $E(\|X^*\|) < \infty$, then there exists an L^* random variable, $E(X^* | \mathfrak{F}')$, which is weak* measurable relative to $\mathfrak{F}'$ and satisfying $\int_\Lambda X^* dP = \int_\Lambda E(X^* | \mathfrak{F}')dP$ for every $\Lambda \in \mathfrak{F}'$. $E(X^* | \mathfrak{F}')$ will be called the conditional expectation of X^* given $\mathfrak{F}'$. The elementary properties of $E(X^* | \mathfrak{F}')$ are derived and martingales of L^* random variables are investigated. (Received April 10, 1957.)

614. H. G. Tucker: *Limit theorems for interpolated random variables.* Preliminary Report.

Sometimes it is difficult, impossible, or costly to observe all of n independent, identically distributed random variables or even to observe the class intervals of a histogram into which all n random variables fall. The case considered here is that in which the number of random variables which fall into alternate class intervals can be counted; the number of random variables which fall into intervals not observed are then interpolated. It is assumed that the common distribution function $F(x)$ possesses a density which is bounded and continuous almost everywhere. By means of this partly observed, partly manufactured histogram, an estimate $F^*(x)$ of $F(x)$ is constructed. By taking limits in an appropriate way, the following limit theorems are proved: (i) $F^*(x)$ converges to $F(x)$ uniformly in x with probability one, (ii) if $\int x dF(x)$ exists and is finite, then $\int x dF^*(x)$ converges to $\int x dF(x)$ with probability one, and (iii) if $\int x^2 dF(x)$ is finite, then $\int x dF^*(x)$ is asymptotically normal. (Received April 29, 1957.)

615t. Jacob Wolfowitz: *The coding of messages subject to chance errors.*

A "channel probability function" p with memory m is defined on the set of all sequences of $(m+1)$ zeros or ones; $0 \leq p \leq 1$. Theorem 1: Let $X_1, X_2, \cdots$ be a stationary, metrically transitive Markov chain with two states, zero and one, and let $Y_1, Y_2, \cdots$ be a sequence of chance variables such that Y_i is 0 or 1 and $P\{Y_i = 1 | X_1, X_2, \cdots, Y_1, \cdots, Y_{i-1}\} = p(X_i, X_{i+1}, \cdots, X_{i+m})$. Let $H(Y)$ be the entropy of the Y-process, $H_X(Y)$ be its entropy relative to the X-process, and $\lambda > 0$.

There is a $K>0$ and, for any n, an error correcting code of length $2^{n(H(Y)-H_X(Y))-Kn^{1/2}}$. The probability that any message (of n symbols) will be incorrectly received is $<\lambda$. Theorem 2. Let $m=0$ and C_0 be the capacity of the channel. There exists a $K'>0$ such that, for any n, a code such that the probability of transmitting any word incorrectly is $<\lambda$, has length less than $2^{nC_0+K'n^{1/2}}$. These results generalize immediately to alphabets with a finite number of symbols. (Received March 4, 1957.)

Topology

616t. S. K. Stein: *Continuous choice functions and convexity.*

Theorem. An open set A in E_n (n dimensional Euclidean space) is convex iff there exists a continuous choice function on the space of r-dimensional cross-sections of A, $1\leqq r\leqq n-1$. The proof is first provided for the case $n-r=1$ by topological arguments and then proceeds by induction on $n-r$. *Theorem.* An open set A in E^n is convex iff there exists a continuous faithful choice function on the space of caps of A. The proof of this uses the fact that if $\phi: S^{n-1}\rightarrow S^{n-1}$($S^{n-1}=n-1$ dimensional sphere) is not the identity but ϕ^2 is the identity and if $f: S^{n-1}\rightarrow S^{n-1}$ satisfies $f\phi=f$ then f has even degree. These theorems imply and are suggested by some of the work of Peano, Ascoli, Dupin, Krafft, and Steinhaus on centers of gravity. (Received April 18, 1957.)

617t. L. B. Treybig: *Concerning certain locally peripherally separable spaces.*

In this paper on a property related to separability in metric spaces, Journal of the Elisha Mitchell Scientific Society, vol. 70, no. 1, June, 1954, F. Burton Jones indicated that the question whether or not every connected, locally peripherally separable, metric space is separable seems to remain unanswered. In the present paper the author answers the question in the negative. (Received April 4, 1957.)

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BOOK REVIEWS

Einführung in die Operative Logik und Mathematik by Paul Lorenzen, Berlin, Göttingen, Heidelberg, Springer-Verlag, 1955. 298 pp.

This book presents in more detail than earlier articles an important and original approach to the foundations of logic and of mathematics. Both disciplines are reinterpreted to have as subject matter operations according to schematic rules. Hence the name *operative*. For convenience, only rules for operating on symbol strings are considered, each list of such (primitive) rules determining a formal system (*Kalkül*). A (well-formed) rule, metarule, metametarule, . . . of a formal system K can be written in the form $A_1, \cdots, A_n \rightarrow A$, where enough dots are to be added above $\rightarrow$ to indicate the composition, where $\rightarrow$ is to be omitted when $n=0$, and where $A_1, \cdots, A_n, A$ are respectively formulas (i.e., strings consisting of atoms of K and of variables ranging over the atom strings of K), rules, metarules, . . . of K. It is *admissible in* K if respectively the derivable atom strings, admissible rules, admissible metarules, . . . of K are closed under it. *Consequential logic* is then a theory of *universal* admissibility, i.e., of forms of rules, metarules, . . . which are admissible in any system K. For one after another of the symbols $\wedge$, $\vee$, $\wedge_x$, $\vee_x$, $\urcorner$, $\equiv$, and ι_x admissibility in K (or a suitable extension of K) of rules, metarules, . . . containing these symbols is then defined, the technical device used depending on the desired formal properties of the symbol introduced. Intuitionistic and also classical systems of logic for the corresponding symbols are then likewise theories of universal admissibility, each theorem being the form of a universally admissible rule, metarule, ... For any K, x ranges here over the *objects* of K, given as the theorems of some K', $\equiv$ indicates similar shape of objects, and ι_x serves as description operator in forming *terms* with properties similar to objects.

Terms serving as abstractions are introduced in a similar manner. If $\rho(x, y)$ is an *abstract equality relation* in K in the sense that $\rho(x, x)$ and $\rho(x, z) \wedge \rho(y, z) \rightarrow \rho(x, y)$ are admissible in K, then admissibility in K is defined for those rules, metarules, . . . containing $\iota_y\rho(X, y)$ which satisfy a certain requirement of *compatibility* with ρ. $\iota_y\rho(X, y)$ might ordinarily be regarded as a name for the class of symbol strings equivalent under ρ to X, but here it is treated *as* a class, Cantor's notion of class being discarded as too vague. In particular, if "$=$" is some abstract equality relation in K, if $X(z)$ is an object form in the sense that substitution of objects for z yields objects, if $y(z)$ ranges over such object forms, and if $\rho(X, y)$ is the abstract equality relation $\Lambda_z \cdot X(z) = y(z)$, then the term $\iota'_{y}\rho(X, y)$ is called a *function*, and is often written as $\lambda_z X(z)$. Similarly, for systems K in which $\Lambda_x \cdot A(x) \leftrightarrow B(x)$ or $\Lambda_{x_1 \cdots x_n} \cdot A(x_1, \cdots, x_n) \leftrightarrow B(x_1, \cdots, x_n)$ respectively are abstract equality relations $\rho(A, B)$, the term $\iota'_{B}\rho(A, B)$ is called a *class* or ***relation*** respectively, and often written as $\kappa_x A(x)$ or $\kappa_{x_1 \cdots x_n} A(x_1, \cdots, x_n)$ respectively.

Arithmetic is regarded as the body of rules, metarules, . . . admissible in a system such as E below, reflecting only the most elementary processes. First, the system Z whose only two (primitive) rules are $|$ and $k \rightarrow k|$ furnishes the ***numbers***. Next, the system D whose only rules are $| = |$ and $k = l \rightarrow k| = l|$, where k and l range over numbers, yields theorems concerning similar shape of numbers. All of Peano's axioms then are true statements of admissibility in D. E may then be obtained, e.g., by adjoining to D two rules $\alpha(k, |, k|)$ and $\alpha(k, l, m) \rightarrow \alpha(k, l|, m|)$, so that $\iota_m \alpha(k, l, m)$ is a term, which can also be written as $k + l$, and by adjoining two similar rules for multiplication. The theories of rational and of algebraic numbers are obtained in a related manner.

The central topic of the book is the replacement of Cantor's notion of *set* by the notion of *formula of a* [denumerable] *language stratum* (*Schicht*) S_ν, the construction being such that formulas of, e.g., $S_{2\omega}$ can serve in place of Cantor's sets as an interpretation or *operative model* of analysis. If S_0 consists of the strings $X_1, X_2, \cdots$ formed from atoms $u_1, \cdots, u_i$ then the *formulas* of the *elementary language* S_1 *over* S_0 are obtained by adding to the formation rules of a first-order predicate calculus with individual terms $X_1, X_2, \cdots$ and relation signs $\rho_1, \cdots, \sigma_1, \cdots$ a rule for constructing further relation signs $|_{\rho}B$ which may enter into formulas, where B is an ***induction scheme*** for ρ. An induction scheme B for ρ is a conjunction of formulas $A_k \rightarrow \rho(X_{k1}, \cdots, X_{kn})$ in which the occurrences of ρ in A_k, if any, are such that, roughly speaking, if admissibility in a "system" K is a ***definite*** notion for the formulas not containing ρ, then for all $\rho(X_1, \cdots, X_n)$ admissibility in the "system" K' obtained by adding B to K is also definite, so that admissibility of $X_1, \cdots, X_n \in |_{\rho}B$ in K can then be equated with admissibility of $\rho(X_1, \cdots, X_n)$ in K'. Here K and K' may be ***improper*** systems since, e.g., those X_j such that $\Lambda_y \sigma_k(y, X_j)$ is admissible in K may not form a recursively enumerable set so that the rule $\Lambda_y \sigma_k(y, x) \rightarrow \rho(x)$ may be nonconstructive. The formation of $|_{\rho}B$ and, at least vaguely, the idea of replacing sets by formulas was suggested by Weyl's *Das Kontinuum*. New is the iteration of this method, $S_{\nu+1}$ being formed over S_ν as S_1 over S_0, its strings including those of S_ϑ and the formulas of $S_{\vartheta+1}$. When Θ is a limit ordinal, then S_Θ is the union of all S_ϑ, $\vartheta<\Theta$, and not the language over a class of atoms. The class of induction schemes in S_ϑ and that of formulas in S_ϑ can be represented in $S_{\vartheta+1}$, in the sense that, e.g., a relation sign φ can be introduced by induction schemes in $S_{\vartheta+1}$ such that $\varphi(X)$ is admissible in the "system" obtained from the induction schemes if and only if X is a formula of S_ϑ. Among the consequences are the following: There are classes of strings in S_0 which can be represented in $S_{\vartheta+1}$ but not in S_ϑ, so that power set becomes a relative notion. The classes representable in S_ϑ of strings in S_0 can be enumerated in $S_{\vartheta+1}$ but not in S_ϑ, so that enumerability becomes a relative notion. The axiom of choice holds in the sense that for any family M of nonempty sets which is representable in S_ϑ, a function can be represented in S_ϑ assigning to each set in M one of its elements (namely that with the least Gödel number).

Ordinary analysis is largely preserved. While orginally its "well-roundedness" was obtained by enlarging what "exists," it now is due to suitable restrictions of expressive means. The real numbers, obtained by abstraction from certain formulas of S_{Θ_1}, satisfy the Dedekind completeness theorem, while the theory of functions of real numbers in S_{Θ_1} requires an S_{Θ_2}, where Θ_1 and Θ_2 are limit ordinals and $\Theta_1<\Theta_2$. Even though in an absolute sense each set is denumerable, a nontrivial Lebesgue measure can be defined, the role of denumerable sets now being played by those in S_{Θ_1}.—In the concluding part a sketch is given of how set- or model-theoretic considerations or ordinary mathematics concerning axiom systems are to be treated.

The reviewer believes that the book does not contribute directly to the formalist program nor to the intuitionist program but rather presents a legitimate and probably fruitful third approach to foundations. Its evident advantage over intuitionism is the preservation of classical logic and arithmetic and of much larger portions of the rest of mathematics, and over formalism the interpretation of these. Its evident disadvantage is the resort to notions which are not constructive but only *definite*, perhaps in the sense of being expressible in the Kleene hierarchy. An evaluation of the advantage over formalism depends of course on one's answer to these questions: Is an interpretation necessary or even desirable? How "natural" is the given interpretation in terms of operations on symbol strings? What alternative interpretations are available? The interpretation given here seems most convincing in the case of set theory and especially of analysis, where it seems hardly less "natural" than the already "artificial" arithmetized analysis and indeed may be regarded as a further development of it. One should like to know therefore whether, without sacrificing its empirical spirit, one can modify this interpretation of set theory and then wed it to a more common interpretation of elementary logic. (On the other hand, one may also wish to develop a constructive formal system of strata as a strengthening of the ramified theory of types, and then look for a more traditional interpretation of it.)

The book may lead to results outside the field of foundations, since it is often convenient to adjoin "purely symbolic" operations to operations that can be interpreted. Admissibility is then a key concept. However, no results have been produced so far. Thus, while the similarity up to now of operative to ordinary analysis is gratifying, it is also a little disappointing.

The main weakness of the book is its failure to indicate as clearly as a work on foundations should the strength of the methods employed and thus of the underlying assumptions. Claims that only the principles of the chapter on *Protologic* are employed and that no understanding of other logical notions is required are unconvincing, and seem unnecessary. Understanding the notion of an improper system, e.g., requires more than ability to check or even to anticipate the use of rules. A discussion would have been helpful of the extent to which the arguments used could become subjects of direct consistency proofs. This would have shown more clearly the relationship to Hilbert's program, the places where transfinite induction is used, and the places where formalization seems impossible. Arguments concerning constructive *eliminability* seem in general more elementary in nature than those concerning admissibility. Hence a clearer description of this notion for metarules, metametarules, . . . and a clearer labelling of those cases where actual eliminability has been proved would have been desirable. Failure to maintain the distinction between eliminability and admissibility also makes the discussion of the law of excluded middle and of *stability* (§8, §16) puzzling, since $\neg\neg A \rightarrow A$ seems always admissible though not always eliminable.

Other shortcomings of the exposition are the following: It is often too sketchy or condensed to allow careful checking. Examples are §10 and lack of a careful definition of *predecessor*, needed for *fundiert* which is critical for the notion of induction scheme. Other distinctions besides that between admissibility and eliminability are not maintained rigorously, e.g., that between rules, metarules, . . . and their forms (§6) or that between admissibility and derivability (§8, §9). The underlying system K, and indeed the fact that admissibility in K is being examined, is not always clearly indicated. In connection with relative admissibility (§7), two major though probably corrigible errors occur. First, as H. B. Curry discovered, the rule $a, b \rightarrow a \wedge b$ is not relative admissible, e.g., in the system whose only rules are $+$ and $+a \rightarrow 0$. Second, $a \wedge b \rightarrow a$ is not admissible, e.g., in the system whose only rules are $+$ and $+a+ \rightarrow ++a+$ and (D_1^0). Despite all these shortcomings, which probably require a thorough revision to overcome, the book constitutes an essentially sound and indeed outstanding contribution.

WILLIAM CRAIG

Introduction to Mathematical Logic, Vol. I, by Alonzo Church. Princeton, The Princeton University Press, 1956. 10+376 pp. $7.50.

This is a revised edition of the slim, paper-backed volume which appeared in 1944 as one of the Annals of Mathematics Studies. Of the five chapters two are devoted to propositional calculus, two to functional calculi of first order, and the last to second-order calculi, so that the plan of the original edition is followed to treat the most basic formal systems of mathematical logic. But the material of the original has been so greatly altered and expanded that it seems best to report the important features of the new work directly, rather than to compare it in detail with its predecessor.

Several unusual features of the book are apparent even before one begins to read: There is an introduction 68 pages in length; it is divided into 10 sections, which is exactly the number of sections in each of the five chapters; there are 590 consecutively numbered footnotes, at least one of which is a full page in length; there was a lapse of five years between the completion of the writing and the appearance of the published volume. While there are separate sections for historical notes, these spill over liberally into the highly informative footnotes; so that when the author's pre-eminent reputation for painstaking attention to historical detail is considered, it seems manifest that this work will quickly establish itself as a definitive reference volume. The book also contains a great many exercises, ranging from simple illustrations to brief sketches of developments not treated in the text. For some curious reason textbooks in symbolic logic have always evinced a conspicuous paucity of problems suitably challenging to mathematically inclined students, a phenomenon which has tended to place this subject at a competitive disadvantage with most of the other mathematical sciences. The present innovation is thus very welcome, and will greatly enhance the value of the book's use in connection with beginning courses for students with some background of mathematical experience.

The long introduction is essentially a conceptual framework for a general theory of linguistic systems, important features of which, especially the treatment of the concepts of sense and denotation (which are fundamental to Church's analysis of language), the author traces back to Frege. From the strict viewpoint of pure mathematics it is possible to regard the technical development of the subject which is given in the five chapters as quite independent of the material contained in the introduction; but the emphasis throughout the book is on logic as *applied*, rather than pure, mathematics, and from this viewpoint the introductory material becomes relevant as explaining the relation between the mathematical theory of formal systems and the languages (of which these systems are idealized models) to which the theory applies. The linguistic analysis which the introduction affords is the subject of continuing controversy in philosophical circles, but this in no way reflects on the mathematical portions of the book. Despite its non-mathematical treatment the introduction holds much of interest for the working mathematician, as it illuminates many basic concepts applicable to mathematical language, concerning which vague and often erroneous ideas are widespread.

In defining "the logistic method" (p. 47) the author explains how some natural language, such as English, may be used as a metalanguage for describing and referring to a formalized object language. However, he emphasizes the desirability of restricting the use of English to a portion of that language which is "just sufficient to enable us to give general directions for the manipulation of concrete physical objects . . . "; in particular, "those additional portions of English are excluded which would be used in order to treat of infinite classes . . . ". It seems to this reviewer that these restrictions are not strictly observed in the author's practice, and that the question of whether they can in principle be followed is open to doubt. For example, at the beginning of the description of the first logistic system (p. 70) the definition of "well-formed formula" is given by stating three formation rules and then adding: "To complete the definition we add that a formula is well-formed if and only if its being so follows from the three formation rules. In other words, the well-formed formulas are the least class of formulas which contains all the formulas stated in 10i and 10ii and is closed under the operation of 10iii." The mention of this least class of formulas, or of the unspecified notion "follows from," seem rather clearly to be couched in portions of the English language which were proscribed.

The first chapter treats in detail a particular system of propositional calculus, including such standard topics as the deduction theorem, decision problem, duality, consistency, completeness, and independence. In the second chapter other complete systems of propositional calculus are considered more briefly, as well as partial systems (including the intuitionistic) and extended systems. Altogether 40 different systems are described, and each is given a name consisting of the letter "P" with sub- or super-script adjoined. This proliferation reflects the state of the literature of the subject, and has desirable features both from the points of view of textbook and reference work; but an untoward effect is to leave the unaided beginning reader confused by the sheer mass of detail. The mathematician who is seeking a first knowledge of technical logic in this book should be warned that the practice of this subject does not consist simply in the performance of endless (or even forty) variations on a theme.

Many systems of first-order functional calculi are considered in Chapters III and IV. The substitution rules are studied in detail, the prenex and Skolem normal forms are treated, as well as more advanced topics such as Gödel's completeness theorem, the Skolem-Löwenheim result, and the decision problem (solution of special cases, reductions of the general problem). The use of particular first-order calculi as systems for formalizing branches of mathematics, such as number-theory or geometry, by the addition of formal axioms which are not valid in arbitrary domains, is put off to section 55 (Postulate theory), which is imbedded in Chapter V (Functional calculi of second order); the fruitful mathematical applications of the completeness theorem to these systems, which have recently been obtained by several authors, are not mentioned. In this connection the reviewer regrets that Church did not see fit to extend the notion of first-order functional calculus so as to include systems which contain operation symbols in addition to functional (i.e., predicate) symbols. Such systems are particularly natural and useful in formalizing many mathematical theories, and the principal metamathematical results, such as Gödel's completeness theorem, apply to them equally. It is true that operation symbols are in principle dispensable when predicate symbols are available, but this fact is itself worthy of study in a comprehensive treatment such as the present volume, and certainly does not warrant ignoring a kind of symbol so widely used in mathematics.

In the final chapter on second-order calculi attention is principally bestowed on the *simple* calculi which correspond to formalized systems of classical mathematics; but there are also two sections devoted to the predicative and ramified calculi which were devised to meet criticisms developed as an outgrowth of "constructive" view-points in mathematics. For the simple calculi the reviewer's completeness theorem is established and then the addition of formal axioms of infinity, and of well-ordering, is considered.

At various points in the book reference is made to topics which are to appear in a projected second volume. A tentative table of contents for this volume is given, listing chapters on higher-order functional calculi, second-order arithmetic, Gödel's incompleteness theorems, recursive arithmetic, an alternative formulation of the simple theory of types, axiomatic set-theory, and mathematical intuitionism. The appearance of this volume promises to complete a work of great usefulness both for students and scholars, and it is to be hoped that a way can be found to shorten the publication time.

LEON HENKIN

Fundamental Concepts of Higher Algebra, by A. Adrian Albert, University of Chicago Press, 1956. 9+165 pp.

Finite fields are soiled: computing machines are beginning to use them. Dickson's *Linear groups and the Galois theory* is the classical exposition of the subject, but since it was written modern algebra has come into existence; Albert's avowed purpose is therefore to give us a timely, modern version of the theory, setting it within its proper context as part of general field theory. But books behave waywardly while they are being written; in a sort of Tristram Shandy fashion four-fifths of this one is over before we get around to the finite fields, and then what we do read about turns out to be a rather brief and odd assortment of material—perhaps representing what will be most useful to those with practical applications in mind. On the other hand, the main part of the book consists of a good, compact presentation of the essentials of modern algebra. As such, it has a wide potential audience and has in fact been written with a wide class of readers in mind. In short: a nice, lightweight algebra text with an addendum on finite fields.

A great deal is covered in its 150-odd pages. The first two chapters discuss in turn group theory (no operators) through the Jordan-Hölder theorem and basis theorem for abelian groups, then elementary ring and ideal theory, with a discussion of factorization for polynomials in one variable. Chapter three treats vector spaces and matrices; elementary transformations of matrices, the characteristic function, and elementary divisor theory are discussed in some detail and there is in the problems a heavy classical emphasis on matricial computations: triangularization, determination of rank, inverse, and invariant factors. The fourth chapter deals with field extensions and

Galois theory, concluding with the normal basis theorem. The detailed study of particular fields is the only way to learn Galois theory; all algebra students should be directed to the more than two pages of specific computational problems about fields of all sorts that Albert includes here.

After deriving the basic elementary results, the bulk of the final chapter on finite fields is devoted to a somewhat heterogeneous collection of techniques for finding irreducible polynomials. A formula of Dedekind for the product of the irreducible polynomials of degree n over $GF(q)$ is proved; a similar but more refined formula of Dickson is stated later with an indication of proof and used as the basis of a computational procedure for determining all the polynomials. For certain not-too-special cases, Albert obtains more explicit determinations by proving theorems he attributes to Serret, these resting ultimately on the following simple lemma: if $f(x)$ is of degree m and irreducible over $GF(q)$, with roots of multiplicative period e, then $f(x^r)$ will also be irreducible, where r is any prime such that $r|e$, $r\nmid(q^m-1)e^{-1}$. Many specific computations using the methods are given, and again, there are plenty of problems. The chapter concludes in a curious fashion by stating without proof theorems of Dickson on a variety of subjects: specific equations of low degree, substitution polynomials, irreducibility criteria. Apparently, this section is meant as a summary (partial?) of Dickson's work in finite fields. If so, then first this should have been stated, and second, the assembling of results should not have been done in such an obviously hasty and disorganized manner—Dickson deserves better. But if it was meant to give one some idea of the modern theory of finite fields, then the omission of a result of such generality as the Riemann hypothesis is well-nigh incomprehensible. A bibliography and tables of least primitive roots and irreducible polynomials are appended.

The universal aims of this book place it more or less inevitably in the please-all-of-them-some-of-the-time category: everyone is offered a berth, but no one will find the fit quite exact. Those wanting to learn or review algebra will be happy to find that in remarkably few pages all the essentials are treated, and they will (or ought to be) delighted with the problems, which will help self-study greatly. But in the condensation the leisurely inter- and intra-theorem explanations of, say, the author's *Modern higher algebra* have had to be sacrificed, and the proofs have become little pieces of old hen: nourishing, but tough and chewy. Albert is well aware of this, of course; he has elected to write his book in a certain style and in so doing he is not without plenty of distinguished predecessors. My own feeling is that it is a poor style for a textbook—especially one for the generally undisciplined American. If the book accompanies a course or is being used for review—his own suggestions for its use—then this is perhaps not such a very serious matter, and after all, the theorems do in a sense speak for themselves. I think however that most students welcome toastmasters who are willing to say a little more than the inevitable and perfectly useless cant, "We have the following result . . . we next prove . . . ."

Those interested in finite fields will find the last chapter interesting and presumably useful; on the other hand they might wish the preceding exposition trimmed of the material which is not really relevant (all the matrix theory, for instance) and to see the extra space used to expose the finite fields in a more complete, more organized, and more explanatory fashion. The computers perhaps have no use for it, but I for one was pained by the omission of Chevalley's theorem that a finite field is quasi-algebraically closed ("Demonstration d'une hypothèse de M. Artin," which Albert points out was actually conjectured some 25 years earlier by Dickson—but Artin's reasons were elegant and convincing). It is a theorem which goes significantly into the structure of the finite field, and at the same time it has one of the most beautiful proofs in algebra. It seems that the machines are to crush our daisies, too.

ARTHUR MATTUCK

BRIEF MENTION

Einführung in die höhere Mathematik. vol. 1. By H. v. Mangoldt and K. Knopp. 10th ed. Stuttgart, S. Hirzel, 1956. 16+564 pp. DM 23.

A new edition of this textbook of basic analysis.

Cours de mathématiques. By J. Bass. Paris, Masson, 1956. 11+916 pp. 7800 fr. (paper), 8500 fr. (cloth).

A textbook of analysis through partial differential equations, intended primarily for engineering students.

A dictionary of statistical terms. By M. G. Kendall and W. R. Buckland. New York, Hafner, 1957. 11+493 pp. $4.50.

Prepared at the invitation of the International Statistical Institute, this book includes also glossaries of statistical terms in French, German, Italian and Spanish.

Theory of Lie groups I. By C. Chevalley. Princeton University Press, 1946. 232 pp. $2.75 (paper).

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CONTINUATION AND REFLECTION OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS[1]

FRITZ JOHN

A solution of an ordinary differential equation can be continued as long as its graph stays in the domain, in which the equation is regular. On the other hand a solution of a partial differential equation can have a *natural boundary* interior to the domain of regularity of the equation. Let R be a closed region and S a portion of the boundary of R. Then S is a natural boundary for a solution u defined in R, if there exists no solution defined in a full neighbourhood of a point of S which agrees with u in R. Examples for the occurrence of such natural boundaries are well known from the theory of harmonic functions. Neither the equation nor the solution has to show any very singular behavior on approaching a natural boundary S.

Only in very special situations can one prove that *every* sufficiently regular solution known in a region R can be continued across a portion S of the boundary. This is e.g. the case for solutions of a single differential equation which is hyperbolic with respect to S. It is also the case for solutions of certain *overdetermined* systems of equations, like those associated with analytic functions of several complex variables; other systems with this property have been studied by S. Bochner [1].

In more general cases the only solutions for which one can prove continuibility are those satisfying *suitable boundary* conditions on S. The classical example is furnished by solutions $u(x, y)$ of the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{1a}$$

defined for $y \geqq 0$ and satisfying the boundary condition

$$u(x, 0) = 0. \tag{1b}$$

They can always be continued across $y = 0$ by the formula

$$u(x, -y) = -u(x, y). \tag{1c}$$

An address delivered before the New York meeting of the Society on April 21, 1956, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors January 11, 1957.

[1] A summary of the results of this paper appeared in *Colloques internationaux du centre national de la recherche scientifique*, vol. 71, Nancy, 1956.

In recent years one has succeeded in extending this continuation theorem in various ways, involving either more general boundary conditions or more general differential equations. I mention the following generalizations. Thus a sufficiently regular solution of (1a) defined for $y \geq 0$ can be continued across $y=0$ whenever it satisfies instead of (1a) a boundary condition of the form

$$u_y = A(x, u, v, u_x) \tag{1d}$$

on $y=0$, where A is an analytic function of its arguments and v denotes the conjugate harmonic to u (See H. Lewy [2], R. Gerber [3]). A solution of any linear elliptic equation of any order and in any number of dimensions which has vanishing Dirichlet data on a hyperplane can be continued across that plane, provided the equation has analytic coefficients. (See Morrey [5], Nirenberg and Morrey [4].) A solution of the equation of minimal surfaces that is analytic on an analytic boundary curve can be continued across that curve. (See Lewy [15].)

Continuation of solutions constitutes an important tool in finding an explicit solution of many problems of applied mathematics. It has been used by Helmholtz, Kirchhoff, Shiffman and others in determining motions of liquids possessing a free boundary.[2] In these problems the continuation of the flow is *not explicit* as long as the location of the free boundary is not known; the free boundary and the flow are obtained simultaneously by also taking into account the boundary conditions on fixed boundaries and reducing the problem to one in conformal mapping. Explicit local continuation can be used, when the boundary conditions are sufficiently simple. This situation arises e.g. in the linearized theory of gravity waves, where one is led to consider solutions of the Laplace equation (1a), which satisfy a boundary condition of the form

$$u_y + ku = 0 \tag{2}$$

with constant k on the x-axis.[3] The continuation of solutions of more general second order equations with constant coefficients which satisfy boundary conditions of the type (2) on a plane has been studied by Diaz and Ludford [9].

A few cases in which solutions of higher order equations can be continued by an explicit reflection formula have been established. Poritsky [10] proved that a solution $u(x, y)$ of the bi-harmonic equation

$$\Delta^2 u = 0 \tag{3a}$$

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[2] See Lamb [6], Ch. IV, Milne-Thomson [7].

[3] See Stoker [8].

which satisfies the boundary conditions

$$u = u_y = 0 \qquad \text{for } y = 0 \tag{3b}$$

can be continued across the x-axis using the formula

$$u(x, -y) = -u(x, y) + 2yu_y(x, y) - y^2\Delta u(x, y). \tag{3c}$$

Analogous formulae have been obtained by Duffin [11; 12] for the systems of equations describing static equilibrium of a 3-dimensional isotropic elastic body. Finally Huber [13] has generalized formula (3c) to the case of solutions of a poly-harmonic equation

$$\Delta^p u(x_1 \cdots, x_n) = 0 \tag{4}$$

having vanishing Dirichlet data on a hyper-plane.

It seems desirable to have a more systematic study of the types of boundary conditions that guarantee existence of a continuation of the solution of a differential equation and also of the cases in which such continuation can be achieved by an explicit formula. Of course, such formulae can be expected only for very simple situations, say for linear equations with constant coefficients and for solutions satisfying linear homogeneous boundary conditions with constant coefficients on a plane boundary. I should like to present here some results in this direction for a rather simplified situation: The equation shall be linear and homogeneous and have constant coefficients, the boundary shall be plane, and the boundary conditions shall be of the type that the solution and a certain number of its derivatives vanish on the boundary plane. The results have been extended by R. I. Canavan to the most general equation with constant coefficients in his thesis on *Necessary conditions for continuation and reflection principles for solutions of linear partial differential equations with constant coefficients*, New York University, 1957.

There shall be $n+1$ independent variables $x_0, x_1, \cdots, x_n$. We write x for the n-vector $(x_1, \cdots, x_n)$, and $u(x_0, x)$ for the function $u(x_0, x_1, \cdots, x_n)$. The differential equation shall be of the form

$$P(\xi_0, \xi)u(x_0, x) = 0 \tag{5a}$$

where $P(\xi_0, \xi)$ is a form with constant complex coefficients of degree N in

$$\xi_0 = \partial/\partial x_0 \quad \text{and} \quad \xi = (\partial/\partial x_1 \cdots, \partial/\partial x_n).$$

Moreover we make the assumption

$$P(1, 0) = 1, \tag{5b}$$

equivalent to noncharacteristic character of the plane $x_0=0$. The boundary condition shall be of the form

$$(\xi_0^k u)_{x_0=0} = 0 \qquad \text{for } k = 0, \cdots, s-1. \tag{5c}$$

We shall show that in general if s is too small there exist solutions of (5a, c) which are in C_∞ for $x_0 \geqq 0$ and have the plane $x_0=0$ as natural boundary.

Solutions of partial differential equations with natural boundaries can be obtained in the form of series similar to those used for constructing analytic functions with natural boundaries. We consider the characteristic equation of equation (5a):

$$P(\eta_0, \eta) = 0. \tag{6}$$

For given $\eta = (\eta_1, \cdots, \eta_n)$ the N roots η_0 of (6) may be denoted by $\lambda_1, \cdots, \lambda_N$. Assume now that for a certain real η with $\eta \cdot \eta = 1$ the roots η_0 of (6) satisfy the conditions

$$\operatorname{Im} \lambda_1 \geqq \operatorname{Im} \lambda_2, \cdots, \operatorname{Im} \lambda_{s+1} \geqq 0 \tag{7a}$$

and

$$\operatorname{Im} \lambda_1 > 0. \tag{7b}$$

Let $f(x_0)$ be the solution of the ordinary differential equation

$$\prod_{k=1}^{s+1} (\xi_0 - i\lambda_k) f(x_0) = 0$$

with initial conditions

$$\xi_0^k f(x_0) = \begin{cases} 0 & \text{for } k = 0, \cdots, s-1, \\ 1 & \text{for } k = s \end{cases}$$

for $x_0=0$. Then, because of (7a), $f(x_0)$ and any of its derivatives become infinite at most like a power of x_0 for $x_0 \to +\infty$. The function

$$e^{i\eta \cdot x} f(x_0)$$

is a solution of (5a, c). It follows that

$$u(x_0, x) = \sum_{j=0}^{\infty} e^{-j^2} \exp(i2^j \eta \cdot x) f(2^j x_0) \tag{8}$$

for $x_0 \geqq 0$ is again a solution of (5a, c) and is of class C_∞. We shall prove that this function u has $x_0=0$ as natural boundary.

Since $\eta \neq 0$ there exists a real vector y such that

$$y \cdot \eta = 2\pi.$$

We have for any positive integer r and for $x_0 \geqq 0$ from (8)

$$u(x_0, x + 2^{-r}y) - u(x_0, x) = w_r(x_0, x), \tag{9}$$

where

$$w_r(x_0, x) = \sum_{j=0}^{r-1} e^{-j^2} \exp\ (i2^j\eta \cdot x) f(2^j x_0)(\exp\ (i2^{j-r+1}\pi) - 1) \tag{9a}$$

is an entire analytic solution of (5a).

If $u(x_0, x)$ would not have the plane $x_0 = 0$ as natural boundary there would exist a point $x = X$, a positive number ϵ, and a function $U(x_0, x)$ of class C_N in

$$|x - X| < \epsilon, \qquad |x_0| < \epsilon \tag{10}$$

which satisfies (5a) and agrees with $u(x_0, x)$ for $x_0 \geqq 0$.

The expression

$$V(x_0, x) = U(x_0, x + 2^{-r}y) - U(x_0, x) - w_r(x_0, x)$$

is a solution of (5a) defined for

$$|x - X| < \epsilon/2, \qquad |x_0| < \epsilon$$

provided r is so large that

$$2^r > 2|y|/\epsilon.$$

Moreover $V(x_0, x) = 0$ for $x_0 \geqq 0$. It follows from the uniqueness theorem of Holmgren[4] that there exists a positive ϵ', independent of r, such that $V(x_0, x) = 0$ for all x_0, x with

$$|x - X| < \epsilon', \qquad |x_0| < \epsilon'.$$

In other words, the functional equation (9) satisfied by u must be preserved under continuation.

Let $Q(\xi_0, \xi)$ denote the differential operator

$$Q(\xi_0, \xi) = \prod_{k=1}^{s+1} (\xi_0 - \lambda_k \eta \cdot \xi).$$

Since

$$\prod_{k=1}^{s+1} (\xi_0 - i\lambda_k) f(x_0) = e^{\alpha_1 x_0},$$

---

[4] See John [14].

we have for $|x-X|<\epsilon/2$, $-\epsilon<x_0<0$

$$
\begin{aligned}
&| Q(\xi_0, \xi)[U(x_0, x + 2^{-r}y) - U(x_0, x)]| \\
&\quad = \left| \sum_{j=0}^{r-1} 2^{sj}e^{-j^2} \exp(i2^j x\cdot\eta)(\exp(i2^{j-r+1}\pi) - 1) \exp(i2^j\lambda_1 x_0)\right| \\
&\quad \geqq 2\cdot 2^{s(r-1)} \exp(-(r-1)^2 - 2^{r-1}x_0 \operatorname{Im}(\lambda_1)) \\
&\qquad - 2 \exp(-2^{r-2}x_0 \operatorname{Im}(\lambda_1)) \sum_{j=0}^{\infty} 2^{sj}e^{-j^2}.
\end{aligned}
$$

Hence

$$\lim_{r\to\infty} | Q(\xi_0, \xi)[U(x_0, x + 2^{-r}y) - U(x_0, x)]| = \infty.$$

But this is incompatible with $U(x_0, x)$ belonging to class C_N. Consequently $u(x_0, x)$ has the plane $x_0=0$ as natural boundary, if we can find a real η with $|\eta|=1$ such that the corresponding roots λ_k of the characteristic equation (6) satisfy (7a, b). Since equation (6) is homogeneous in η_0 and η, the restriction $|\eta|=1$ is unessential.

We call boundary conditions for a solution on the plane $x_0=0$ *adequate* for continuation, if every sufficiently regular solution defined on one side of the plane in a neighborhood of the plane can be continued locally across that plane. We can then formulate our result as follows:

For the boundary conditions (5c) *to be adequate for continuation of a solution of* (5a) *it is necessary that for every real* η *the roots* λ_k *are either all real, or that at least* $N-s$ *of them have positive real part and at least* $N-s$ *of them have negative real part.*

Consequently for $s<N/2$, the boundary conditions (5c) can be adequate for continuation only if for every real η the roots λ_k are all real, or, in other words, they are never adequate unless the differential equation (5a) is hyperbolic with respect to the plane $x_0=0$.

If N is even and $s=N/2$, conditions (5c) express that u has vanishing Dirichlet data on $x_0=0$. Then the vanishing of the Dirichlet data can be adequate for continuation only if for any real η the solutions η_0 of (6) are either all real or all imaginary. They are all imaginary when equation (5a) is *elliptic*. In this case vanishing of the Dirichlet data is adequate for continuation by virtue of the result of Morrey and Nirenberg mentioned above. It is possible that the necessary condition for adequacy of boundary data is always sufficient, regardless of the type of the equation.

We apply the condition to the example of the equation of elastic waves in an istropic medium:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\left(\frac{\partial^2}{\partial t^2} - c'^2\Delta\right)u(x, y, z, t) = 0 \tag{11a}$$

where Δ is the Laplace operator in xyz-space. Here the boundary conditions

$$u = \frac{\partial u}{\partial x} = 0 \qquad \text{for } x = 0 \tag{11b}$$

are inadequate for continuation, if the constants c and c' are distinct. For, in that case, there exist real y, z, t such that the 4 roots of the characteristic equation

$$(t^2 - c^2(x^2 + y^2 + z^2))(t^2 - c'^2(x^2 + y^2 + z^2)) = 0$$

are neither all real nor all imaginary. If, on the other hand, $c = c'$, the conditions (11b) are adequate; indeed, then, continuation is achieved by the explicit formula (analogous to (3c))

$$u(-x, y, z, t) = -u + 2xu_x + \frac{x^2}{c^2}\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)u \tag{11c}$$

where on the right hand side the arguments of u are x, y, z, t.

Equation (11a) is hyperbolic with respect to the plane $t = 0$. Thus the boundary conditions

$$u = u_t = 0 \qquad \text{for } t = 0 \tag{11d}$$

are adequate for continuation, even when $c \neq c'$. Here again an explicit "reflection formula" exists, giving the continuation, which will be derived later (see formula (32c)).

We next turn to the question: Can continuation be achieved by a simple formula of the type (1c) or (3c), if the data are adequate for continuation? Instead of trying to give directly a more precise meaning to what constitutes a "simple formula" it is convenient to take notice of a more qualitative feature of formulae (1c), (3c), namely that they constitute "reflection principles." The values of the continued function at one point P are expressible in terms of the values of the original functions and of its derivatives in points on the line through P perpendicular to the boundary plane. We shall restrict ourselves to the case, where the order N of the differential equation is even, $N = 2m$, and where the boundary conditions on the plane $x_0 = 0$ consist in the vanishing of the Dirichlet data $(s = m)$:

$$(\overset{k}{\xi_0} u)_{x_0=0} = 0 \qquad \text{for } k = 0, \cdots, m - 1. \tag{12}$$

We are led to the following definition: Equation (5a) is said to possess a reflection principle (with respect to the Dirichlet data on the plane $x_0=0$), if there exists a number $\epsilon>0$ (depending only on P) such that every solution $u(x_0, x)$ of (5a), which is analytic in a cylindrical set which has the domain D in x-space as base

$$x \text{ in } D, \qquad 0 \leqq x_0 < h \tag{13a}$$

can be continued as a regular analytic solution into the set

$$x \text{ in } D, \ -\epsilon h < x_0 \leqq 0, \tag{13b}$$

provided u satisfies the boundary conditions (12) for x in D.

It turns out that all equations having a reflection principle in this sense can be characterized rather simply, and that existence of a *reflection principle* implies existence of an *explicit formula* for continuation of a solution.

It follows from the definition that an equation (5a) possesses no reflection principle, if there exists an analytic solution $u(x_0, x)$ of (5a) with vanishing Dirichlet data on $x_0=0$, which is regular for $x=0$, $x_0\geqq 0$, and has a singularity in some point of the negative x_0-axis. Let now η be any *complex* n-vector$\neq 0$, and let $\lambda_1, \cdots, \lambda_{2m}$ be the corresponding roots η_0 of (6). Then

$$\begin{aligned} u(x_0, x) &= \frac{x_0^m}{\prod_{k=1}^{m+1} (\eta\cdot x + \lambda_k x_0 + \lambda_1)} \\ &= \frac{(-1)^{m+1}}{2\pi i} \oint_C \frac{d\lambda}{(\eta\cdot x + \lambda x_0 + \lambda_1) \prod_{k=1}^{m+1} (\lambda - \lambda_k)}, \end{aligned}$$

(where C is a small path about the point $\lambda = -(\eta\cdot x+\lambda_1)/x_0$) is easily seen to be an analytic solution of (5a), (12) which is singular at the point

$$x = 0, \qquad x_0 = -1.$$

If this solution is regular for $x=0$, and $x_0\geqq 0$, there is no reflection principle. This is the case, when

$$\lambda_k x_0 + \lambda_1 \neq 0 \quad \text{for real } x_0 \geqq 0 \text{ and } k = 1, \cdots, m+1.$$

Thus necessary for the existence of a reflection principle is that whenever one root λ_1 is different from 0 then at least one out of any m of the remaining $2m-1$ roots lies on the ray opposite to that containing λ_1. In other words, it is necessary that for any complex η either the roots λ_k of (6) all vanish, or that none of them vanishes and that they all lie on the same line through the origin, half of them on either side of the origin.

We are led to look for the forms $P(\eta_0, \eta)$ whose roots $\eta_0 = \lambda_k$ lie on a straight line through the origin for any complex η. There would have to exist a real θ and real c_k such that

$$\lambda_k = c_k e^{i\theta} \qquad \text{for } k = 1, \cdots, 2m.$$

Let

$$P(\eta_0, \eta) = \eta_0^{2m} + q_1(\eta)\eta_0^{2m-1} + q_2(\eta)\eta_0^{2m-2} + \cdots + q_{2m}(\eta),$$

where $q_k(\eta)$ is a form of degree k in $\eta = (\eta_1, \cdots, \eta_n)$; $q_k(\eta)e^{-ik\theta}$ would have to be real for any $k = 1, \cdots, 2m$. Hence for any complex η and any $k, j = 1, \cdots, 2m$

$$\frac{(q_k(\eta))^j}{(q_j(\eta))^k} \tag{14}$$

would have to be real, whenever $q_j(\eta) \neq 0$. It follows that the expression (14) must be constant, since an analytic function of η which is real for all complex η in the neighbourhood of a point in η-space must be a constant.

Let S be the subset of the set of integers $j = 1, \cdots, 2m$, for which $q_j(\eta) \not\equiv 0$. Unless $P \equiv \eta_0^{2m}$, the set S is not empty. Let μ be the greatest common divisor of all the numbers in S. We can represent μ in the form

$$\mu = \sum_{j \subset S} j n_j$$

with integral coefficients n_j. Let $Q = Q(\eta)$ be the rational function defined by

$$Q = \prod_{j \subset S} q_j(\eta)^{n_j}.$$

Then for any k in S

$$\frac{q_k(\eta)^\mu}{Q(\eta)^k} = \prod_{j \subset S} \frac{q_k(\eta)^{jn_j}}{q_j(\eta)^{kn_j}} = \text{constant} = (\gamma_k)^\mu$$

or

$$q_k(\eta) = \gamma_k(\eta) Q(\eta)^{k/\mu} \tag{15}$$

where $(\gamma_k)^\mu$ is independent of η and $\neq 0$. Since the polynomial $q_k(\eta)$

is bounded for all bounded η, the same holds for the rational function $Q(\eta)$, which consequently must be a polynomial in η. The number k/μ is an integer for k in S, and thus, by (15), γ_k is independent of η. It follows that P is of the form

$$P = \sum_{0\leq r\leq 2m/\mu} \eta_0^{2m-r\mu} c_r Q(\eta)^r. \tag{16}$$

Here the c_r are constant, and Q is a form of degree μ. Whenever $\eta_0=\lambda$ is a root of the polynomial P given by (16), $\epsilon\lambda$ is also a root, where ϵ is any μth root of unity. For $\lambda\neq 0$, $\mu>2$, the various $\epsilon\lambda$ do not all lie on a straight line. Hence P is either given by η_0^{2m} or by (16), where Q is a form of degree μ with $\mu=1$ or $\mu=2$. In either case we can factor P, and conclude that a reflection principle is possible only if P is of one of the following 3 types:

$$P = \eta_0^{2m}, \tag{17a}$$

$$P = \prod_{k=1}^{2m} (\eta_0 - a_k L(\eta)), \tag{17b}$$

$$P = \prod_{k=1}^{m} (\eta_0^2 - a_k^2 q(\eta)). \tag{17c}$$

Here the a_k are constant, and L and q denote, respectively, a linear or a quadratic form in η which is independent of k. Since the roots η_0 must either vanish or lie on one line through the origin, half on each side, we can change L or q by a constant factor, so that in (17b) the a_k are all real, $a_1, \cdots, a_m$ positive, $a_{m+1}, \cdots, a_{2m}$ negative, and in (17c) so that all the a_k are real and positive.

It remains to show that for any equation of one of the types (17a, b, c) with real a_k of the correct sign there actually exists a reflection principle. This is achieved by constructing an explicit formula that represents a continuation of the desired type. We omit the case of an equation of type (17a) in which continuation is trivial from further consideration. To arrive at reflection formulae for equations of type (17b, c), we first derive such a formula for the case of an ordinary differential equation with constant coefficients.

Let $p(\lambda)$ be a polynomial of degree $2m$ with roots $a_1, \cdots, a_{2m}$, where $a_k\neq a_j$ for $k=1, \cdots, m$ and $j=m+1, \cdots, 2m$, and all the a_i are different from 0. We consider a solution $u(x_0)$ of the equation

$$P(\xi_0)u(x_0) = 0 \tag{18a}$$

for which

$$(\xi_0^k u(x_0))_{x_0=0} = 0 \qquad \text{for } k = 0, 1, \cdots, m-1. \tag{18b}$$

We put

$$A(\lambda) = \prod_{k=1}^{m} (\lambda - a_k), \qquad B(\lambda) = \sum_{k=m+1}^{2m} (\lambda - a_k), \tag{18c}$$

so that $p(\lambda) = A(\lambda)B(\lambda)$. Let C' and C'' be two closed paths in the λ-plane not containing the origin which are exterior to each other, and such that C' contains $a_1, \cdots, a_m$ and C'' contains $a_{m+1}, \cdots, a_{2m}$. We define for any λ, μ the polynomial $R(\lambda, \mu)$ by

$$R(\lambda, \mu) = \frac{p(\lambda) - p(\mu)}{\lambda - \mu}.$$

Then for any integer $j \geqq 0$ the identity

$$\begin{aligned} \xi_0^j u(x_0) &= (2\pi i)^{-2} \oint_{C'} d\alpha \oint_{C''} d\beta \\ &\quad \frac{\beta^j R\left(\frac{\alpha}{\beta}\xi_0, \alpha\right) u\left(\frac{\beta}{\alpha}x_0\right) - \alpha^j R\left(\frac{\beta}{\alpha}\xi_0, \beta\right) u\left(\frac{\alpha}{\beta}x_0\right)}{(\beta - \alpha)A(\alpha)B(\beta)} \end{aligned} \tag{19}$$

is satisfied for every solution u of (18a, b). The expression on the righthand side of (19) is clearly defined since any solution u of (18a) is analytic, and $(R(\alpha/\beta)\xi_0, \alpha)$ is a differential operator depending regularly on the parameters α, β.

Identity (19) can be verified most easily by observing that the general solution u of (18a, b) is of the form

$$u(x_0) = \frac{1}{2\pi i} \oint_C \frac{h(\lambda)e^{\lambda x_0}}{p(\lambda)} d\lambda, \tag{20}$$

where $h(\lambda)$ is an arbitrary polynomial in λ of degree $< m$, and C is a path containing all roots of p. We may assume that C contains both C' and C'' in its interior. Then

$$\begin{aligned} R\left(\frac{\alpha}{\beta}\xi_0, \alpha\right) u\left(\frac{\beta}{\alpha}x_0\right) &= \frac{1}{2\pi i} \oint_C \frac{p(\lambda) - p(\alpha)}{(\lambda - \alpha)p(\lambda)} h(\lambda) e^{\beta\lambda x_0/\alpha} d\lambda \\ &= h(\alpha)e^{\beta x_0} - \frac{p(\alpha)}{2\pi i} \oint_C \frac{h(\lambda)}{(\lambda - \alpha)p(\lambda)} e^{\beta\lambda x_0/\alpha} d\lambda. \end{aligned}$$

The second term makes no contribution, since

$$\frac{p(\alpha)}{(\beta - \alpha)A(\alpha)(\lambda - \alpha)} e^{\beta\lambda x_0/\alpha}$$

is a regular function of α for α inside C', β on C'', λ on C and the point $\alpha=0$ outside C'. Moreover, from the theory of residues

$$\frac{1}{2\pi i}\oint_{C'} \frac{h(\alpha)}{(\beta-\alpha)A(\alpha)}\,d\alpha = \frac{h(\beta)}{A(\beta)},$$

since h is of lower degree than A, and $\alpha=\beta$ is the only pole of the integrand outside C'. It follows that

$$(2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta\, \frac{\beta^j R\left(\dfrac{\alpha}{\beta}\xi_0, \alpha\right) u\left(\dfrac{\beta}{\alpha}x_0\right)}{(\beta-\alpha)A(\alpha)B(\beta)} = \frac{1}{2\pi i}\oint_{C''} \frac{\beta^j e^{\beta x_0} h(\beta)}{p(\beta)}\,d\beta.$$

Evaluating the remaining portion of the right hand side of (19), in a similar way, the validity of the identity (19) becomes obvious if we compare it with the expression for $\xi_0^j u(x_0)$ arising from (20).

In principle the right hand side of identity (19) can be evaluated by the theory of residues. It then yields a representation of $\xi_0^j u(x_0)$ in terms of u and its derivatives taken for the arguments $a_k x_0/a_j$ and $a_j x_0/a_k$ where $k=1, \cdots, m$ and $j=m+1, \cdots, 2m$. If all a_k are positive all a_j negative, this represents a reflection principle, since all a_k/a_j are then negative and since for $x_0<0$ we can express $\xi_0^j u(x_0)$ in terms of u and its derivatives for positive arguments. If all a_k, a_j are distinct we have, for example, the formula

$$\xi_0^j u(x_0) = \sum_{k=1}^{m}\sum_{r=m+1}^{2m} \frac{a_r^j R\left(\dfrac{a_k\xi_0}{a_r}, a_k\right) u\left(\dfrac{a_r x_0}{a_k}\right) - a_k^j R\left(\dfrac{a_r\xi_0}{a_k}, a_r\right) u\left(\dfrac{a_k}{a_r}x_0\right)}{(a_r-a_k)A'(a_k)B'(a_r)}. \tag{20a}$$

Here

$$R\left(\frac{a_k\xi_0}{a_r}, a_k\right) u\left(\frac{a_r x_0}{a_k}\right) = \left[\frac{p(d/ds)}{d/ds-a_k}u(s)\right]_{s=a_r x_0/a_k}.$$

If we differentiate identity (19) k times with respect to x_0 and put $x_0=0$ we obtain the relation

$$[\xi_0^{j+k} u(x_0)]_{x_0=0} = \left[F_{jk}\left(\frac{d}{ds}\right)u(s)\right]_{s=0} \tag{21}$$

where $F_{jk}(\lambda)$ is a polynomial of degree $\leqq 2m-1+k$:

$$(22)\quad F_{jk}(\lambda) = (2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta \frac{\lambda^k[\beta^{k+j}\alpha^{-k}R(\lambda, \alpha) - \alpha^{k+j}\beta^{-k}R(\lambda, \beta)]}{(\beta - \alpha)A(\alpha)B(\beta)}.$$

There exist polynomials $G_{jk}(\lambda)$ and $H_{jk}(\lambda)$ such that

$$(23)\qquad \lambda^{j+k} = F_{jk}(\lambda) + G_{jk}(\lambda) + H_{jk}(\lambda)p(\lambda)$$

where G_{jk} is of degree $<2m$. Then from (21) we have

$$[G_{jk}(\xi_0)u(x_0)]_{x_0=0} = 0$$

for any solution u of (18a, b). Since for such a u the values of $[\xi_0^k u(x_0)]_{x_0=0}$ vanish for $k=0, \cdots, m-1$ and are arbitrary for $k=m, m+1, \cdots, 2m-1$, it follows that G_{jk} must be of degree $<m$. Hence there exists an identity of the form (23) with a polynomial G_{jk} of degree $<m$.

We are now in a position to write down a reflection principle for an equation of the type (17b). Let $u(x_0, x)$ be a solution of the differential equation

$$(24)\qquad P(\xi_0, \xi)u(x_0, x) = \prod_{k=1}^{2m} (\xi_0 - a_k L(\xi))u(x_0, x) = 0$$

which is analytic for real x_0, x in a set (13a), and satisfies the Dirichlet conditions (12) for x in D. Here the a_k shall be real numbers, $a_1, \cdots, a_m$ positive, $a_{m+1}, \cdots, a_{2m}$ negative, and L shall be a linear form with constant (real or complex) coefficients.

Formally the conditions on u are of the type (18a, b), only with a_k replaced by $a_k L(\xi)$. For $j=2m-1$, we are then led to the identity

$$(25)\qquad \xi_0^{2m-1}u(x_0, x) = v(x_0, x)$$

where

$$(26\text{a})\quad v(x_0, x) = (2\pi i)^{-2}\oint_{C'} d\alpha \oint_{C''} d\beta \frac{S}{(\beta - \alpha)A(\alpha)B(\beta)},$$

$$(26\text{b})\qquad \begin{aligned} S = {} & \beta^{2m-1}\frac{P\left(\dfrac{\alpha}{\beta}\xi_0, \xi\right) - P(\alpha L(\xi), \xi)}{\dfrac{\alpha}{\beta}\xi_0 - \alpha L(\xi)}\, u\left(\frac{\beta}{\alpha}x_0, x\right) \\ & - \alpha^{2m-1}\frac{P\left(\dfrac{\beta}{\alpha}\xi_0, \xi\right) - P(\beta L(\xi), \xi)}{\dfrac{\beta}{\alpha}\xi_0 - \beta L(\xi)}\, u\left(\frac{\alpha}{\beta}x_0, x\right). \end{aligned}$$

Here the polynomials $A(\alpha)$, $B(\beta)$ are defined by (18c); the path C' can be chosen close to the smallest interval on the positive real axis containing the points $a_1, \cdots, a_m$ and C'' close to the smallest interval on the negative real axis containing the points $a_{m+1}, \cdots, a_{2m}$. Then α/β and β/α are always close to a negative real number whose absolute value does not exceed the largest of the values $-a_k/a_j$. Let ϵ be the positive number defined by

$$\frac{1}{\epsilon} = \operatorname*{Max}_{k,j=1,\cdots,2m} \frac{-a_k}{a_j}.$$

Then for α on C', β in C'' the quantities α/β and β/α will be close to negative real numbers lying in the interval from $-1/\epsilon$ to 0. Furthermore, for x in D and $-\epsilon h < x_0 \leqq 0$ the quantities $u((\alpha/\beta)x_0, x)$ and $u((\beta/\alpha)x_0, x)$ will be defined as well as S which represents a differential expression on u and depends regularly on the parameters α, β. (The corresponding formula for $j < 2m-1$ would still contain the factor $L(\xi)^{j-2m+1}$, which would not be a *differential* operator.) It follows that the function $v(x_0, x)$ is defined and analytic for real x_0, x in the set (13b). In order to prove that it represents a continuation of $\xi_0^{2m-1}u$ it is sufficient to compare the derivatives with respect to x_0 at $x_0=0$. Comparison with formula (22) shows that this amounts to proving the identity

$$(27)\quad [\xi_0^{2m-1+k}u(x_0, x)]_{x_0=0} = \left[L^{2m-1+k}(\xi)F_{2m-1\,k}\left(\frac{1}{L(\xi)}\frac{d}{ds}\right)u(s, x)\right]_{s=0}$$

for $j=2m-1$, $k=0, 1, 2, \cdots$ and any solution u of (24), (12). By (23) we have

$$\begin{aligned}\xi_0^{2m-1+k} &= L^{2m-1+k}F_{2m-1\,k}(L^{-1}\xi_0) + L^{2m-1+k}G_{2m-1\,k}(L^{-1}\xi_0)\\ &\quad + L^{k-1}H_{2m-1\,k}(L^{-1}\xi_0)L^{2m}p(L^{-1}\xi_0)\end{aligned}$$

identically in ξ_0 and L. For $L=L(\xi)$ both sides represent differential operators. We apply this identity to a solution of (24), (12) obtaining (27) since

$$L^{2m}(\xi)p(L^{-1}(\xi)\xi_0)u(x_0, x) = P(\xi_0, \xi)u(x_0, x) = 0$$

and since also

$$[L^{m-1}(\xi)G_{2m-1\,k}(L^{-1}(\xi)\xi_0)u(x_0, x)]_{x_0=0} = 0,$$

$L^{m-1}(\xi)G_{2m-1\,k}(L^{-1}(\xi)\xi_0)$ being a polynomial of degree $<m$. This completes the proof of the fact that $v(x_0, x)$ represents an analytic continuation of $\xi_0^{2m-1}u(x_0, x)$.

The function v is given by a double integral in the complex α- and β-planes; the integrand has poles at the zeros of $A(\alpha)$ and $B(\beta)$. The integral can be expressed by the residues at these poles, which in turn can be written as differential operators acting on the integrand. In principle then $v(x_0, x)$ can be expressed as a combination of derivatives of u taken at the points

$$\left(\frac{a_k}{a_j}\, x_0, x\right) \quad \text{and} \quad \left(\frac{a_j}{a_k}\, x_0, x\right)$$

with $k=1, \cdots, m$ and $j=m+1, \cdots, 2m$. The resulting formula involves only u and a finite number of its derivatives in a number of real points, all lying on the same perpendicular to the plane $x_0=0$. In case all the a_i are distinct the formula is of the form (20a):

$$\xi_0^{2m-1} u(x_0, x) = \sum_{k=1}^{m} \sum_{r=m+1}^{2m} \frac{S_{kr}}{(a_r - a_k)A'(a_k)B'(a_r)},$$

$$S_{kr} = a_r^{2m-1} \left[\frac{P\left(\frac{d}{ds}, \xi\right) - P(a_k L(\xi), \xi)}{\frac{d}{ds} - a_k L(\xi)}\, u(s, \xi) \right]_{s=a_r x_0/a_k}$$

$$- a_k^{2m-1} \left[\frac{P\left(\frac{d}{ds}, \xi\right) - P(a_r L(\xi), \xi)}{\frac{d}{ds} - a_r L(\xi)}\, u(s, \xi) \right]_{s=a_k x_0/a_r}.$$

The identity (25) gives a continuation v of $\xi_0^{2m-1}u$. If we define a function $U(x_0, x)$ in (13b) by the conditions

$$(28) \qquad \xi_0^{2m-1} U = v, \qquad (\xi_0^k U)_{x_0=0} = (\xi_0^k u)_{x_0=0} \qquad \text{for } k = 0, \cdots, 2m-2,$$

U and u will agree with all their derivatives at $x_0=0$, and hence U will represent an analytic continuation of u. The determination of U in terms of v involves only repeated integration along the line $x_0=$ constant. This establishes the existence of a reflection principle in the sense defined here for equations of the type (17b). It is easily seen that the resulting formulae yield a continuation of a solution u of (24), (12) not only when u is analytic, but whenever u is sufficiently often differentiable.

Reflection principles for equations of type (17c) are obtained from the same basic formula (19), taken for $j=2m-2$. If $p(\lambda)$ is an *even* polynomial, with roots

$$\lambda = \pm a_k \qquad \text{for } k = 1, \cdots, m,$$

we have $B(\lambda)=(-1)^m A(-\lambda)$. Replacing β by $-\beta$ in (19) we can choose the same path of integration on C' for α and β. We then arrive at the formula

$$\xi_0^{2m-2} u(x_0) = \frac{2(-1)^m}{(2\pi i)^2}\oint_{C'} d\alpha \oint_{C'} d\beta \, \frac{\alpha\beta^{2m-2}}{(\alpha+\beta)A(\alpha)A(\beta)} \cdot \frac{p\left(\frac{\alpha}{\beta}\xi_0\right) - p(\alpha)}{\frac{\alpha^2}{\beta^2}\xi_0^2 - \alpha^2}\, u\left(\frac{-\beta}{\alpha}x_0\right). \tag{29}$$

Here for positive real $a_1, \cdots, a_m$ the path C' can be taken as a set of small circles about the points $a_1, \cdots, a_m$.

Let now $u(x_0, x)$ be a function which is analytic in the set (13a), satisfies the differential equation

$$P(\xi_0, \xi)u = \prod_{k=1}^{m} (\xi_0^2 - a_k^2 q(\xi))u = 0 \tag{30}$$

and the Dirichlet conditions (12), and let

$$\begin{aligned} v(x_0, x) &= \frac{2(-1)^m}{(2\pi i)^2}\oint_{C'} d\alpha \oint_{C'} d\beta \, \frac{\alpha\beta^{2m-2}S}{(\alpha+\beta)A(\alpha)A(\beta)}; \\ S &= \frac{P\left(\frac{\alpha}{\beta}\xi_0, \xi\right) - P(\alpha(q(\xi))^{1/2}, \xi)}{\frac{\alpha^2}{\beta^2}\xi_0^2 - \alpha^2 q(\xi)}\, u\left(\frac{-\beta}{\alpha}x_0, x\right), \end{aligned} \tag{31}$$

then v is defined and analytic in the set (13b), where

$$\epsilon = \min_{k,j=1,\cdots,m} \frac{a_j}{a_k}.$$

By the same argument as before, comparing derivatives at $x_0 = 0$, one verifies that

$$\xi_0^{2m-2} u(x_0, x) = v(x_0, x).$$

Repeated integration with respect to x_0 then furnishes an explicit reflection formula for $u(x_0, x)$ itself.

A typical example is given by the case $m = 2$ with distinct a_k. Let $u(x_0, x)$ be a solution of the equation

$$(\xi_0^2 - a_1^2 q(\xi))(\xi_0^2 - a_2^2 q(\xi))u(x_0, x) = 0, \tag{32a}$$

where q is a quadratic form and a_1, a_2 are distinct real positive quantities. Let u satisfy the initial conditions

$$u = \xi_0 u = 0 \quad \text{for } x_0 = 0. \tag{32b}$$

The residues at the simple poles of the integrand in the expression for v are easily evaluated, and one finds the formula

$$\begin{aligned}(a_1 + a_2)(a_1 - a_2)^2 \xi_0^2 u(x_0, x) &= \left[(a_1 + a_2)\left((a_1^2 + a_2^2)\frac{\partial^2}{\partial s^2} - 2a_1^2 a_2^2 q(\xi)\right)u(s, x)\right]_{s=-x_0} \\ &\quad - 2a_1 a_2^2\left[\left(\frac{\partial^2}{\partial s^2} - a_2^2 q(\xi)\right)u(s, x_0)\right]_{s=-a_2x_0/a_1} \\ &\quad - 2a_1^2 a_2\left[\left(\frac{\partial^2}{\partial s^2} - a_1^2 q(\xi)\right)u(s, x_0)\right]_{s=-a_1x_0/a_2}.\end{aligned} \tag{32c}$$

In the case of an equation of the type (32a), for which $a_1 = a_2 = 1$, one obtains a reflection formula either directly from (31) by evaluating residues at double poles, or from (32c) by passing to the limit with a_1 and a_2. The resulting formula is

$$\begin{aligned}\xi_0^2 u(x_0, x) = \Bigg[\bigg(\frac{\partial^2}{\partial s^2} + 2x_0\frac{\partial^3}{\partial s^3} - x_0^2\frac{\partial^4}{\partial s^4} + 2q(\xi) - 4x_0 q(\xi)\frac{\partial}{\partial s} \\ + x_0^2 q(\xi)\frac{\partial^2}{\partial s^2}\bigg)u(s, x)\Bigg]_{s=-x_0}.\end{aligned}$$

Taking into account the initial conditions (32b) we can obtain a simpler formula for u itself, which does not involve any quadratures:

$$u(x_0, x) = \left[\left(-1 - 2x_0\frac{\partial}{\partial s} - x_0^2\frac{\partial^2}{\partial s^2} + x_0^2 q(\xi)\right)u(s, x)\right]_{s=-x_0}. \tag{33}$$

Formula (33) contains formula (3c) for the bi-harmonic equation as the special case $q = -\xi_1^2$. Reflection formulae for the more general polyharmonic case can be obtained from (31).

One observes that the reflection formulae for partial differential equations of the types (17b, c) have been obtained formally from the formula (19) which applies to ordinary differential equations, by treating $P(\xi_0, \xi)$ as an ordinary differential operator in ξ_0. For general forms $P(\xi_0, \xi)$ the *formal* expressions have no immediate concrete interpretation, since the formal roots ξ_0 of the equation $P(\xi_0, \xi)=0$ are only algebraic functions of differential operators. For equations of the special type (17b, c) meaningful formulae are obtained, due to the fact that in that case the quotients of roots are ordinary numbers instead of operators.

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NEW YORK UNIVERSITY

THE SUMMER MEETING IN UNIVERSITY PARK

The sixty-second Summer Meeting of the American Mathematical Society was held at Pennsylvania State University, University Park, Pennsylvania, on Tuesday through Friday, August 27–30, 1957, in conjunction with meetings of the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and Pi Mu Epsilon. The registration was 649 including 493 members of the Society.

Professor N. E. Steenrod of Princeton University delivered the Colloquium lectures, entitled *Cohomology operations, and obstructions to extending continuous functions* on Tuesday at 2:00 P.M. and Wednesday, Thursday, and Friday at 9:00 A.M. Presiding officers were Professors Salomon Bochner, Samuel Eilenberg, Leo Zippin, and G. W. Whitehead.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professor A. P. Calderòn of the Massachusetts Institute of Technology addressed the Society on *Singular integrals and differential equations* on Wednesday at 11:00 A.M., and Professor M. A. Rosenlicht of Northwestern University delivered an hour address, entitled *Rationality questions on algebraic groups*, on Thursday at 10:30 A.M. Presiding officers were Professors Tibor Radó and A. M. Gleason, respectively.

There were fifteen sessions for contributed papers. The presiding officers were Professors R. D. Anderson, J. H. Barrett, H. P. Evans, Orrin Frink, Jr., M. R. Hestenes, G. W. Mackey, N. H. McCoy, H. T. Muhly, E. D. Nering, Dr. W. H. Pell, Professor J. H. Roberts, Dr. R. L. San Soucie, Professors E. V. Schenkman, I. M. Sheffer, and Dr. W. C. Taylor.

A reception in the Hetzel Union Building was held on Wednesday afternoon. There was a chicken barbecue on Thursday evening in Horticulture Woods.

A banquet was held in the Hetzel Union Building on Wednesday evening. Professor H. B. Curry acted as toastmaster. The speakers were Dean Ben Euwema, Professor Orrin Frink, and President Eric A. Walker of Pennsylvania State University, Professors E. J. McShane, G. B. Price, J. S. Frame, and Dr. T. H. Southard. A resolution of thanks to Pennsylvania State University and to the Arrangements Committee, of which Professor Evan Johnson, Jr., was the chairman, was presented by Professor W. L. Hart.

The Council met on Tuesday afternoon and evening, August 27, 1957.

The Secretary announced the election of the following eighty-nine persons to ordinary membership in the Society:

Mrs. Muriel S. Alexander, Rumson, New Jersey;
Dr. J. S. Aronofsky, Magnolia Petroleum Co., Dallas, Texas;
Mr. Changmo Band, Choongang High School, Seoul, Korea;
Mr. Milton Bauman, International Business Machines Corporation, Philadelphia, Pennsylvania;
Mr. H. M. Beck, Naval Research Laboratory, Washington, D. C.;
Mr. L. A. Blount, Naval Air Development Center, Johnsville, Pennsylvania;
Mr. J. E. Bogar, Haller, Raymond and Brown, Inc., State College, Pennsylvania;
Dr. A. M. Bomberault, Institute of Mathematical Sciences, New York University;
Mr. J. L. Borden, University of California, Los Angeles;
Mr. G. L. Borman, University of Wisconsin, Milwaukee;
Mr. J. E. Boyd, Western Electric Company, Kearny, New Jersey;
Mr. T. S. Bright, The Girdler Company, Orlando, Florida;
1st Lt. T. A. Brown, U. S. Air Force, Wright-Patterson Air Force Base, Ohio;
Mr. R. T. Carty, Service Bureau Corporation, International Business Machines Corp., St. Louis, Missouri;
Miss Inge F. Christensen, Catholic University of America;
Dr. John Christopher, Burroughs Corporation, Pasadena, California;
Mr. J. E. Conboy, Providence, Rhode Island;
Mr. A. W. Coutris, Columbia University;
Professor William Craig, Pennsylvania State University;
Mr. J. W. Creely, Wallerstein Company, Staten Island, New York;
Mr. R. A. DeSantis, Marquardt Aircraft Co., Van Nuys, California;
Mr. Alfred Descloux, Bell Telephone Laboratories, New York, New York;
Mr. John W. Drane, University of Florida;
Mr. Ronald E. Drummond, Rutgers University;
Captain R. C. Eckerlin, U. S. Air Force, Kirtland Air Force Base, New Mexico;
Mr. S. E. Elliott, Phillips Petroleum Company, Bartlesville, Oklahoma;
Mr. F. D. Farley, University of Michigan;
Mr. L. A. Forrest, Vickers Incorporated, Detroit, Michigan;
Mrs. Amy E. A. Griffin, U. S. Naval Ordnance Test Station, China Lake, California;
Miss Kathryn B. Hanchon, General Electric Company, Cincinnati, Ohio;
Mr. Hershel I. Harrison, Wayne State University;
Mr. Paul Helt, Prudential Insurance Company of America, Newark, New Jersey;
Dr. Jack Heller, Institute of Mathematical Sciences, New York University;
Mr. Marvin H. Hewitt, Glenn L. Martin Company, Middle River, Maryland;
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Mr. G. F. Kottler, General Astronautics Corporation, New York, New York;
Mr. Andrew Kraus, National Bureau of Standards, Boulder, Colorado;
Mr. Ora K. Krienke, Jr., Seattle Pacific College;
Mr. R. H. Kupperman, New York University;

Dr. Herbert Kurss, New York University;
Professor W. M. Laird, University of Pittsburgh;
Professor Lester H. Lange, Valparaiso University;
Mr. H. S. Lauson, University of California, Los Alamos;
Dr. W. S. McCulloch, Massachusetts Institute of Technology;
Mr. W. G. Madison, Avco Manufacturing Corporation, Boston, Massachusetts;
Professor Felina G. Mapa, University of the Philippines;
Miss Bertha W. Mather, Pennsylvania State University;
Mr. George J. Minty, University of Michigan;
Mr. R. A. Moore, University of Texas;
Miss Vivian Morgan, North American Aviation, Canoga Park, California;
Mr. H. W. Nace, Lawrence Institute of Technology;
Mr. Samuel Orlinick, Scientific Library Service, New York, New York;
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Mr. G. B. Parrish, Office of Ordnance Research, Durham, North Carolina;
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Miss Carol B. Peters, Hoover Microwave Company, College Park, Maryland;
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Mr. W. C. Rheinbolt, National Bureau of Standards, Washington, D. C.;
Professor David Rosenblatt, American University;
Dr. Joan R. Rosenblatt, National Bureau of Standards, Washington, D. C.;
Mr. R. S. Rosenzweig, Allstate Insurance Company, Skokie, Illinois;
Professor J. E. Rowe, University of Michigan;
Mr. Dominick A. Russo, Sperry Gyroscope Co., Great Neck, New York;
Professor D. A. Rux, Wisconsin State College, Oshkosh;
Mr. Peter F. Stebe, Queens College;
Mr. E. C. Throndsen, General Electric Company, New York, New York;
Miss Eugenia I. Trapp, International Business Machines Corporation, Houston, Texas;
Mr. William F. Trench, Philco Corporation, Philadelphia, Pennsylvania;
Mr. T. E. Truan, Crucible Steel Company of America, Pittsburgh, Pennsylvania;
Mr. L. H. Turner, Purdue University;
Mr. W. E. Watts, Reynolds Metals Company, Listerhill, Alabama;
Mr. David M. Waxman, Nuclear Development Corporation of America, White Plains, New York;
Mr. M. S. Weinstein, Employees' Retirement System, Albany, New York;
Miss Miriam Weissner, AEC Computing Facility, New York University;
Mr. C. S. West, Jr., International Fidelity Insurance Company, Dallas, Texas;
Miss Edith S. Windsor, New York University;
Professor Izaak Wirszup, University of Chicago;
Miss Monica J. Wyzalek, University of Illinois;
Mr. Floyd E. Young, Western Life Insurance Company, Helene, Montana;
Mr. J. M. Zimmerman, Rocketdyne, North American Aviation Corporation, Canoga Park, California.

It was reported that the following eight persons had been elected to membership on nomination of institutional members as indicated:

Montana State University: Mr. J. A. Peterson.
University of North Carolina: Mr. L. L. Bumgarner, Mr. J. H. Keller, Mr. J. P. McAllister, Mr. M. B. Smith, Jr., Mr. D. P. Stanford.
Purdue University: Mr. R. E. Dowds.
University of Rochester: Mr. G. C. Branche.

The Secretary reported that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker-Vereinigung: Dr. Detlef Laugwitz, Dr. Hanfried Lenz; Société Mathématique de France: Professor André Lentin, Mr. Jean Riss, Mr. Aimé Sestier, Mr. Weishu Shih; London Mathematical Society: Professor Geoffrey Bosson, Dr. Ronald Harrop, Professor Patrick B. Kennedy; Schweizerische Mathematische Gesellschaft: Dr. Heinz T. Renggli.

The following appointments by the President were reported: as a Committee to Nominate Officers and Members of the Council for 1958: C. B. Allendoerfer, Chairman, R. P. Boas, G. W. Mackey, S. Ulam, A. D. Wallace; as an arrangements Committee for the November, 1958, meeting in Coral Gables, Florida: W. L. Strother, C. E. Capel, Georgia K. Del Franco, J. H. Roberts; as a Committee to Recommend Gibbs Lecturers for 1958 and 1959: S. Ulam, Chairman, Lipman Bers, Wilfred Kaplan; as a Committee to Award the Bôcher Prize in 1958: S. Bochner, Chairman, J. L. Doob, D. V. Widder; as a Program Committee for a Symposium on Combinatorial Designs and Analysis: Marshall Hall, Chairman, Richard Bellman, S. A. Schelkunoff, Claude Shannon, and C. B. Tompkins; as a Committee to Recommend Recipients of National Science Foundation Travel Grants to the International Congress of Mathematicians in Edinburgh, Scotland, 1958: R. L. Wilder, Chairman, E. Artin, G. W. Mackey, S. S. Chern, N. Levinson, K. O. Friedrichs, O. Zariski; as a member of the Committee to Select Hour Speakers for Western Sectional Meetings: R. C. Buck (until December 31, 1958, to fill out the unexpired term of A. C. Schaeffer); as a delegate to the inauguration of Richard Palmer as the tenth President of Morningside College, Dr. Marjorie H. Beaty.

The Secretary reported that the following had accepted invitations to deliver hour addresses: A. H. Clifford at the University of Miami, November 1957; W. H. Gottschalk, Washington, D. C., October 1957; Leo Sario at Corvallis, Oregon, June 1958; Daniel Zelinsky at Columbia, Missouri, November 1957.

The following dates and places of meetings were set: April 25 and 26, 1958 at Columbia University; June 20, 1958, at the Oregon State College; and August–September 1959 at the University of Utah.

The Council voted that beginning with 1958, abstracts shall be published in the Notices instead of in the Bulletin. The space saved in the Bulletin will be used for short announcements of the sort that have appeared in the Proceedings of the National Academy of Sciences. Further details concerning these new arrangements will appear shortly.

A business meeting of the Society was held at 10:00 A.M., August 28, with Vice President Bochner presiding. The Secretary reported briefly on the affairs of the Society.

The following amendments to the by-laws were adopted:

Article VII, Section 1. The last sentence was amended to read: All terms of office shall begin on January 1 and terminate of December 31 with the exception that the officials specified in Articles I, II, III, IV, and V (excepting the President Elect and Ex-President) shall continue to serve until their successors have been duly elected and qualified.

Article VII, Section 2. The article was amended to read: The President Elect, the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Trustees, the members of the Publications Committees, and the members-at-large of the Council shall be elected by written ballot. An official ballot shall be sent to each member of the Society by the Secretary on or before October 10, and such ballots if returned to the Secretary in envelopes bearing the name of the voter and received within 30 days shall be counted. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast shall be necessary for election. In case of failure to secure a plurality for any office, the Council shall choose by written ballot among the members having the highest number of votes.

Article VII, Section 4. The first sentence was amended to read: On or before January 15, the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee.

A new section was added to Article VII as follows:

Section 9. In case any officer dies or declines to serve between the time of election and the time at which he was to assume office, the vacancy shall be filled in the same manner as if he had served one day of his term.

Article IX, Section 5. This article was amended to read: The minimum annual dues of a corporate member shall be one thousand dollars.

Article X, Section 1. The first sentence was amended to read: The Annual Meeting of the Society shall be held between the fifteenth of December and the tenth of February next following.

Article X, Section 4. The article was amended to read: The Board of Trustees shall hold at least one meeting in each calendar year. Meetings of the Board of Trustees may be called by the President, the Treasurer, or the Secretary of the Society upon three days' notice of such meetings mailed to the last known post office address of each Trustee. The Secretary of the Society shall call a meeting upon the receipt of a written request of two of the Trustees. Meetings may also be held by common consent of all the Trustees.

The abstracts of the papers follow. Those having the letter "*t*" following the abstract number were read by title. Where a paper, presented in person, has more than one author, the symbol (p) follows the name of the author who presented it. Mr. Appling was introduced by Professor H. S. Wall, Mr. Crowe by Dr. J. S. Griffin, Mr. Ito by Professor Hidehiko Yamabe, and Mr. Sublette by Professor Seymour Sherman.

ALGEBRA AND THEORY OF NUMBERS

618*t*. S. B. Akers, Jr.: *On a theory of Boolean functions.*

Given a Boolean function, $F(x_1, x_2, \cdots, x_m)$, the *partial derivative* of F with respect to a basic variable, x_i, $(1 \leq i \leq m)$, is denoted by F_{x_i} and defined as $F_{x_i} = F(x_1, \cdots, x_i, \cdots, x_m) \oplus F(x_1, \cdots, x_i', \cdots, x_m)$ where $\oplus$ denotes exclusive disjunction. It follows that a necessary and sufficient condition that F be a dependent function of x_i is that $F_{x_i} \neq 0$ where 0 is the null element. This notion of a Boolean derivative is applied to such concepts as series expansions, integration, functional decomposability, and functional invariance. Many concepts of ordinary function theory such as Jacobians, constants of integration, Taylor and Maclaurin series, chain rule of partial differentiation, etc., are found to have counterparts in the resulting theorems. The application of the results to switching circuit analysis and synthesis is discussed. In particular, given $F(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ and $G(y_1, y_2, \cdots, y_n)$ a necessary and sufficient condition that F be a dependent function of G is derived in terms of their partial derivatives. An existence theorem is also proved—i.e., given F, under what conditions does there exist a function G such that F is a dependent function of G? (Received May 23, 1957.)

619. H. L. Alder: *A generalisation of the Euler ϕ-function.*

The function $\phi(n, m)$ with $m \geq n$ is defined as the number of ordered pairs $\langle x, y \rangle$ for which $x+y=m$, $1 \leq x \leq n$, and x and y both relatively prime to n. Since, for $m=n$, this function obviously reduces to the Euler ϕ-function, $\phi(n, m)$ is a generalization of the latter. A simple formula for the calculation of $\phi(n, m)$ is given. (Received July 10, 1956.)

620. D. R. Anderson: *On the estimation of the Kloosterman sum.*

This paper deals with the estimation by analytic methods of the magnitude of the Kloosterman sum $K(m_1, m_2; n)$ in the case where n is a prime $p>2$. This is defined by the following relation: $K(m_1, m_2; n)=\sum_{1\leq h\leq n;(h,n)=1} \exp\,[(2\pi i/p)(m_1h+m_2\bar{h})]$, where m_1, m_2 and n are positive integers and $h\bar{h}\equiv 1 \pmod{p}$. (For basic properties see Acta Math. vol. 49 (1926) pp. 407–464.) The best previous estimate was due to A. Weil. (See Proc. Amer. Acad. Sci. vol. 34 (1938) pp. 204–207.) This estimate is: $|K(m_1, m_2; p)|\leq 2p^{1/2}$. The estimate of the present paper results as follows. By way of a method similar to that used by Mordell in obtaining the formula for the Gauss sum, $K(m_1, m_2; p)$ is expressed in terms of several contour integrals. The latter integrals are re-expressed in the form from which the estimate is made through analytic continuation and Mellin transform theory. The result obtained is: $|K(m_1, m_2; p)| \leq C(\log p)^2$, C being a constant relative to all variables. (Received August 28, 1957.)

621. Rafael Artzy: *Cross-inverse loops.*

A cross-inverse (C.I.) loop is defined as a loop satisfying the identity $xy\cdot x'=y$, where x' is the right inverse of x. Isotopic C.I. loops are isomorphic. A principal isotope of a C.I. loop G, determined by $xg\circ fy=xy$, is C.I. if and only if $fx\cdot gy=fg\cdot xy$ for all x and y in G. Every autotopism (U, V, W) of a C.I. loop determines uniquely a C.I. principal isotope with $1U=f$, $1V=g$. The f's and g's of all autotopisms of a C.I. loop form a normal subloop of G which is commutative and Moufang. For each C.I. principal isotope of a C.I. loop, fg^{-1} is in the centre of G. (Received July 3, 1957.)

622*t*. S. K. Berberian: *$n\times n$ matrices over an* AW*-*algebra.*

Theorem: if A is any AW*-algebra, the algebra A_n of $n\times n$ matrices over A is an AW*-algebra. (Received July 10, 1957.)

623. H. E. Campbell: *On Whitehead's second lemma.*

An alternative algebra is called *nondegenerate* if for right multiplications R_a and R_b, trace (R_aR_b) is a nondegenerate bilinear form. A direct proof of the second Whitehead lemma, which was discussed by Schafer (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 444–451), for nondegenerate alternative algebras of arbitrary characteristic is obtained by using the Casimir operator defined in the author's *On the Casimir operator* (submitted to Pacific J. Math.). This implies the Wedderburn principal theorem (Schafer, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 604–614) and simplifies its proof for the class of nondegenerate algebras. (Received July 12, 1957.)

624*t*. Eckford Cohen: *Representations of even functions* (mod r).

This paper is concerned with the theory of even arithmetical functions of n (mod r), defined by $f(n, r)=f((n, r), r)$, $r\geqq 1$ [Proc. Nat. Acad. Sci., vol. 41 (1955) pp. 939–944]. The paper is divided into five parts, summarized as follows. (1) The fundamental theorem on even functions (mod r) is reformulated in a manner useful for applications; (2) a new proof of the Anderson-Apostol theorem on functions satisfying a Hölder relation is given; (3) an inversion formula for even functions (mod r) is proved; (4) completely even functions (mod r), defined by $f(n, r)=g((n, r))$, are arithmetically characterized; (5) a special class of functions of the form $\sum_{d|r} h(d, r/d)$, $(d, n)=1$, is studied. The theory is illustrated with several arithmetical functions, and numerous identities between various functions are deduced as special cases. The paper provides, in fact, a general method for establishing certain types of arithmetical identities. (Received July 8, 1957.)

625*t*. Eckford Cohen: *Trigonometric sums in elementary number theory.*

Let n, r, s be integers, $r>0$, $s\geqq 0$. Denote by $\sigma_s^*(n, r)$ the number of x_i, y_i (mod r) such that $n=x_1y_1+\cdots+x_sy_s$ (mod r). Also, let $\phi_s^*(n, r)$ denote the number of x_i (mod r) such that $n=x_1+\cdots+x_s$ (mod r), under the restriction $((x_1, \cdots, x_s), r)=1$. Finite trigonometric and arithmetical expansions of the functions $\sigma_s^*(n, r)$ and $\phi_s^*(n, r)$ are deduced. By applying a special limiting process to the results obtained for $\sigma_s^*(n, r)/r^s$ and $\phi_s^*(n, r)/r^s$, Ramanujan's well-known series expansions for $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$ $(n>0, s>0)$ are shown to result, respectively. (Received August 15, 1957.)

626. Haskell Cohen: *Convexity of the kernel in admissible mobs.*

A mob S is a Hausdorff topological semigroup. An admissible mob is in addition a compact convex subset of a real linear topological space satisfying $x\cdot[\alpha y+(1-\alpha)z]=(x\cdot y)+(1-\alpha)(x\cdot z)$ and $[\alpha y+(1-\alpha)z]\cdot x=\alpha(y\cdot x)+(1-\alpha)(z\cdot x)$ for all $x, y, z\in S$ and all α such that $0\leqq\alpha\leqq 1$ (where "$\cdot$" is the semigroup operation). The kernel K of a mob is its minimal ideal. The following results are obtained. (i) If the dimension of $S\leqq 2$, K is convex. (ii) if dim $S\leqq 3$ and S has a unit, K is convex. In addition examples are given of a three dimensional mob with nonconvex kernel, and of a four dimensional mob with unit having a nonconvex kernel. (Received July 10, 1957.)

627. H. S. Collins: *Admissible mobs.* I. Preliminary Report.

A semigroup S is *affine* if S is a convex subset of a real linear space E and multiplication is, separately, an affine mapping of S into S. S is an *admissible mob* if, in addition, E is a Hausdorff linear topological space, S is compact, and multiplication is continuous. In this paper, all finite dimensional affine semigroups with unit and zero are characterized as matrix semigroups; more concrete characterizations are given for one-dimensional affine semigroups with unit, for two-dimensional admissible mobs with unit and no zero, and for any two-dimensional disc with unit and with zero at its center. (Received July 10, 1957.)

628*t*. R. P. Dilworth: *A representation theorem for universal algebras.*

Let $\mathfrak{A}$ denote a universal algebraic system. If $\{A_\sigma | \sigma\in\Sigma\}$ is a collection of realizations of this system, let πA_σ denote the Cartesian product of algebras, that is, the set of functions f on Σ such that $f(\sigma)\in A_\sigma$. Operations and relations in πA_σ are defined componentwise. An algebra A is said to be the *direct union* of the algebras A_σ if there exists a one-to-one mapping ϕ from A into πA_σ such that (1) the operations and relations in A are preserved, and (2) if $\sigma_1, \cdots, \sigma_n$ is a finite subset of Σ and $a_1, \cdots, a_n$ are elements of $A\sigma_1, \cdots, A\sigma_n$ respectively, then there exists $a\in A$ such that $\phi(a, \sigma_i)=a_i$ for $i=1, \cdots, n$. A is *completely indecomposable* if whenever A is a direct union of algebras A_σ, the mapping $\phi(a, \sigma)$ is an isomorphism for some $\sigma\in\Sigma$. It is proved that every algebra is a direct union of completely indecomposable algebras. Applications to lattice theory and group theory are given. (Received July 11, 1957.)

629. Philip Dwinger: *Completeness of quotient algebras of Boolean algebras.*

Let A be a Boolean algebra and I an ideal in A. If $X\subset A$, let $\kappa(X)$ denote the power of X and $\delta(X)$ the least cardinal number ρ, such that the power of every disjointed subset of X is at most ρ. A is α-complete, α a cardinal number, if for every $X \subset A$, $\kappa(X) \leqq \alpha$, $\sum X$ exists. I is α-complete if for every $X \subset I$, $\kappa(X) \leqq \alpha$, $\sum X$ exists and $\in I$. Tarski and Smith (Trans. Amer. Math. Soc. vol. 84 (1957) pp. 230–257) have proved that if (i) A is β-complete, β transfinite, (ii) I is α-complete for every $\alpha < \beta$, (iii) $\delta(A/I) \leqq \beta$, then A/I is complete. In the present paper it is proved, that if one replaces (iii) by (iii)′: $\delta(A/I) > \beta$ and adds the condition (iv) I dense and $\sum X \in A - I$ if $X \subset I$ and $\kappa(X) \geqq \beta$, then A/I is not β-complete. This result is a corollary of a more general theorem which is proved for lattices satisfying a weak infinite distributive law. Another result is the following: if A is σ-complete (A not finite), I dense and $\sum X \in A - I$ if $X \subset I$, X not finite, then A/I is not σ-complete and there does not exist a nontrivial finitely additive measure μ on A, such that $I = \{x \mid x \in A,\ \mu(x) = 0\}$. (Received June 17, 1957.)

630*t*. Murray Gerstenhaber: *Nilalgebras and linear varieties of nilpotent matrices* I.

Let A be a commutative but not necessarily associative algebra of characteristic zero; A may be of infinite dimension. If there exists an integer t such that the tth power of every a in A in every association is zero, then the linear transformation $R_a: x \rightarrow xa$ is nilpotent for all a in A. If A is a commutative and power associative algebra of characteristic zero but not necessarily finite dimension, then a nilpotent implies R_a nilpotent. (Received May 13, 1957.)

631. Murray Gerstenhaber (p) and S. A. Hoffman: *Nilalgebras and linear varieties of nilpotent matrices* II.

Let V be a linear variety of nilpotent matrices of dimension k over the complex numbers. Then $k \leqq n(n-1)/2$, and if $k = n(n-1)/2$ then there exists a nonsingular matrix X such that XVX^{-1} is in fact the set of all triangular matrices. Since for $n = 2, 3$ one has $n = n(n-1)/2$ it is a corollary that if A is an algebra (not necessarily associative) of dimension 2 or 3 over the complex numbers in which R_a is nilpotent for all a in A, then A is nilpotent. (Received May 13, 1957.)

632*t*. Emil Grosswald, Anne Calloway and Jean Calloway: *The proper representation of integers as sums of* 3 *squares.*

The equation $(^*) n = u^2 + v^2 + w^2$ has integral solutions, unless $n = 4^a \cdot n_1$, $4 \nmid n_1$, $n_1 \equiv 7 \pmod 8$; for some n, however, all such solutions satisfy $u \cdot v \cdot w \cdot = 0$. Any integer n, for which $(^*)$ has some integral solution with $u \cdot v \cdot w \neq 0$, is said to be properly represented by $(^*)$. The problem arises to characterize the integers n with proper representations $(^*)$ and to find the number $N(x)$ of such integers $n \leqq x$. Theorem: The integer $n = 4^a \cdot n_1$ has a proper representation $(^*)$, unless either $n_1 \equiv 7 \pmod 8$, or else $n_1 \in S$, where S is a certain finite set of m integers. Furthermore, $N(x) = 5x/6 - (\log 4)^{-1} \sum \log (x/b_i) + R$, the sum being extended over all $b_i \in S$ and where $-(m - 7/24) \leqq R \leqq R_M = (\log 4)^{-1} \log (x/7) + 7/24$. Conjecture: $S = \{1, 2, 5, 10, 13, 25, 37, 58, 85, 130\}$, $m = 10$; hence, $N(x) = 5x/6 - 10 (\log 4)^{-1} \log x + a + R$, $a = (\log 4)^{-1} \cdot \sum \log b_i = 19{,}7 \cdots$ and $-233/24 \leqq R \leqq R_M$. Besides classical results (Landau, Hurwitz), the proof is based mainly on Bateman's formula for the number of representations of an integer as a sum of three squares (Trans. Amer. Math. Soc. vol. 71 (1951) pp. 70–101), combined with the Siegel-Tatuzawa estimate of the class-number (Jap. J. Math. vol. 21 (1951) pp. 163–178). These permit to show that for sufficiently large n_1, the number of representations of n by three squares exceeds the (properly counted) number of representations by two squares and the conclusion follows. (Received June 11, 1957.)

633*t*. D. K. Harrison: *Partial cohomology of modules.* I *Abelian groups*. Preliminary report.

Satellites and homology of a functor are taken, not with respect to all exact sequences, but only with respect to special sequences. If a sequence splits, it would be foolish to use only that it is exact. Similarly, if an exact sequence is "tight," this paper attempts to utilize that "tightness" with a theory aimed explicitly at that type of sequence. Several types of sequences are shown to be such that the main theorems of homological algebra apply with exact sequences replaced by sequences of their type. For abelian groups we consider in this sense the cohomology of pure subgroups. The pure projective groups are the direct sums of cyclic groups. The pure injective groups are the sums of a divisible group with a direct summand of a product of finite cyclic groups. These concepts can be described in terms of the degree of completeness of the uniformity of the group G defined by the multiples nG. It is proved using these methods that: an abelian group can be made into a compact topological group if and only if it is algebraically isomorphic to a product of p-adic integers, finite cyclic groups, the reals, and groups of type $Z(p^\infty)$, where the number of copies of the reals exceeds or is equal to the number of $Z(p^\infty)$ for any p; if A and B are torsion groups, $A \otimes B$ is a direct sum of cyclic groups, and Hom (A, C) is a direct summand of a product of finite cyclic groups; and H is a pure subgroup of G if and only if $0 \rightarrow H \otimes A \rightarrow G \otimes A$ is exact for all A. (Received July 11, 1957).

634. Juris Hartmanis: *A note on the lattice of topologies.*

The collection of all topologies on a fixed set S forms a complete point lattice under its natural ordering, i.e. $R_1 \leq R_2$ if every set closed under R_1 is also closed under R_2. It is shown that the lattice of all topologies has only trivial homomorphisms if S consists of more than two elements. On the other hand, this is not true for the lattice of all T_1 topologies on S and the construction of all homomorphisms is given in this case. It is further shown that the group of automorphisms for the lattice of topologies is (a) isomorphic to a symmetric group if S is infinite, or consists of one or two elements, (b) isomorphic to the direct product of a symmetric group with the two element group otherwise. Finally it is observed that the lattice of topologies is complemented if and only if it is finite. (Received July 8, 1957.)

635*t*. J. H. Hodges: *Scalar polynomial equations for matrices over a finite field.*

Let $GF(q)$ denote the finite field of order $q = p^n$. If $E = E(x)$ is a monic polynomial over $GF(q)$, let $N(E, m)$ denote the number of matrices θ of order m over $GF(q)$ such that $E(\theta) = 0$. In this paper, the classical theory of the scalar polynomial equation for matrices over a general field is combined with a theorem of L. E. Dickson (*Linear Groups*, Leipzig, 1901, p. 235) on commutativity of certain matrices over $GF(q)$ to give an explicit formula for $N(E, m)$. The formula is illustrated for the case $E(x) = x^3 - 1$. (Received July 11, 1957.)

636*t*. Nathan Jacobson and L. J. Paige: *On Jordan algebras with two generators.*

An associative algebra $\mathfrak{A}$ over a field of characteristic $\neq 2$ with multiplication composition ($\times$) defines a Jordan algebra $\mathfrak{A}^+$ relative to the Jordan multiplication $ab=(a\times b+b\times a)/2$. A Jordan algebra is called special if it is isomorphic to a subalgebra of an algebra $\mathfrak{A}^+$, $\mathfrak{A}$ associative. A. I. Shirshov (*On special J-rings*, Mat. Sbornik vol. 38 (1956)) proved that the free Jordan algebra with two generators is special. The authors give a comparatively simple alternative proof of this result and at the same time determine a simple basis for the algebra. Briefly, a basis for the free special Jordan algebra $\mathfrak{J}_0$ (the subalgebra relative to Jordan multiplication generated by two generators in a free associative algebra F) is defined inductively using Jordan multiplication. Imitating this procedure in the free Jordan algebra $\mathfrak{J}$ with two generators, a set S in $1-1$ correspondence with the basis in $\mathfrak{J}_0$ is defined. To prove that S is a basis for $\mathfrak{J}$, three variable identities of independent interest are used. The results on the basis of $\mathfrak{J}_0$ are applicable to Jordan algebras of characteristic 2. Although it is known that any Jordan algebra (of characteristic $\neq 2$) with two generators is special, the corresponding result fails for characteristic two even for one generator. (Received June 24, 1957.)

637*t*. Harry Kesten: *Symmetric random walk on groups.*

Let G be a group generated by elements $a_1, \cdots, a_k$. Consider a random walk on G in which a step consists of multiplication on the right by one of the generators $a_1, \cdots, a_k$ or its inverse, each possibility having the same probability $(2k)^{-1}$. This random walk defines a Markov chain, the transition probability from g_1 to g_2 ($g_1, g_2 \in G$) being the probability that g_2 is reached in one step from g_1. Since G is countable this Markov chain can be represented by its matrix M of transition probabilities. The following result is proved. The group G is a free group with independent generators $a_1, \cdots, a_k$ if and only if the largest eigenvalue of M is $(2k-1)^{1/2}k^{-1}$. In this case the spectrum of M is the interval $[-(2k-1)^{1/2}k^{-1}, +(2k-1)^{1/2}k^{-1}]$. (Received June 25, 1957.)

638. Naoki Kimura: *On some existence theorems in a multiplicative system.*

Let $A \in (\pi)$ denote the proposition that a multiplicative system A satisfies the property π. Here we consider only properties which are satisfied by a multiplicative system consisting of only one element. Then the theorems are as follows: I. Let π satisfy the following three conditions: (1) $A \in (\pi)$ implies every subsystem $\in (\pi)$, (2) If $A, B \in (\pi)$, then $A\times B \in (\pi)$ (or (2′) If $A, B \in (\pi)$, then $A \times^C B \in (\pi)$ for any nontrivial system C), (3) If $A_\gamma \in (\pi)$, then its inverse limit inv lim $A_\gamma \in (\pi)$. Then for any given system there exists a greatest (or maximal) decomposition so that its quotient system satisfies the property π. II. Let π satisfy the following three conditions: (1) $A \in (\pi)$ implies every quotient system $\in (\pi)$, (2) If $A, B \in (\pi)$, then $A * B \in (\pi)$ (or (2′) If $A, B \in (\pi)$, then $A * {}_C B \in (\pi)$ for any system C), (3) If $A_\gamma \in (\pi)$, then its direct limit dir lim $A_\gamma \in (\pi)$. Then for any given system there exists a greatest (or a disjoint family of maximal) subsystem(s), (any of) which satisfies the property π. The notion of a direct product with amalgam, $A \times^C B$, and a free product with amalgam, $A * {}_C B$, are considered. A generalization of the above theorems are also stated. Further several applications will be given. (Received June 28, 1957.)

639*t*. L. A. Kokoris: *Some nodal noncommutative Jordan algebras.*

Let A be a noncommutative Jordan algebra over a field F with an identity e and let $A=eF+N$ where every element of N is nilpotent. If N is not a subalgebra of A, A is called a nodal algebra by R. D. Schafer. Schafer has proved that nodal noncommutative Jordan algebras of characteristic zero do not exist (see *Noncommutative Jordan algebras of characteristic* 0, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 472–475). In this paper simple nodal algebras are constructed for every characteristic different from zero. These examples have the property that the associated algebras A^+ are associative. (Received May 6, 1957.)

640*t*. George Kolettis, Jr.: *A generalization of Ulm's theorem.*

If G is a reduced primary abelian group, then its Ulm invariants characterize the group in case G is countable. (Ulm's theorem, cf. Kaplansky, Infinite Abelian groups). It is shown that the Ulm invariants are still sufficient to characterize the group in the uncountable case, provided G can be expressed as a direct sum of countable groups. This is accomplished by a reduction to the case where G has cardinality $\aleph_1$. Necessary and sufficient conditions are obtained for a set of cardinals to be the Ulm invariants of a direct sum of countable groups. (Received July 12, 1957.)

641*t*. R. C. Lyndon: *On special Horn sentences.*

For terminology see two abstracts presented at the April 1957 meeting in Berkeley. A ***special Horn sentence*** is a sentence obtained by universal quantification and conjunction from formulas of the form $P \to a$ where P is a positive formula and a an atomic formula. Let S and T be two sentences, and form $S(z)$ from S by replacing each part $\phi_i(t_1, \cdots, t_{r_i})$ of S by $\pi_i(z, t_1, \cdots, t_{r_i})$, where z is a new variable and the π_i are distinct new predicates. Form T' from T by replacing each part $\phi_i(t_1, \cdots, t_{r_i})$ by $\forall z \cdot \pi_i(z, t_1, \cdots, t_{r_i})$. Then S ***strongly implies*** T if $\forall z \cdot S(z)$ implies T'. Theorem 1. If M is a special Horn sentence, then M strongly implies M. Theorem 2. If S strongly implies T, then there exists a special Horn sentence M such that S implies M and M implies T. The main tool of the proof is a detailed application of the Gentzen calculus. (Received July 10, 1957.)

642*t*. R. C. Lyndon: *On subdirect products of relational systems.*

For terminology see preceding abstract. A subsystem $\mathfrak{A}'$ of the direct product $\mathfrak{A}$ of systems $\mathfrak{A}_i$ is a ***subdirect product*** of the $\mathfrak{A}_i$ if each natural homomorphism $\mathfrak{A}$ onto $\mathfrak{A}_i$ maps $\mathfrak{A}'$ onto $\mathfrak{A}_i$. For a class K of systems, let PK be the class of all systems isomorphic to a subdirect product of systems from K. Lemma. $\mathfrak{A} = \langle A, R, \cdots \rangle$ is isomorphic to a subdirect product of systems $\mathfrak{A}_i$ if and only if there exist systems $\mathfrak{B}$ and $\mathfrak{B}_i$ with congruences C, C_i such that $\mathfrak{A} \cong \mathfrak{B}/C$, each $\mathfrak{A}_i \cong \mathfrak{B}_i/C_i$, where each $B_i = B$ and $\bigcap C_i = C$, $\bigcap R_i = R$, $\cdots$. (Operations add no complication.) Theorems. 1. If a special Horn sentence holds in K, then it holds in PK. 2. If, for all K such that S holds, T holds in PK, then there exists a special Horn sentence M such that S implies M and M implies T. 3. If K is an arithmetical class, then APK is characterized by those special Horn sentences that hold in K. 4. If K is arithmetical, then $K = PK$ if and only if K is characterized by special Horn sentences. 5. There exists arithmetical K such that PK is not arithmetical. (Received July 10, 1957.)

643. H. B. Mann: *On integral bases.*

Let J be a Dedekind domain, F its quotient field. Let F' be a quadratic extension of F and let J' be the domain of elements of F which satisfy monic equations with coefficients in J. The domain J' will be called an algebraic extension of J. If F has characteristic different from 2 then J' has an integral basis over J if and only if $F' = F(D^{1/2})$ where D is the discriminant of F' over F. If F has characteristic 2 then the different of F' over F is an ideal of J and J' has an integral basis over J if and only if this ideal is principal. If J contains an ideal which is not principal then it is possible to construct a quadratic extension J' over J which has no integral basis. Hence every algebraic extension of J has an integral basis if and only if every ideal of J is principal. (Received July 2, 1957.)

644*t*. M. D. Marcus, B. N. Moyls and R. Westwick: *Some extreme value results for indefinite Hermitian matrices*. II.

Let H be an n-square Hermitian matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$; and let $E_r[a_1, \cdots, a_k]$ be the rth elementary symmetric function of the numbers $a_1, \cdots, a_k$. It is shown that the maximum and minimum values of $E_r[(Hx_1, x_1), \cdots, (Hx_k, x_k)]$, as $x_1, \cdots, x_k$ range over all sets of k orthonormal vectors ($1 \leqq r \leqq k \leqq n$), have the form $E_r[b_1, \cdots, b_{k_1}, \cdots, b_{k_q}]$, $0 = k_0 < k_1 < \cdots < k_q = k$, where $b_{k_j+1} = \cdots = b_{k_{j+1}} = (\mu_{k_j+1} + \cdots + \mu_{k_{j+1}})/(k_{j+1} - k_j)$ for $0 \leqq j \leqq q-1$, and $\mu_1, \cdots, \mu_k$ are the first s and last $k-s$ of the λ's ($0 \leqq s \leqq k$). (Received July 1, 1957.)

645*t*. Anthony Mardellis: *The monodromic group and the Picard-Vessiot theory*. Preliminary report.

The author's purpose is to show the place occupied by the monodromic group of a differential equation in the Picard-Vessiot theory as presented by E. R. Kolchin (Ann. of Math. (2) vol. 49 (1948) pp. 1–42). Consider the differential equation of order n: $\sum_{i=0}^{n} \mathfrak{A}_i(z) d^{n-i} w(z)/dz^{n-i} = 0$, $\mathfrak{A}_0(z) = 1$, with coefficients in the field of rational functions $\mathfrak{F}$. Let $w^1, w^2, \cdots, w^n$ be a fundamental system of solutions. Furthermore, let $\mathfrak{G} = \mathfrak{F}\langle w^1, w^2, \cdots, w^n\rangle$, and let g denote the group of all automorphisms of $\mathfrak{G}$ over $\mathfrak{F}$; $\mathfrak{F}_1$ the subfield of all one-valued functions contained in $\mathfrak{G}$; $g(\mathfrak{F}_1)$ the subgroup of g consisting of all automorphisms which leave invariant every element of $\mathfrak{F}_1$. The monodromic group M is a group of automorphisms of $\mathfrak{G}$ over $\mathfrak{F}_1$, and $g(\mathfrak{F}_1)$ is the smallest algebraic subgroup of g containing M. $g = M \leftrightarrow$ all solutions $w^1, w^2, \cdots, w^n$ in a fundamental system are algebraic. If $\mathfrak{F} = \mathfrak{F}_1$, then M can replace g in the following two theorems: "The equation is reducible over $\mathfrak{F} \leftrightarrow g$ is reducible"; "All integrals of the equation are algebraic $\leftrightarrow g$ is finite." The class of equations for which $\mathfrak{F} = \mathfrak{F}_1$ includes all equations which are everywhere of the Fuchs type. If two equations belong to this class, and have the same monodromic group $M = M'$, then $g = g'$. (Received July 3, 1957.)

646. E. D. Nering: *An application of results on the reduction of an algebraic function field to extensions of the field of constants of an algebraic function field.*

This paper develops the application of a previous paper, *Reduction of an algebraic function field modulo a prime in the constant field* (unpublished), to the extensions of the field of constants of an algebraic function field. In the earlier paper a formula of the form $g - 1 = \sum_{i=1}^{s} r_i e(H_i)(g_i - 1) + \rho$ was obtained in which g is the genus of a given algebraic function field K and the g_i are the genera of fields obtained by reducing K modulo extensions of a prime in the constant field. In the present paper an algebraic extension of the constant field is obtained by first extending the field by adjoining a transcendental element Λ, and then reducing this field modulo a prime corresponding to the irreducible polynomial $h(\Lambda)$ for the algebraic element λ. For such a prime $s = 1$ and $e(H_1) = 1$. If either λ or K is separable over the constant field, then $r_1 = 1$. If λ is separable, $\rho = 0$. 2ρ is the degree of the divisor of a conductor. Tate

(Proc. Amer. Math. Soc. vol. 3 (1952) pp. 400–406) has discussed the inseparable case, and in his results the term corresponding to the 2ρ is the degree of the divisor of a different. The two methods are entirely analogous; the conductor is determined in a way which is, in a sense, "dual" to the way in which the different is determined. The conductor is defined in an invariant manner. (Received July 10, 1957.)

647t. Anil Nerode: *Composita, equations, and freely generated algebras.*

Define a V-compositum T to consist of a set T, a nonempty subset V of T, and a set S of maps on T to T such that: S is closed under composition and contains the identity; every map on V to T has exactly one extension in S. Relative to suitable definitions every compositum has a regular representation as a subcompositum of a concrete compositum. In logic this theory yields a generalization to composita of the semantic completeness of the calculus of identities of Birkhoff. In algebra this theory yields: rigorous existence and uniqueness proofs for freely generated abstract algebras implying the equivalence of characterizations either in terms of endomorphism structure or mapping properties or true identities; and a subdirect sum representation for freely generated algebras analogous to Birkhoff's for general abstract algebras. Further, these imply what appears to be a solution to problem 68 of Birkhoff. If G has α elements, the free distributive lattice on α generators is the sublattice of $P(P(G))$ generated by W_g for g in G, where W_g consists of all subsets of G containing g. (Received July 8, 1957.)

648t. H. A. Osborn: *Lie algebras of alternating forms.*

Let E be a finite-dimensional vector space over a field of characteristic zero, E' the dual of E, and S a nondegenerate symmetric bilinear form on $E'\times E'$. If A and B are any two alternating bilinear forms on $E\times E$ then one can define a new alternating bilinear form $[A, B]_S$ on $E\times E$ such that, in terms of an arbitrary basis, $([A, B]_S) = (A)(S)(B)-(B)(S)(A)$, where parentheses indicate the appropriate matrix representations. Using the correspondence $(A)\leftrightarrow(S)(A)$, which can also be defined independently of the basis, any such Lie algebra of alternating forms is isomorphic to a Lie algebra of endomorphisms of E. There exist many Lie algebras of endomorphisms of E which cannot be obtained in this way, however, so that the following result is a considerably sharpened form of Ado's theorem. THEOREM: *Any Lie algebra over a field of characteristic zero possesses a faithful representation by an algebra of alternating bilinear forms.* COROLLARY: *Any Lie algebra over an algebraically closed field of characteristic zero possesses a faithful matrix representation (in the usual sense) by a Lie algebra of skew-symmetric matrices.* (Received June 5, 1957.)

649. Martin Pearl: *On normal matrices.*

Let F be an arbitrary field and let $a\leftrightarrow\bar{a}$ be an involutory automorphism of F. The conjugate transpose $A^* = [b_{ij}]$ of the matrix $A = [a_{ij}]$ is defined by $b_{ij} = \bar{a}_{ji}$. A is said to be normal if $AA^* = A^*A$ and is said to be an EPr matrix if A has rank r and if $A\xi = 0$ implies $A^*\xi = 0$ (H. Schwerdtfeger, *Introduction to linear algebra and the theory of matrices*). By means of EPr matrices, some of the classical results concerning real and complex normal matrices are generalized to F. It is shown that a necessary and sufficient condition that a matrix A which has the same rank as AA^* be normal is that there exist a unitary matrix (i.e. $UU^* = U^*U = I$) U such that $A^* = UA = AU$. Let K be a field containing F and all of the characteristic roots of A. Furthermore, let $A-\lambda I$ and $(A-\lambda I)(A-\lambda I)^*$ have the same rank for all $\lambda\in K$ and let $A-\lambda I$ be an

EPr_λ matrix, where r_λ is the rank of $A-\lambda I$, for all $\lambda \in K$. Then A is normal and there is a polynomial f such that $A^* = f(A)$. (Received July 11, 1957.)

650. E. C. Posner: *Differentiably simple rings.*

Let R be a commutative ring and D a family of derivations of R such that R has no ideals invariant under every derivation in D. Then R is an algebra with unit such that every zero divisor is nilpotent; if R is of characteristic zero, then R has no divisions of zero. The ring of differential polynomials in D over R made faithful on R is simple; if the elements of D commute and are linearly independent over R, the ring of commutative differential polynomials is this faithful ring. (Received July 10, 1957.)

651. G. B. Preston: *Representations of semigroups.*

In a recent paper M. P. Schützenberger (C. R. Acad. Sci. (Paris) vol. 244 (1957) pp. 1994–6) showed that every elementary (loc. cit.) $\mathfrak{D}$-class D in a semigroup with identity S determines a pair of left-right dual semigroups of matrices $M(D)$ and $N(D)$, say, and a pair of homomorphisms $f_D: S \to M(D)$ and $g_D: S \to N(D)$ of S onto these semigroups. We show that the restriction to elementary $\mathfrak{D}$-classes is unnecessary. Let $\mathfrak{D}_S$ be the set of all $\mathfrak{D}$-classes in S and denote by $M(N)$ the direct product of all the $M(D)(N(D))$ for $D \in \mathfrak{D}_S$. Then M and N can also be regarded as semigroups of matrices. Define $f: S \to M$ by $(sf)_D = sf_D$, for $s \in S$, where for $m \in M$, m_D denotes the component of m in $M(D)$. Similarly define $g: S \to N$ by $(sg)_D = sg_D$, and $f \times g: S \to M \times N$ by $s(f\times g) = (sf, sg)$. We find necessary and sufficient conditions for f, g and $f \times g$ to be faithful, i.e. to be one-to-one. For regular semigroups $f \times g$ is faithful. For inverse semigroups f and g are each faithful. (Received July 11, 1957.)

652t. J. B. Roberts: *A new proof of a theorem of Lehmer.*

The following theorem is proved. Let $\alpha_0, \cdots, \alpha_k$ be an arbitrary set of $k+1$ complex numbers (distinct or not) and let b be an integer $\geqq 2$. Let C be the collection of all numbers of the form $j_0\alpha_0 + \cdots + j_k\alpha_k$ where the j_i are integers satisfying $0 \leqq j_i \leqq b-1$. Further let C_j, $0 \leqq j \leqq b-1$, be the collection of elements of C for which $j_0 + \cdots + j_k \equiv j \pmod{b}$. Then if $P(x)$ is a polynomial of degree smaller than or equal to k, $\sum_{n \in C_i} P(x+n)$ is equal to a constant independent of i, $0 \leqq i \leqq b-1$. This theorem is equivalent to a theory of Lehmer (Scripta Mathematica vol. 13 (1947) pp. 37–41). The proof given here, while in some respects similar to that of Lehmer, makes use of difference operators while his proof makes use of differentiation. We also give a quick method for the calculation of the classes C_j in the special case where $\alpha_i = b^i$. (Received June 24, 1957.)

653t. J. B. Roberts: *Some classes of orthogonal functions.*

The author has proved two identities of the form $\sum_{n=0}^{b^{k+1}-1} c_k(n)P(x+n) = 0$ for all polynomials of sufficiently small degree. One is proved in the paper discussed in the preceding abstract and the other in the Amer. Math. Monthly vol. 64 (1957) pp. 317–322. We shall call these identities I and II respectively. For each $n \geqq 0$ define the function $w_n(x)$ to be of period 1, constant over intervals of length $1/b^{n+1}$, and taking the value of the ith coefficient in I on the ith interval of length $1/b^{n+1}$ in $[0, 1)$. Define $t_n(x)$, $n \geqq 0$, in the same way using the coefficients in II. Then it is shown that $w_n(x) = \prod_{j=0}^{n} r_j(x)$ where $r_j(x)$ is the $j+1$st Rademacher function of order b. Hence $\{w_n(x)\}$ is a lacunary subsequence of the Walsh functions of order b and constitutes an orthonormal set. Further there is a class of functions $s_j(x)$, $j \geqq 0$, such that: (a) If

$Q(x)$ is a power product of the $s_j(x)$ with at least one exponent 1 then $\int_0^1 Q(x)dx = 0$; $t_n(x) = \prod_{j=0}^{n} s_j(x)$. Hence $\{t_n(x)\}$ is an orthogonal set. However while the Walsh functions of order b are the completion of the Rademacher functions of order b one cannot complete the set of $s_j(x)$ in the same way. Their completion would be of interest. (Received June 24, 1957.)

654t. Alex Rosenberg and Daniel Zelinsky: *Tensor products of semiprimary algebras.*

Theorem: Let A and B be algebras over a field F. If $A \otimes B$ is semiprimary then either A is semiprimary or B is radical. Here "semiprimary" means that A modulo a radical ideal is semisimple with minimum condition; "radical" may be taken to mean either quasi-regular, nil or nilpotent. Under the same hypotheses, if B is the opposite of A and "radical" means "nilpotent," then A modulo its maximal nilpotent ideal is finite over F. (Received May 20, 1957.)

655. N. J. Rothman: *Characters of mobs.* Preliminary report.

If S is an abelian mob ($=$topological semigroup), S^*, the set of all bounded continuous homomorphisms to the multiplicative complex plane, is called the characters of S. S^* with the compact open topology and multiplication defined by $fg(x) = f(x)g(x)$ is a mob with uniformly continuous multiplication. For $A \subset S$, $(S^*:A) = [f \in S^*: f|A = 0]$ is a closed ideal in S^*. If I is an ideal in S then $(S^*:I)$ is topologically isomorphic to $(S/I)^* \backslash \{1\}$ (S/I is the Rees quotient); also I^* is algebraically isomorphic to $S^*/(S^*:I)$. If $\{S_i\}_{i=1}^{n}$ is a finite collection of mobs with the identities e_i then the character mob of their product is topologically isomorphic to a Rees quotient of the product of the character mobs S_i^*. An abelian mob S with 0 possesses nontrivial characters if and only if S contains an ideal I and there is a closed RST submob R of $I \times I$ with I/R isomorphic to a nondegenerate submob of the usual unit interval (Cf. Wallace: Tulane, *Notes on mobs*). (Received July 11, 1957.)

656t. R. D. Schafer: *On noncommutative Jordan algebras.*

A power-associative algebra A with 1 over a field F is called a ***nodal algebra*** in case every element of A is of the form $\alpha 1 + z$ where $\alpha \in F$ and z is nilpotent, and A is not of the form $A = F1 + N$ for N a nil subalgebra of A. Let K be the algebraic closure of F. If A is a noncommutative Jordan algebra (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 472–475) over F of characteristic $\neq 2$ such that A_K is without nodal subalgebras, then A is trace-admissible. A structure theory for such algebras is obtained. The paper concludes with some theorems about nodal noncommutative Jordan algebras (which are necessarily of characteristic p). That nodal noncommutative Jordan algebras exist has recently been shown by L. A. Kokoris (see abstract 639t). (Received May 27, 1957.)

657. R. D. Schafer: *Restricted noncommutative Jordan algebras of characteristic p.*

Commutative Jordan algebras of characteristic $p > 2$ satisfy the identity $R_x^p = R_{x^p}$, where R_x denotes a right multiplication. Noncommutative Jordan algebras of characteristic p do not in general satisfy this identity. Those which do are called ***restricted noncommutative Jordan algebras of characteristic p*** (by analogy with the terminology for Lie algebras). It is proved that this class does not contain nodal algebras, so the results of the preceding abstract may be applied to obtain a structure theory. (Received May 27, 1957.)

658*t*. A. Sharma: *q-Bernoulli and Euler numbers of higher order.*

The object of this note is to extend the recent results of L. Carlitz [Duke Math. J. vol. 15 (1948) pp. 987–1000 and Trans. Amer. Math. Soc. vol. 76 (1954), pp. 332–350] on q-Bernoulli and Eulerian numbers to such numbers of higher order. For want of a natural generating function for the numbers introduced by Carlitz such an extension is not immediately obvious. We define numbers $\beta_m^{(k,h)}$ for k, h integers ≥ 1, $m \geq 0$ such that $\beta_m^{(k,h)} = \beta_m^{(k-1,h)} + (q-1)\beta_{m+1}^{(k-1,h)}$ and $\beta_m^{(k,h+1)} = (m-h)/[m-h]\beta_m^{(k,h)}$ where $[x] = (q^x - 1)/(q-1)$. The numbers $\beta_m^{(k,h)}$ and the corresponding polynomials in q^x reduce for $q=1$ and $h=k$ to the $\beta_m^{(k)}$—the Bernoulli numbers, etc. of higher order. Euler numbers and polynomials of higher order and their interrelations are also studied. A sort of von Staudt Clausen theorem also holds when $h > k = 1$. (Received July 8, 1957.)

659*t*. Abraham Spitzbart: *A generalisation of Hermite's interpolation formula.*

Hermite's interpolation formula (cf. Z. Kopal, *Numerical analysis*, New York, 1955) provides an explicit representation of a polynomial of minimum degree which assumes given functional and first derivative values at specified points. The present paper generalizes this result to include the values of derivatives up to specified, arbitrary orders at the given points. The desired polynomial can be written in the form $\sum_{j=0}^{n} \sum_{k=0}^{r_j} A_{jk}(x) f_j^{(k)}$ where x_j, r_j, $f_j^{(k)}$, $j = 0, 1, \cdots, n$; $k = 0, 1, \cdots, r_j$ are given. If $P_j(x) = \prod_{i=0}^{\prime n} (x - x_i)^{r_i+1}$, where the prime indicates the omission of $(x - x_j)^{r_j+1}$, and $g_j(x) = [p(x)]^{-1}$, the result is $A_{jk}(x) = p_j(x)(x - x_j)^k (k!)^{-1} \sum_{t=0}^{r_j-k} (t!)^{-1} g_j^{(t)}(x_j)(x - x_j)^t$. This result then enables one to give an explicit formula for the general divided difference with repeated arguments, of any order. (Received July 5, 1957.)

660. Seth Warner: *Ulam's Axiom and Cartesian products of fields.*

Ulam's Axiom is the assertion that there exists no (countably additive) nonzero measure m defined on the set of all subsets of some set, taking on only the values 0 and 1, such that $m(F) = 0$ for all finite subsets F. If K is a field and A a set, let $K(A)$ be the Cartesian product of (K_α), $\alpha \in A$, where $K_\alpha = K$ for all α. We call a homomorphism from the K-algebra $K(A)$ into K a multiplicative linear form on $K(A)$. 1. If the cardinality of A does not exceed that of K, every nonzero multiplicative linear form on $K(A)$ is a projection. 2. Ulam's Axiom implies that if K is an infinite field, every nonzero multiplicative linear form on $K(A)$ is a projection. 3. Let K be a countably infinite field. Then Ulam's Axiom is equivalent to the assertion that for any set A, every nonzero multiplicative linear form on $K(A)$ is a projection. 4. If K is a finite field, there exists a canonical one-to-one correspondence between the nonzero multiplicative linear forms on $K(A)$ and the ultrafilters on A; in particular, if A is infinite, there exist nonzero multiplicative linear forms on $K(A)$ which are not projections. (Received July 5, 1957.)

661*t*. F. B. Wright: *Some remarks on Boolean duality.*

The duality of homomorphisms between Boolean algebras with continuous functions and Boolean spaces was established by M. H. Stone (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 37–111). This has been extended recently to a duality of hemimorphisms with Boolean relations, by P. R. Halmos (Comp. Math. vol. 12 (1955) pp. 217–249). The purpose of this note is to show that Halmos' extension has essentially reached the limits of this sort of duality. Let f be any mapping of one

Boolean algebra into another, and let ϕ be any relation from one Boolean space to another. Following Halmos, the duals f^* and ϕ^* can be defined; f^* is a relation and ϕ^* is a mapping. (1) A necessary and sufficient condition that $f=f^{**}$ is that f be a hemimorphism. (2) A necessary and sufficient condition that $\phi=\phi^{**}$ is that ϕ be a Boolean relation. In both cases, the sufficiency of these conditions is due to Halmos, and it requires only slightly more delicate considerations to establish the necessity of these conditions. (For example, the ranges of f^* and ϕ^* must be handled with care.) This analysis serves to extend the Jónsson-Tarski duality theorem (Amer. J. Math. vol. 73 (1951) pp. 217–249), and yields an alternative proof of the McKinsey-Tarski representation of closure algebras (Ann. of Math. vol. 45 (1944) pp. 891–939). (Received July 8, 1957.)

ANALYSIS

662*t*. H. A. Antosiewicz: *Reducible linear differential systems*, I.

The main result proved is the following theorem for complex n-vector equations $\dot{x}=[C+B(t)]x$ where $C=\text{diag}\,(\lambda_1, \cdots, \lambda_n)$ and $B(t)=(b_{ij}(t))$, $1\leqq i, j\leqq n$, is defined and continuous on the (real) half-line $L=(0, \infty)$: If for some index q, $1\leqq q\leqq n$, (1) $\text{Re}\,(\lambda_i-\lambda_q)\neq 0$ for all $i\neq q$, (2) $\sup\left|\int^t b_{qq}(s)ds\right|<\infty$ on L, and (3) $\int^\infty |b_{ij}(s)|^\gamma ds<\infty$ for some γ, $1\leqq\gamma\leqq 2$, and all $(i, j)\neq(q, q)$, there is a solution $x_q(t)=(x_{1q}(t), \cdots, x_{nq}(t))$ which, as $t\to\infty$, satisfies $x_{iq}(t)\exp\,(-\lambda_q t-\delta_{iq}\int_q^t b_{qq}(s)ds)\to\delta_{iq}$, $1\leqq i\leqq n$. This solution is unique provided the set P_q of indices p such that $\text{Re}\,(\lambda_q-\lambda_p)>0$ is empty. If $P_q\neq 0$, $x_q(t)$ may be determined so as to satisfy, at any $t_0>0$ sufficiently large, $x_{pq}(t_0)=c_p$, $p\in P_q$, with arbitrary constants c_p. It follows that if the hypotheses hold for all q, $1\leqq q\leqq n$, the equation $\dot{x}=[C+B(t)]x$ is reducible in Lyapunov's sense [Annals of Mathematics Studies, no. 17, Princeton University Press, 1947, p. 242], i.e., a base of solutions, $X(t)$, has the form $X(t)=Z(t)e^{Ct}$ where $Z(t)$ is, on L, nonsingular, continuously differentiable and bounded together with $Z^{-1}(t)$. This generalizes a theorem due to Erugin [Trudy Mat. Inst. V. A. Steklov vol. 13, 1946, 93 pp.] and complements results of Yakubovic [Dok. Akad. Nauk SSSR vol. 66 (1949) pp. 577–580] and of Wintner [Amer. J. Math. vol. 77 (1955) pp. 45–86]. (Received July 3, 1957.)

663*t*. W. D. L. Appling: *Concerning integrals*. Preliminary report.

Suppose each of f and g is a real function on $[a, b]$, each of k and J a number and, if $h>k$, there is a chain, C from a to b such that, for each refinement $(x_i)_{i=0}^n$ of C, $\left|J-\sum_{i=1}^n (1/2)(f(x_i)+f(x_{i-1}))(g(x_i)-g(x_{i-1}))\right|<h$. There is a least such k and, for this k, only one J, denoted by $\int_a^b fdg(=fg|_a^b-\int_a^b gdf)$. Properties of this integral are developed. Classes of functions f are characterized by means of the integral, e.g., (1) every bounded f is integrable with respect to any g of bounded variation, and if every continuous f is g-integrable, then g is of bounded variation and (2) g is constant on no subinterval and bounded if and only if every g-integrable f is bounded. (Received June 24, 1957.)

664. R. W. Bass: *A global structure theorem for arbitrary dynamical systems in E^n*.

Notations: Bull. Amer. Math. Soc. Abstracts 63-3-358 through 63-3-361 and 63-3-385, 63-3-386. *Def.* The phase-space G of an *elementary system* is the product of E^1 with a connected region of an $(n-1)$-dimensional manifold; E^1 acts on G as a translation group. The flow in G is *P-irrotational*, *wandering* (G. D. Birkhoff), has no *improper saddle-point* (Nemickii), hence is *dispersive* (Montgomery-Zippin). *Defs.* (Birkhoff) The *center* C of S is the largest closed set possessing recurrence of relative domains, i.e. the closure of all Poisson-stable orbits. The *hull* $H = S - C$. *Def.* An (asymptotic) orbit of the hull is a *separatrix* if not "orbitally asymptotically stable." *Def.* W (for *walls*) is the (closed) set of all separatrices. For compact conservative S's, $H = \phi$; but analysis of H for dissipative S's is important. *Th.* $S = C \cup H$, $C \cap H = \phi$; $H = W \cup G$, $W \cap G = \phi$; $G = \bigcup_\alpha G_\alpha$, $G_i \cap G_j = \delta_{ij}$, where each G_α is an elementary system. *Cor.* (a) NASC's for global asymptotic stability of equilibrium ($C = x_0$); (b) inversion of Lyapunov's *Direct Theorems*; (c) planar decomposition theory of Poincaré, Bendixson, and Markus (Trans. Amer. Math. Soc. vol. 76 (1954) pp. 127–148.) (Received July 12, 1957.)

665. Errett Bishop: *A generalisation of a theorem of Mergelyan.* Preliminary report.

Let C be a compact set of the complex plane, whose complement is connected. We investigate the structure of a Borel measure μ, on the boundary of C, having the property that $\int f(z)d\mu(z) = 0$ for all polynomials f. The measure μ can be written as an infinite sum, $\mu = \sum_{i=1}^{\infty} \mu_i$, with convergence in the norm topology for measures, where each of the Borel measures μ_i has the following structure. There exists a component U_i of the interior of C, such that μ_i is a measure on the boundary of U_i and such that the analytic function ϕ_i on U_i defined by $\phi_i(z) = (1/2\pi i)\int d\mu_i(\zeta)/(\zeta - z)$ has the following property: there exists a sequence $\{\gamma_j\}$ of simple closed rectifiable curves lying in U_i such that $\int_{\gamma_j} g(z)\phi_i(z)dz \to \int g(z)d\mu_i(z)$ as $j \to \infty$, for each continuous function g on C. We say that μ_i is the boundary measure of the differential $\phi_i(z)dz$. As a corollary of this theorem, one obtains the theorem of Mergelyan, that any continuous function on C which is analytic at interior points can be uniformly approximated by polynomials. (Received July 22, 1957.)

666. R. C. Buck: *Converse forms of the Hadamard product theorem.*

If $F(z)$ and $G(z)$ are regular everywhere, except for isolated singularities at α and β, respectively, then their Hadamard product $H(z)$ is regular everywhere except at $\alpha\beta$. The following partial converse is obtained: Let $H(z)$ be regular everywhere except at γ, and be the Hadamard product of functions $F(z)$ and $G(z)$; if F and G are required to have their singularities confined to some bounded set in the right half plane, and γ is either a pole or an essential singularity of finite exponential order, then F and G must be of the same sort. This does not hold if γ is unrestricted. (June 24, 1957.)

667t. R. C. Buck: *Derivations in operator algebras.*

Let E be a locally convex linear space, and $\mathfrak{L}(E)$ be the algebra of continuous linear transformations, $E \to E$. A derivation of $\mathfrak{L}(E)$ is a linear mapping ∂ of $\mathfrak{L}(E)$ into itself which obeys $\partial(AB) = \partial(A)B + A\partial(B)$. Any transformation $D \in \mathfrak{L}(E)$ gives rise to an inner derivation of $\mathfrak{L}(E)$ defined by $\partial(A) = [D, A] = DA - AD$. Theorem 1. The inner derivation ∂ obtained from D is continuous in the bounded-open topology on $\mathfrak{L}(E)$. Theorem 2. Every continuous derivation of $\mathfrak{L}(E)$ is inner. The result extends also to transitive subalgebras of $\mathfrak{L}(E)$ which contain minimal left ideals. The proof depends upon an explicit construction of D from ∂ which has been obtained jointly by the author and C. W. Curtis for the algebraic case, without topology. (Received June 24, 1957.)

668*t*. R. C. Buck: *Automorphisms of operator algebras.*

With notation as in the preceding abstract, it is shown that every automorphism k of $\mathfrak{L}(E)$ which is continuous in the bounded-open topology has the form $k(T) = D^{-1}TD$ for some nonsingular continuous $D \in \mathfrak{L}(E)$. If $\mathfrak{A}$ is a transitive subalgebra of $\mathfrak{L}(E)$ with a minimal left ideal, then every continuous automorphism of $\mathfrak{A}$ can be extended to a continuous (inner) automorphism of $\mathfrak{L}(E)$. (Received June 24, 1957.)

669*t*. R. C. Buck: *A Stone-Weierstrass theorem for* $C^*[X]$.

Let X be a space that is locally compact, but not compact, and let $C^*[X]$ be the set of *all* bounded continuous real valued functions on X. Introduce into $C^*[X]$ the strict topology β. (See Proc. Amer. Math. Soc. vol. 3 (1952) pp. 681–687 and Bull. Amer. Math. Soc. Abstract 62-3-310.) It is now known that β is locally convex, nonmetrizable, and topologically complete. Theorem: If X is σ-compact, then any subalgebra of $C^*[X]$ which separates points of X, is strictly dense. When X is not required to be σ-compact, the same is true for subalgebras of $C^*[X]$ whose closures (strict) contain a nonvanishing function. This condition is not necessary, for the subalgebra $C_\infty[X]$ of functions with compact carrier is strictly dense in $C^*[X]$. (Received July 3, 1957.)

670*t*. C. C. Camp: *An integral equation whose kernel involves finite discontinuities along both diagonals.*

The former theory of the equation $u(x) = \lambda\int_0^1 K(x, s)u(s)ds$ with a discontinuity of 1 in $\partial K/\partial x$ on the line $s = x$ and of -1 along the line $s = -x$ has been extended so as to remove the restriction requiring symmetry to the line $x = 1/2$. The present paper considers $u(x) = \lambda\int_{-1}^1 K(x, s)u(s)ds$ with a unit jump along the main diagonal $s = x$ but a discontinuity of $-k(-x) < 0$ along the other diagonal $s = -x$. With the usual restrictions of continuity of $k(x)$, $K(x, s)$ and its derivatives, if $[k(-x)K_{xs}(-x, s) - K_{xs}(x, s)]/p(x)$ where $p(x) = k(x)k(-x) - 1 \neq 0$ possesses a Fredholm reciprocal kernel the problem may be reduced to an integro-differential functional system and then to a regular integro-differential system. The Green's function for the latter may be used to establish an expansion for an arbitrary function $f(x)$ with piecewise continuous derivative. If $p(x) = 0$ the problem can still be solved in a variety of cases. (Received July 15, 1957.)

671*t*. J. W. Carr, III: *Generalized functional round-off error analysis.*

Generalized error bounds for (a) implicit iteration or functional solutions and (b) explicit iterations are obtained for the round-off error generated during the course of a computation. (a) An extension is made of the work of Weissinger and Collatz in applying classical functional analysis to determine truncation error bounds for functional iteration, to include round-off error at each stage. This is done for a general system of nonlinear functions of an arbitrary number of variables, using arbitrary norms; conditions are found to guarantee stability of the solution under round-off, and upper limits for the total error obtained. (b) For explicit iteration, a similar round-off error analysis provides error bounds for a general system of nonlinear equations, for arbitrary norms. In both cases if the norm of the iteration operator is bounded less than one, the total round-off error is bounded independent of the number of iterations. Particular applications of the theory include the Jacobi diagonalization procedure, previously analyzed in an unpublished work of Goldstine, the

Gauss Elimination solution methods, and various techniques of numerical integration of ordinary and partial differential equations. (Received June 18, 1957.)

672. Lamberto Cesari and W. R. Fuller (p): *Existence theorems for periodic solutions of weakly nonlinear Lipschitzian autonomous differential systems.*

Consider the system (*) $\ddot{x}_j+\sigma_j^2x_j=\epsilon q_j(x, \dot{x}, \epsilon)$, $j=1, \cdots, n$, where $x=(x_1, \cdots, x_n)$, $('=d/dt)$, $\sigma_j>0$, $m\sigma_j+\sigma_k\neq 0$, $j\neq k$, j, $k=1, \cdots, n$; $m=0, \pm 1, \cdots$, and each $q_j(x, \dot{x}, \epsilon)$ defined in $Q=[|x_j|\leq R, |\dot{x}_j|\leq R, 0\leq\epsilon\leq\epsilon_0]$ is bounded in Q, is Lipschitzian with respect to x, $\dot{x}$ and continuous with respect to ϵ in Q, and $q_j(-x, \dot{x}, \epsilon)=-q_j(x, \dot{x}, \epsilon)$. Then for every complex number a, $0<|a|<R$, and for all ϵ, $0\leq\epsilon\leq\epsilon_1$ $(0<\epsilon_1\leq\epsilon_0)$ and every $l=1, \cdots, n$, system (*) has a solution of the form $x_l=|a|\sigma_l^{-1}\sin(\tau t+\phi)+\epsilon W_l(\tau t+\phi, \epsilon)$, $x_k=\epsilon W_k(\tau t+\phi, \epsilon)$, $k\neq l$, $k=1, \cdots, n$, ϕ arbitrary, where all W_s are continuous functions of $\tau t+\phi$, ϵ, periodic of period 2π in $\tau t+\phi$, where $\tau=\tau_j(a, \epsilon)$ is continuous in a, ϵ, and $\tau_j(a, 0)=\sigma_j$. This statement and others give existence theorems for periodic solutions under the sole requirements of a Lipschitz condition and conditions of symmetry. These statements have been obtained by a new proof of the convergence of the casting out method of successive approximations first used by L. Cesari [Atti Accad. Italia (6) vol. 11 (1940) pp. 633–692], and then developed by L. Cesari, J. K. Hale, R. A. Gambill, H. R. Bailey [see, e.g., Bull. Amer. Math. Soc. Abstracts 60-1-118, 60-1-119, 60-4-487, 62-6-668, 62-6-669] for analytic linear and nonlinear, autonomous and periodic, differential systems, and used by W. R. Fuller for differential-difference systems [Bull. Amer. Math. Soc. Abstract 63-4-520]. (Received July 10, 1957.)

673. V. F. Cowling: *On series of Legendre and Laguerre Polynomials.*

In this paper we are concerned with the problem of determining a region of regularity for functions defined by series of Laguerre and Legendre polynomials. Such regions are determined in terms of the rate of growth, in certain directions, of analytic functions which generate the coefficients at the positive integers. The complicated nature of the results prohibits their presentation here. (Received July 15, 1957.)

674. S. P. Diliberto: *A new proof of C. L. Siegel's center theorem.*

In his Vorlesungen Uber Himmelsmechanik (Springer-Gottingen 1956) C. L· Siegel states and proves this remarkable theorem: *Let* $dx/dt=f(x)$ $\cdots$ (*) *be analytic in* $x=(x_1, \cdots, x_{2n})$ *about* $x=0$, $f(0)=0$, *and Hamiltonian. If* $f(x)=Ax+$*higher order terms*, let $\lambda_1, \cdots, \lambda_n$ *be the characteristic roots of A. If the* λ_i *are pairwise different,* λ_1 *purely imaginary, and* $\lambda_i/\lambda_1\neq$*integer* $i=2, \cdots, n$ *then the system* (*) *has a one parameter family* (*analytic*) *of periodic solutions centering* $x=0$. The author gives a simpler proof and drops the requirement that λ_i be pairwise distinct. The possibility of a zero root (i.e. $\lambda_i=0$ for some $i>1$) is included with additional hypothesis. (Received July 15, 1957.)

675*t*. W. F. Eberlein: *An integral over function space.*

Real functions $x(t)=\sum_0^\infty x_nt^n$ $(-1\leq t\leq 1)$ with $\|x\|_1=\sum_0^\infty |x_n|<\infty$ may be identified with elements $x=(x_0, x_1, x_2, \cdots)$ of the sequence space l_1. Since the unit sphere S of l_1 is compact under the weak* topology = topology of coordinatewise convergence, a countably additive measure on S is induced by a positive linear functional E (integral) on $C(S)$, the weak* continuous functions on S. There exists a "natural" integral over S reducing to $E(f) = 2^{-1}\int_{-1}^{1} f(x_0)dx_0$ when f is a function of x_0 alone. This integral has numerous applications to harmonic analysis and the theory of numerical integration, in particular. (Received July 11, 1957.)

676. Albert Edrei (p) and W. H. J. Fuchs: *On the lower order of meromorphic functions with deficient values.*

Let $f(z)$ be meromorphic and $\sigma > 1$. Using the standard notations of the theory of R. Nevanlinna: $T(r) \leq \{4/(\sigma - 1)\} T(\sigma r) + \max\ \{N(\sigma r, 0) + N(\sigma r, \infty)\} + O(\log\ r)$ for $r > 2$. It follows that if $f(z)$ has two deficient values α and β and if $\kappa = \max\ (1 - \delta(\alpha), 1 - \delta(\beta))$ then the lower order μ of $f(z)$ satisfies $\mu \geq -\log\ \{\kappa(2 - \kappa)\}/\log\ (1 + 4/\kappa(1 - \kappa))$. In particular, the lower order of a function which has at least two deficient values must be positive. (Received July 10, 1957.)

677. J. M. G. Fell: *Weak containment of group representations.* Preliminary report.

Let G be a locally compact group. For each x in the group algebra $L_1(G)$, let $\|x\|_* = \sup \|T_x\|$, where T runs over all unitary representations of G. The completion of $L_1(G)$ under $\|\ \|_*$ is a C^*-algebra, called $C^*(G)$. A function ϕ of the form $\phi(g) = (\xi, T_g \xi)$, where ξ is in the space of the representation T, is *associated with* T. Now let W be a family of unitary representations of G, and T^0 a unitary representation of G. Then W *weakly contains* T^0 if each function of positive type associated with T^0 is a uniform-on-compacta limit of sums of functions of positive type associated with representations in W. It is shown that W weakly contains T^0 if and only if the kernel of T^0 contains the intersection of the kernels of the T in W, T^0 and W being considered as acting on $C^*(G)$. In particular, the regular representation weakly contains all representations if and only if $C^*(G)$ is the completion of $L_1(G)$ as an algebra of operators acting on $L_2(G)$. This fails for the 2×2 complex unimodular group; here the regular representation does not weakly contain the supplementary series. (Received July 2, 1957.)

678. Tomlinson Fort: *Linear difference and differential equations satisfying conditions at more than one point.*

The author establishes sufficient conditions in terms of the nonvanishing of certain determinants of the coefficients that a system of ordinary linear differential equations or a system of ordinary linear difference equations have one and only one solution with prescribed values, some at one point and some at other points. The method of proof is to establish the theorems for difference equations and to pass to the limit for differential equations. (Received June 24, 1957.)

679. W. C. Fox: *Existence of harmonic functions with prescribed singularities.*

Let R be an annulus and let A be a closed disk (in the Riemann surface X) whose boundary circle is also a boundary curve of R. If ϕ is harmonic on R, then a function u is said to have the singularity ϕ on the disk $B = A \cup R$ provided u is harmonic on $X - A$ and $u - \phi$ has a harmonic extension to B. If (u_n) is a sequence of functions with the singularity ϕ on B, and if each u_n is bounded on $X - B$ by the maximum of $|u_n|$ on the boundary of B, then the sequence is uniformly bounded provided that at some point x of A the values $u_n(x)$ all coincide. This fact is instrumental in showing that for open X and arbitrary ϕ, R, and A, a function u always exists which has the singularity ϕ. When X is compact such a u is known to exist iff ϕ is the real part of a single-valued function analytic on R. For any X such a u exists iff one of the various equivalent extremal problems involving the Dirichlet integral has a solution. (Received July 9, 1957.)

680*t*. Evelyn Frank: *On the equivalence of certain algorithms for the generation of continued fraction expansions.*

The equivalence is shown between the author's algorithms [Bull. Amer. Math. Soc. vol. 52 (1946) pp. 144–157, 890–898] and the quotient-difference algorithm of H. Rutishauser [Mitteilungen aus dem Institut für angewandte Mathematik no. 7, Birkhäuser Verlag, Basel, Stuttgart, 1957] for the construction of J-fractions and S-fractions corresponding to rational functions and to power series. The author's stability criterion is also interpreted in terms of Rutishauser's parameters. (Received July 12, 1957.)

681*t*. K. O. Friedrichs: *Symmetric positive linear differential equations.*

Let $u=\{u_1, \cdots, u_k\}$ be a system of functions of the points x of an m-dimensional manifold R with boundary B. With respect to local coordinate systems $x=\{x_1, \cdots, x_m\}$, let $\nabla_\rho=\partial/\partial x_\rho$, and introduce k by k matrices α^ρ and κ with the properties I: α^ρ is symmetric, $\rho=1, \cdots, m$; II: $\kappa+\kappa'$ is positive definite. The differential operator $K=\alpha^\rho\nabla_\rho+\nabla_\rho\alpha^\rho+\kappa$ is called "symmetric positive." On B with interior normal $\{n_1, \cdots, m\}$ one finds matrices μ with III: $\mu+\mu'$ non-negative and IV: $u=u_+ + u_-$ with $(\mu-\beta)u_+=(\mu+\beta)u_-=0$. Set $M=\mu+\beta$. Then the boundary value problem $Ku=f$, $Mu=0$ with $f\in\mathfrak{L}_2$ has a unique strong solution.—Uniqueness follows from $c\|u\|\leq\|Ku\|$, where $\|u\|^2=(u, u)$ is an appropriate integral over R. A system of operators $D_\sigma=\eta_\sigma^\rho\nabla_\rho+\eta_\sigma$, $\sigma=0, \cdots, s$ is so chosen that $n_\rho\eta_\sigma^\rho=0$ on B, and $[D_\sigma, D_\tau]=r_{\sigma\tau}^\rho D_\rho$, $[D_\sigma, K]=p_\sigma^\tau D_\tau-t_\sigma K$, $[D_\sigma, M]=q_\sigma D_\tau-t_\sigma M$ on B. Then $Du=\{D_\sigma u\}$ satisfies an equation $K_1Du=Df$, $M_1Du=0$. Assuming K_1 satisfies II$_1$, III$_1$ we have a priori $c_+\|u\|_+\leq\|Ku\|_+$, where $\|u\|_+^2=\sum_\sigma\|Du_\sigma\|^2$. With the associated negative norm $\|\;\|_-$ the dual inequality $c_-\|u\|_-\leq\|Ku\|$ holds. Adaptation of the method of P. Lax [Comm. Pure Appl. Math. vol. 8 (1955) pp. 615–633] yields the existence of a solution with $\|u\|_+<\infty$. Assumptions II$_1$, III$_1$ are removed by a continuation argument. Problems of elliptic, hyperbolic, and mixed equations, such as Tricomi's, are included as special cases. (Received May 27, 1957.)

682. R. E. Fullerton: *An intersection property for cones in a linear space.*

Let C be a cone in a linear space X. If x, $y\in X$, let $C'=(x+C)\cap(y+C)$. It is shown that if C' is itself a cone, there exists an element $z\in X$ such that $C'=z+C$. Various examples of cones having the property that there exist points x, $y\in X$ with $(x+C)\cap(y+C)$ a cone are given and discussed. (Received July 11, 1957.)

683*t*. I. S. Gál: *On uniformisable spaces with a unique structure.*

Let X be a uniformizable Hausdorff space. The purpose of this note is to give a n.a.s.c. that X has a unique structure. Let $C(X)$ denote the space of all real valued continuous functions on X topologized by the topology of uniform convergence and let $A(X)$ be the set of real continuous functions on X having compact support. The closed sets C_1 and C_2 are called normally separable if there exists an $f \in C(X)$ vanishing on C_1 and identically 1 on C_2. Theorem: *Any two of the following statements are equivalent*: (i) *X has a unique structure*; (ii) *if C_1 and C_2 are normally separable closed sets in X then at least one of them is compact*; (iii) *the set $A(X)$ is dense in the space $C(X)$*. Condition (ii) shows that compactness is sufficient and local compactness is necessary for uniqueness. Condition (ii) is due to R. Doss [R. Doss, *On uniform spaces with a unique structure*, Amer. J. Math. vol. 71 (1949) pp. 19–23]. Condition (iii) can be proved without using Doss' result by considering certain ideals in $[f^2 : f \in C(X)]$. These are generalizations of filters on X. (Received June 28, 1957.)

684t. F. W. Gehring: *Asymptotic values for analytic functions of bounded characteristic.*

Let $f(z)$ denote a complex-valued function defined in the unit disc D and let P denote a point on the unit circle. We say that $a \in \Gamma(P, f)$ if there exists a Jordan arc α contained, except for P, in D such that $f(z) \to a$ as $z \to P$ along α; $a \in \Gamma_\Delta(P, f)$ if, in addition, α can be chosen so that it lies in some Stolz angle with vertex at P. THEOREM 1. Suppose that $f(z)$ is analytic and has bounded characteristic in D. Then $\Gamma(P, f)$ contains at most two finite values and $\Gamma_\Delta(P, f)$ contains at most one finite value. THEOREM 2. Suppose that $f(z)$ is analytic and in H_p for some $p > 0$. Then $\Gamma(P, f)$ contains at most one finite value. THEOREM 3. Suppose that $f(z)$ is analytic and has bounded characteristic in D and that $f(z)$ omits one finite value, say 0. Then $\Gamma(P, f)$ contains at most one finite value. Moreover, if $\Gamma(P, f)$ contains $a \neq 0$, ∞, it contains only that value. Simple examples show that all these results are best possible. (Received July 12, 1957.)

685t. F. W. Gehring and A. J. Lohwater: *On the Lindelöf theorem.*

Let D denote the unit disc and let P denote a point on the unit circle. The theory of cluster sets is used to derive results, related to a well-known theorem due to Lindelöf [Acta Soc. Sci. fenn. vol. 35, Nr. 4 (1915)], of which the following is typical. THEOREM. Suppose that $f(z) = u(z) + iv(z)$ is analytic and bounded in D and suppose there exist two Jordan arcs α and β contained, except for P, in D such that $u(z) \to a$ and $v(z) \to b$ as $z \to P$ along α and β respectively. Then $f(z) \to a + ib$ uniformly as $z \to P$ in each Stolz angle with vertex at P. (Received July 12, 1957.)

686. A. W. Goodman: *On the critical points of a multivalent function.*

If the critical points of $f(z)$, analytic in $E: (|z| < 1)$, are too close to the origin, the function cannot be p-valent in E. Precisely it is conjectured that if $C_1, C_2, \cdots, C_k$ $(k > p)$ are critical points of $f(z)$, with $|C_j| \leq (2p - 1 - 2(p^2 - p)^{1/2})^{1/k}$ for $j = 1, 2, \cdots, k$ and if the strict inequality holds for at least one index then $f(z)$ cannot be analytic and p-valent in E. The bound if true would be sharp as the function $F(z) = z(1 - z^k)^{(2p-1)/k}/(1 + z^k)^{2p/k}$ shows. If $f(z) \neq 0$ in $0 < |z| < 1$, then the condition $k > p$ can be dropped. Using variational methods properties of the extremal functions for this problem are obtained. A relation between this problem and a coefficient problem for p-valent functions is obtained and a new conjecture on the coefficients of p-valent functions is suggested. (Received June 28, 1957.)

687t. Hugh Gordon: *Integrals defined by abstract L_p spaces.*

Let E be a set; let V be the set of all real-valued functions on E. Let V_0 be a linear subspace of V such that: (a) if $f \in V$, $g \in V_0$ and $|f| \leq |g|$, then $f \in V_0$; and (b) if $f_1, f_2, \cdots \in V_0$ and $f_n \to f \in V$ pointwise, then $f \in V_0$. Let L be a subspace of the

Riesz space V/V_0 and $p \geq 1$ be a fixed real number such that: (a) L is a Banach lattice; (b) the class of the constant function 1 is in L; (c) $\| |\bar{f}| \| = \|\bar{f}\|$ for all $\bar{f} \in L$; (d) $\|\bar{f}+\bar{g}\|^p \geq \|\bar{f}\|^p + \|\bar{g}\|^p$ if $\bar{f}, \bar{g} \in L$ are positive; and (e) $\|\bar{f}+\bar{g}\|^p \leq \|\bar{f}\|^p + \|\bar{g}\|^p$ if $\bar{f}, \bar{g} \in L$ are positive and $\bar{f} \wedge \bar{g} = 0$. Then there are a σ-ring $\mathfrak{M}$ of subsets of E, a Riesz space $V_1 \subset V$ and a positive linear functional F on V_1 with the following properties: (1) F on V_1 satisfies the theorem of Lebesgue. (If $g, f_1, f_2, \cdots \in V_1, f \in V, M \in \mathfrak{M}$, $f_n(x) \to f(x)$ for each $x \in E - M$ and $|f_n(x)| < g(x)$ for each $x \in E - M$ and each n; then $f \in V_1$ and $Ff_n \to Ff$.) (Thus F is an integral determined by a σ-additive measure.) (2) Each element of L is contained in an element of the space L_p with respect to F and V_1, and L and L_p are isomorphic, as Banach lattices, in the natural way. Acknowledgement is made to the Nat. Sci. Found. under NSF G 1981. (Received July 12, 1957.)

688t. Hugh Gordon: *The spaces L_p for generalized integrals.*

Let $\mathfrak{A}$ be a collection of filter bases on a set E. We suppose that for each $x \in E$ there is a filter base $\mathcal{A} \in \mathfrak{A}$ such that $x \in A$ for each $A \in \mathcal{A}$. Let V be the set of all real-valued functions on E. Let $f, f_1, f_2, \cdots \in V$; we write $f_n \to f$ if each filter base $\mathcal{A} \in \mathfrak{A}$ contains an A on which $\{f_n\}$ converges to f uniformly. Let V_1 be a subspace of V and F be a positive linear functional on V_1 such that: (1) V_1 is a Riesz space; (2) if $g, f_1, f_2, \cdots \in V_1, f_n \to f$, and $|f_n| < g$ for all n, then $f \in V_1$ and $Ff_n \to Ff$; (3) for each $f \in V_1$ there are bounded $f_1, f_2, \cdots \in V_1$ such that $f_n \to f$; and (4) $1 \in V_1$. (For example, F may be an $\mathcal{A}$-integral; see abstract of E. R. Lorch below.) For each real number $p > 1$, let V_p be the set of $f \in V$ such that $(f^+)^p, (f^-)^p \in V_1$. Let V_0 be the set of $f \in V_1$ such that $F|f| = 0$. Then $V_0 \subset V_p$ for all $p \geq 1$; let $L_p = V_p/V_0$. Each L_p is a Riesz space and a normed linear space with the norm $\|\bar{f}\|_p = (F|f|^p)^{1/p}$ where f is any representative of the class $\bar{f} \in L_p$. If p, q are positive real numbers such that $p^{-1} + q^{-1} = 1$ and $\bar{g} \in L_q$ is positive, then the mapping $\bar{f} \to F(fg)$ (where f, g represent $\bar{f}, \bar{g}$) is defined and is a bounded positive linear functional on L_p. Acknowledgement is made to the Nat. Sci. Found. under NSF G 1981. (Received July 12, 1957.)

689. Felix Haas: *A formula relating the index and the sectors of a singular point in N-space.*

The index of a nonvanishing vector field, V, in E_N with respect to a surface, S, is defined to be the degree of the map $f: S \to$ unit sphere which the vector field, V, induces. The index of an isolated singular point, p, of V is the index of V with respect to sufficiently small spheres enclosing p. The index of a smooth closed curve, C, is the index of V with respect to sufficiently small tori enclosing C. All these definitions reduce to the usual definitions for the plane. If V defines a differential equation with analytic coefficients, the following theorems hold: (1) The index of a singular point $= (2 + 2^{N-2}(N_1 - N_2))/2$ where N is the dimension of the space, N_1 is the number of elliptic sectors, and N_2 is the number of hyperbolic sectors. (This generalizes a well-known theorem of Bendixson for the plane.) (2) The index of a closed curve solution is 0. (3) The index of a closed surface equals the sum of the indices of the singular points inside the surface. (Received July 11, 1957.)

690. J. K. Hale: *Sufficient conditions for the existence of periodic solutions of systems of weakly nonlinear first and second order differential equations.*

Suppose that a vector function $f(x, t)$, $f = (f_1, \cdots, f_n)$, of the real vector

$x=(x_1, \cdots, x_n)$, and the real variable t belongs to the class $A[\omega]$, $\omega>0$, if for every t, $-\infty<t<+\infty$, each component f_j of f is analytic in a neighborhood U of $x=(0, \cdots, 0)$ independent of t, and the power series expansion of f_j in $x_1, \cdots, x_n$ is convergent in U, and its coefficients are periodic functions of t of period $T=2\pi/\omega$. Consider the system of differential equations (1) $y''+Dy=\epsilon f(y, w, y', \epsilon, t)$, $w'=\epsilon g(y, w, y', \epsilon, t)$, where $y=(y_1, \cdots, y_\mu)$, $w=(w_1, \cdots, w_\nu)$, $D=\operatorname{diag}(\sigma_1^2, \cdots, \sigma_\mu^2)$, ϵ, $\sigma_1, \cdots, \sigma_\mu$ are real positive numbers, and f, $g\in A[\omega]$. Let $a=(a_1, \cdots, a_\mu)$, $b=(b_1, \cdots, b_\mu)$ where each a_j, b_j, is a positive integer, $\sigma=(\sigma_1, \cdots, \sigma_\mu)$, $c=(c_1, \cdots, c_{\mu+\nu})$. By employing a method of successive approximations which has been successively developed by L. Cesari, R. A. Gambill and J. K. Hale it is shown there exist real functions $H_j(a, b, c, \sigma, \omega, \epsilon)$, $j=1, 2, \cdots, \mu$, analytic for $|\epsilon|<\epsilon_0$, $\epsilon_0>0$, such that the following holds: *Theorem.* If $f(y, w, -y', \epsilon, -t)=f(y, w, y', \epsilon, t)$, $g(y, w, -t', \epsilon, -t)=-g(y, w, y', \epsilon, t)$ and if, for $|\epsilon|$ sufficiently small, the system of equations (2) $a_j\omega/b_j+\epsilon H_j=\sigma_j$, $j=1, 2, \cdots, \mu$, has a solution for some real, nonzero c, σ and ω, then (1) has a periodic solution $y(\epsilon, t)$, $\omega(\epsilon, t)\in A[\omega/b_1 \cdots b_\mu]$ which is even in t. Under mild restrictions, there is thus a ν parameter family of periodic solutions. (Received May 29, 1957.)

691t. J. K. Hale: *Linear systems of first and second order differential systems with periodic coefficients.*

Consider the vector system of differential equations (1) $u''+Au=\lambda f(u, v, w, u', v', t)$, $v''+Bv=\lambda g(u, v, w, u', v', t)$, $w'=\lambda h(u, v, w, u', v', t)$, where λ is a real parameter, u, v, w are vectors $A=\operatorname{diag}(\sigma_1^2, \cdots, \sigma_\mu^2)$, $B=\operatorname{diag}(\sigma_{\mu+1}^2, \cdots, \sigma_n^2)$ and the vector functions f, g, h are linear functions of u, v, w, u', v', with coefficients periodic in t of period $2\pi/\omega$, L-integrable in $[0, T]$ with mean value zero. Further, suppose $\sigma_j\neq\sigma_k$ (mod ωi), $j\neq k$, j, $k=1, 2, \cdots, n$, $\sigma_k\neq 0$ (mod ωi) $k=1, 2, \cdots, n$. By employing a method of successive approximations which has been successively developed by L. Cesari, R. A. Gambill and J. K. Hale, the following *Theorem* is proved: If $f(u, -v, w, -u', v', -t)=f(u, v, w, u', v', t)$, $g(u, -v, w, -u', v', -t)=-g(u, v, w, u', v', t)$, $h(u, -v, w, -u', v', -t)=-h(u, v, w, u', v', t)$, then for $|\lambda|$ sufficiently small all the absolutely continuous solutions of (1) are bounded. (Received May 29, 1957.)

692. P. C. Hammer: *Subadditive functions in general.* Preliminary report.

The inadequate generality of previous work on subadditive functions is perhaps a result of the subject being considered primarily for limited kinds of applications in analysis. We define *subadditivity* and *reverse subadditivity* of functions mapping a set S with a binary operation onto a partially ordered set G with a binary operation. Theorems are proved under as general assumption as possible or expedient. In particular, for example, if S is a semigroup and G a partially ordered group the theory of generating subadditive functions from others is stated in better *detail* than now given for real functions of a real variable. (Received July 1, 1957.)

693. John Harton, Jr.: *Distortion theorems for real star mapping.*

The maximum and minimum values of $|f'(z_0)|$ and $|\phi'(\zeta_0)|$ are studied, where $f(z)=z+a_2z^2+a_3z^3+\cdots$ is a simple star mapping of $|z|<1$ for which $a_2, a_3, \cdots$ are real and $\phi(\zeta)=\zeta+b_0+b_1/\phi+\cdots$ is a simple star mapping of $|\zeta|>1$ for which $b_0, b_1, \cdots$ are real. In the interior case the extremal functions map $|z|<1$ onto the ω-plane with one or two radial slits. Explicit formulas are given. In the exterior case, every extremal function must be a function which maps $|\zeta|>1$ onto the ω-plane with one-, two-, or three-line segments. In a certain region of $|\zeta| > 1$ every extremal function leaves out exactly three line segments and in other regions one or two line segments. These regions and also a region of nonuniqueness where any function of a certain form is an extremal function are given. The methods used are similar to those used by Robinson (Proc. Amer. Math. Soc. vol. 6 (1955) pp. 364–377), and Goodman (Proc. Amer. Math. Soc. vol. 4 (1953) pp. 278–286). (Received July 8, 1957.)

694. L. J. Heider: *Measures on Boolean algebras and abstract (L)-spaces.*

The measures on an arbitrary abstract Boolean algebra are in $1-1$ onto, linear, normed-lattice, isomorphic correspondence with the Baire measures on the Stone representation space of the algebra, with countably additive measures corresponding to Baire measures vanishing on Baire sets of the first category while purely finitely additive measures correspond to Baire measures vanishing outside a Baire set of the first category. In consequence, the abstract (L)-space of all measures on the algebra is the cross product of the (L)-space of countably additive measures with the (L)-space of purely finitely additive measures. In particular, the second adjoint space of an abstract (L)-space with weak unit is the cross product of the given (L)-space with a second newly introduced (L)-space. (Received July 8, 1957.)

695. Henry Helson: *Continuous functions as Fourier transforms.*

Given a closed set of points E on the line and a positive weight-function $p(x)$, it is a general problem to determine which continuous functions $F(t)$ defined on E are the restrictions of Fourier transforms of summable functions $f(x)$ satisfying $\int_{-\infty}^{\infty}|f(x)|^2 \cdot p(x)dx < \infty$. If E is a discrete set $\{\lambda_1, \lambda_2, \cdots\}$, the problem requires the determination of the largest and smallest eigenvalues of the infinite matrix $(A(\lambda_j-\lambda_k))$, where $A(t)$ is the Fourier transform of $1/p(x)$. Some results have been obtained for particular $p(x)$ and E. (Received July 8, 1957.)

696. C. S. Herz: *A theorem on Fourier-Stieltjes transforms.*

Let Γ be a locally compact abelian group and $\widehat{\Gamma}$ its character group,—the character of Γ corresponding to $t \in \widehat{\Gamma}$ at the point $x \in \Gamma$ is written as $e(tx)$. Let G be a dense subset of $\widehat{\Gamma}$ which is closed under the group operations in $\widehat{\Gamma}$. Further let Ω be an open subset of $\widehat{\Gamma}$, $\overline{\Omega}$ its closure. THEOREM 1. A necessary and sufficient condition that a bounded continuous function, λ, defined on Ω, be representable in the form $\lambda(t) = \int e(tx)\mu(dx)$, $t \in \Omega$, where μ is a Borel measure on Γ with variation $\leq M$ is that $|\sum c_i\lambda(t_i)| \leq M \sup_x |\sum c_i e(t_i x)|$ for each finite collection of pairs $\{(c_i, t_i)\}$ where c_i is a complex number and $t_i \in G \cap \Omega$. THEOREM 2. Let ϕ be a continuous function on Γ with $\sup_x |\phi(x)| \leq N$. Suppose the spectrum, $\Lambda(\phi)$ is contained in Ω, or more generally that ϕ is synthesizable from $\overline{\Omega}$. Then ϕ is the limit, uniformly on compact sets, of finite sums $\sum c_i e(t_i x)$ where $\sup_x |\sum c_i e(t_i x)| \leq N$ and $t_i \in G \cap \Omega$. Theorem 1 generalizes a result of Bochner (Bull. Amer. Math. Soc. vol. 40 (1934) pp. 271–276) where Γ is the real line, $\Omega = \widehat{\Gamma}$, and G is the rationals or the reals. (Received July 2, 1957.)

697. Seizô Itô: *A boundary value problem of diffusion equations.*

D denotes a (generally unbounded) domain in R^m with its boundary B of C^3 hypersurface. A diffusion equation $Lu = (1/a^{1/2})(a^{ij}a^{1/2}u_j)_i + cu - u_t = 0$ with mixed-type boundary condition $(B_\phi): \alpha u + \beta u_n = \phi$ on B, where $u_i = \partial u/\partial x^i$, $a = \det(a^{ij})^{-1}$, a^{ij}'s are

C^2 and c is nonpositive and Hölder-continuous on $\bar{D}$, u_n denotes normal derivative, $\alpha, \beta \geqq 0$, and α, β and ϕ are C^2 on B. Results: (1) There exists a function $U(t, x, y)$ satisfying $LU=0$ and (B_0): $\alpha U+\beta U_n=0$, in (t, x) for fixed y, and also in (t, y) for fixed x. (2) Given f, h and ϕ, $u(t, x)=\int_D U(t, x, y)f(y)dy+\int_0^t ds\int_D U(t-s, x, y)h(y)dy+\int_0^t ds\int_D[U(t-s, x, y)-\partial U(t-s, x, y)/\partial n_y]\phi(y)dy$ satisfies $Lu+h=0$, $u(0, x)=f(x)$ and the boundary condition (B_ϕ). (3) If c or $\phi\neq 0$, then $G(x, y)=\int_0 U(t, x, y)dt$ exists and is locally summable in x and in y separately, and $G(x, y)$ is Green's function of $(1/a^{1/2})(a^{ij}a^{1/2}u_j)_i+cu+h=0$ with the boundary condition (B_ϕ). Here a^{ij} is assumed to be positive definite throughout this paper. (Received June 13, 1957.)

698t. N. D. Kazarinoff: *Complete asymptotic solutions for ordinary second order linear differential equations with two simple turning points. I.*

The differential equation $d^2u/ds^2-\lambda^2Q(s, \lambda)u=0$, $Q(s, \lambda)=\sum_0^\infty q_j(s)\lambda^{-j}$, $|\lambda|>N$, is considered for s in a domain $\mathfrak{D}$ in which q_0 has exactly two simple zeros and all the q_j's are analytic. The matter at issue is the behavior of the solutions of this equation for all $s\in\mathfrak{D}$ as $|\lambda|\rightarrow\infty$. The natural first approximating equation is the modified Weber equation $d^2y/dx^2-\lambda^2(x^2-1)y=0$. The asymptotic behavior of its solutions has been examined by Erdélyi, Kennedy, and McGregor [J. of Rat. Mech. and Anal. vol. 3 (1954) pp. 459–485]. A *related* equation is constructed by means of a variant of a method due to McKelvey [Trans. Amer. Math. Soc. vol. 79 (1955) pp. 103–123], the construction being based upon the first approximating equation. The asymptotic behavior of solutions of the related equation is studied, and in doing this the results of Erdélyi, Kennedy, and McGregor are refined. The related equation is of the form $d^2y/ds^2-[\lambda^2Q(s, \lambda)+\lambda^{-n}R(s, \lambda)]y=0$, where n is an arbitrary positive integer and $R(s, \lambda)$ is analytic for $s\in\mathfrak{D}$ and $|\lambda|>N$. (Received May 31, 1957.)

699. Bruce Kellogg: *Growth of solutions of a certain hyperbolic equation.*

The equation $u_{tt}-\Delta u+u=0$ is considered, where Δ is the Laplacian in n space variables. An energy integral calculation shows that if the initial data and a number of its derivatives are bounded, then the square integral of the solution and a number of its derivatives over spheres of radius r in the $t=$constant hyperplanes is $O((t+r)^n)$, and from this it follows that the square of the solution is $O(t^n)$ as $t\rightarrow\infty$. Solutions having this growth are constructed. (Received July 12, 1957.)

700t. W. M. Kincaid: *Two-point procedures for the numerical solution of systems of equations.*

A family of methods has been developed for the numerical solution of a system of equations of the form $f_i(x_1, x_2, \cdots, x_n)=0$, $i=1, 2, \cdots, n$. The methods are iterative, but terminate after a finite number $[O(n^2)]$ of steps when the equations are linear. They may be regarded as generalizations of the classical method of false position. At each step, a linear function g agreeing with one of the functions f_i (or with a linear combination of them) at two points P and Q is constructed, and the point R on the line PQ at which $g=0$ is determined. Another point S not on PQ is then paired with R, and the process is repeated with another f_i. Convergence for nonlinear systems appears to be of the second order in general. The methods are well adapted for use with high-speed digital computers. This research was supported by Department of the Army Contract DA-36-039-SC52634 with the Engineering Research Institute of

The University of Michigan, operating under a Tri-Service Charter, administered by the U. S. Signal Corps. (Received July 12, 1957.)

701*t*. Joseph Kist: *Locally o-convex spaces*. I.

A *locally o-convex space* is a partially ordered vector space E together with a locally convex topology for which there exists a fundamental system of neighborhoods of the origin consisting of o-convex sets. (A subset S of E is called o-convex if it is convex, and if $x \in S$ whenever there exist elements $s, s' \in S$ such that $s \geqq x \geqq s'$.) The class of locally o-convex spaces includes the class of partially ordered normed spaces whose cone of positive elements is a normal cone [cf. M. Krein C.R. (Doklady) Acad. Sci. U.R.S.S. (N.S.) vol. 28 (1940) pp. 13–17]. Several of Krein's results have analogues in the more general setting. As an illustration, let E be a partially ordered locally convex space with closed positive cone P. Then, of the following statements, (2) and (3) are equivalent and are implied by (1); if E is a normed space, then all statements are equivalent: (1) E is a locally o-convex space. (2) $E' = P' - P'$, where P' is the cone of all continuous positive linear functionals (p.l.f.) on E. (3) With the weak topology, E is a locally o-convex space. The equivalence of (1) and (2) for normed spaces was proved by Krein. (Received July 10, 1957.)

702*t*. Joseph Kist: *Locally o-convex spaces*. II.

It is known [cf. e.g., Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) pp. 536–538] that a compatible Banach space topology on a vector lattice is the finest compatible topology. The following result has been obtained (terminology as in the preceding abstract): If a vector lattice E is endowed with a sequentially complete compatible topology T, then the following statements are equivalent: (a) T is the finest compatible topology. (b) Any homomorphism (i.e., order preserving linear transformation) of E with range in a locally o-convex space is continuous. (c) Any bounded homomorphism of E with range in a locally o-convex space is continuous. (d) $T = \tau_q(E, E')$, and every p.l.f. on E is continuous. ($\tau_q(E, E')$ is the l.u.b. of all locally o-convex topologies which are coarser than $\tau(E, E')$, the Mackey topology.) (e) $T = \tau_q(E, E')$, and every bounded p.l.f. on E is continuous. (f) Every o-convex, symmetric, and absorbing subset of E is a neighborhood of the origin. (g) Every o-convex and symmetric subset which absorbs all bounded subsets of E is a neighborhood of the origin. (h) E is the o-inductive limit of Banach lattices $\{E_\alpha\}$ with respect to homomorphisms $\{f_\alpha\}$ such that $E = \bigcup_\alpha f_\alpha(E_\alpha)$. (Received July 10, 1957.)

703. M. S. Klamkin (p) and D. J. Newman: *On the reducibility of some linear differential operators*.

The differential equation, $[x^nD^{2n} - 1]y = 0$, has been solved previously by Lommel but not in a direct way. By first showing that the operator x^nD^{2n} is reducible to $[xD^2 - (n-1)D]^n$, the problem is reduced to solving the Bessel equation $[xD^2 - (n-1)D]y = a_ry$ where $a_r^n = 1$. Whence, $y = x^{n/2}\sum_{r=0}^{n-1} [A_rJ_n(2(-xa_r)^{1/2}) + B_rY_n(2(-xa_r)^{1/2})]$. We also effect the reducibility of the following operators: $x^{rn}D^{(r+1)n}$, $x^{(r+1)n}D^{rn}$, $D^{(r+1)n}x^{rn}$, $D^{rn}x^{(r+1)n}$, $(xD^{1-r})^pD^{pr}$, and $x^{pr}(x^{1-r}D)^p$. (Received July 9, 1957.)

704. V. L. Klee, Jr.: *Extremal structure of convex sets*. I.

The Krein-Milman theorem and its converse are extended to certain noncompact convex sets with the aid of the notion of *extremal ray*, an extremal ray of a convex set C being an open half-line $\rho \subset C$ such that $C \sim \rho$ is convex and the endpoint of ρ is an extreme point of C. The principal results are as follows: (A) If C is a finite-dimensional closed convex set and C contains no line, then C is the convex hull of (ext C)$\cup$(rext C), where ext C is the set of all extreme points of C and rext C is the union of all extremal rays of C. (B) If C is a locally compact closed convex subset of a locally convex Hausdorff linear space and C contains no line, then C is the closed convex hull of (ext C)$\cup$(rext C). More generally, if $X\subset C$, then cl conv $X=C$ if and only if cl $X\supset$ext C and X is asymptotically codirectional with each extremal ray of C. (Received April 15, 1957.)

705t. Jacob Korevaar: *Local characterization of Fourier transforms.*

Let (a, b) be an interval of length $<2\pi$. Wiener (*The Fourier integral*, Cambridge, 1933, p. 80) proved that a function f defined on (a, b) is the restriction to (a, b) of the Fourier transform of a function F of $L_1(-\infty, \infty)$ if and only if f may be given on (a, b) by an absolutely convergent Fourier series $\sum_{-\infty}^{\infty} a_k e^{ikt}$. The following generalization is proved. Let c be real and let $1\leqq\alpha\leqq\infty$. A distribution ϕ on (a, b) is the restriction to (a, b) of the Fourier transform of a function F such that $(|t|+1)^c F(t)\in L_\alpha(-\infty, \infty)$ if and only if ϕ may be given on (a, b) by a trigonometric series $\sum a_k e^{ikt}$ with the property that the sequence $\{(|k|+1)^c a_k\}$ is of class l_α. Now let $(|t|+1)^c F(t)\in L_\alpha$, $(|t|+1)^d G(t)\in L_\beta$ where $c+d\geqq 0$, $\alpha\geqq 1$, $\beta\geqq 1$, $\alpha+\beta\geqq\alpha\beta$. Define tempered distributions Φ and Ψ by $\Phi=[F]^{(p)}$, $\Psi=[G]^{(q)}$ where p and q are integers $\geqq 0$. Then the above characterization enables one to prove that $T(\Phi*\Psi)=T\Phi\cdot T\Psi$. Here $\Phi*\Psi=[F*G]^{(p+q)}$ while the product $T\Phi\cdot T\Psi$ is defined locally with the aid of representing trigonometric series. (Received June 17, 1957.)

706t. Jacob Korevaar: *Completeness theorems for sets of translates as cancellation theorems.*

Wiener (*The Fourier integral*, Cambridge, 1933, p. 97) proved that the set of translates $F(t+\lambda)$, $-\infty<\lambda<\infty$, of a function $F\in L_1$ is fundamental in L_1 or complete with respect to L_∞ if and only if the Fourier transform $f=TF$ has no (real) zeros. The following proof is based on the preceding note. The set $\{F(t+\lambda)\}$ is complete if and only if $F*G=0$, G bounded and locally integrable, implies $G=0$. Setting $TG=\psi$ the equation $F*G=0$ is equivalent to $f\cdot\psi=0$. Here f is given locally by an absolutely convergent trigonometric series, ψ by one with bounded coefficients. If $f(t)\neq 0$ everywhere then $f\cdot\psi=0$ implies $\psi=0$ and hence $G=0$. A similar proof may be given for the following theorem of Boas and Bochner (Ann. of Math. vol. 39 (1938) pp. 287–300). Let C denote the Banach space of all continuous functions F on $[-\infty, \infty]$ with the norm max $|F(t)|$, C_0 the subspace of all functions vanishing at $\pm\infty$. The set of translates $\{F(t+\lambda)\}$ of $F\in C$ is fundamental in C_0 if and only if there is no interval on which the Fourier transform $\phi=TF$ is equal to zero. (Received June 17, 1957.)

707t. Jacob Korevaar: *Fourier transforms and the linear harmonic oscillator.*

Let the set $u_0, u_1, \cdots$ be orthonormal on $(-\infty, \infty)$ and complete with respect to a (large) vector space of functions C. Every $f\in C$ is uniquely determined by its expansion $\sum d_k u_k$ where $d_k=(f, u_k)$. One wishes to define a generalized Fourier transformation $[T]$ by the formula $[T]\sum c_k u_k=\sum c_k T u_k$. When will the Tu_k be simple linear combinations of the u's, for example $Tu_k=a_k u_k$? Answer: if the u_k are eigenfunctions of a boundary value problem $Hy=ly$, (B) such that H commutes with Fourier transformation while Tf satisfies the boundary condition (B) whenever

f does. For if λ, u is a characteristic pair of such a b.v. problem then so is λ, Tu, and if every eigenvalue λ has multiplicity one then $Tu=\alpha u$. One is thus led to the b.v. problem of the linear harmonic oscillator $Hy=(x^2-D^2)y=ly$, $0<\int_{-\infty}^{\infty}|y(x)|^2dx<\infty$. The paper employs a rigorous version of the factorization method devised by Schrödinger (Proc. Roy. Irish Acad. A vol. 46 (1940) pp. 9–16) to obtain the characteristic pairs in the form $2k+1$, $v_k(x)=l_kN^k\exp(-x^2/2)$. Here $N=x-D$; the v_k are the normalized Hermite functions. One has $Tv_k=(-i)^kv_k$. (Received June 17, 1957.)

708t. Jacob Korevaar: *Hermite expansions.*

It is shown that the normalized Hermite functions v_k, $k=0, 1, \cdots$ are complete with respect to the class A of functions f for which there exists a number $a<1/2$ such that $f(x)\exp(-ax^2)\in L_1$. If $f\in A$ then $\{x\}f\in A$. If f is absolutely continuous and its derivative $Df\in A$ then $f\in A$. If f and Df are absolutely continuous and $D^2f\in A$ then $Hf=(x^2-D^2)f\in A$. The Hermite expansions of $\{x\}f$, Df and Hf are expressed in terms of the expansion coefficients d_k of f, similarly the expansion coefficients of the Fourier transform Tf when $f\in L_1$. It is noted that every expansion coefficient of each of these four transforms of f involves only a finite number of d_k's. As an application a simple proof is given of the Fourier inversion theorem in the case where f and Tf are in L_1. The order of magnitude of the expansion coefficients is studied for several classes of functions. For f of at most polynomial growth $d_k=O(k^a)$, for f of at most exponential growth $d_k=O\{\exp(ck^{1/2})\}$, while for $f\in A$ one has $d_k=O(c^k)$. (Received June 17, 1957.)

709t. Jacob Korevaar: *Pansions.*

A pansion ϕ is a formal series $\sum_{k=0}^{\infty}c_kv_k$ of normalized Hermite functions. If ϕ is the expansion of a function f of class A (preceding abstract) or a suitable larger class one identifies ϕ with f. The global derivative $[D]\phi$, the global product $[x]\phi$ and $[H]\phi$ are defined for all pansions ϕ in accordance with the rules valid for the Hermite expansions of the corresponding transforms of functions f of A. It is shown that $[D]\phi=0$ if and only if $\phi=\{c\}$, a constant function. Similarly $[x]\phi=0$ if and only if $\phi=c\delta$ where δ is the Dirac "function" (global derivative of the unit step function). One also defines $[E(D)]\phi$, $[E(x)]\phi$ and $[E(H)]\phi$ where E is an element of a class of entire functions appropriate to ϕ. The operator $[e^{aD}]$ plays the role of a translation operator: for every function $f\in A$ and real a one has $[e^{aD}]f=\{f(x+a)\}$ ("Taylor's formula"). Global multiplication of two pansions is considered, as well as convolution. One has $(2\pi)^{1/2}\delta*\phi=\phi$, $(2\pi)^{1/2}[D]\delta*\phi=[D]\phi$ for every pansion ϕ. (Received June 17, 1957.)

710t. Jacob Korevaar: *Fourier transforms of pansions.*

Let ϕ be the pansion $\sum c_kv_k$ (preceding abstract). By definition the Fourier transform $[T]\phi$ is the pansion $\sum c_kTv_k=\sum(-i)^kc_kv_k$. Similarly the conjugate Fourier transform $[T]\phi=\sum i^kc_kv_k$. If ϕ is a function f of L_1 then $[T]\phi$ equals the ordinary Fourier transform Tf. For pansions the Fourier inversion formula holds without restriction: if $\psi=[T]\phi$ then $[T]\psi=\phi$. As an application one has a very simple derivation of Plancherel's theory of Fourier transforms for L_2. This derivation shows some similarity with Wiener's treatment (*The Fourier integral*, Cambridge, 1933, p. 46) based on Hermite expansions, but it is considerably shorter. The following rules are valid without restriction: $[T][D]\phi=i[x][T]\phi$, $[T][x]\phi=i[D][T]\phi$, $[T][H]\phi=[H][T]\phi$. The rules $[T][E(D)]\phi=[E(ix)][T]\phi$, $[T][E(x)]\phi=[E(iD)]\cdot[T]\phi$ are valid for a large class of pansions ϕ and entire functions E. Finally

$[T](\phi * \psi) = [T]\phi \cdot [T]\psi$ whenever one of the two sides exists. One has $[T]\{1\} = (2\pi)^{1/2}\delta$. (Received June 17, 1957.)

711*t*. Jacob Korevaar: *Fourier transform theory for pansions of polynomial growth.*

Using global properties of the solutions of the differential equation $Hy = (x^2 - D^2)y = f$ it is shown that every function of at most polynomial growth on $(-\infty, \infty)$ can be represented in the form $H^n f_0$ where f_0 is a sufficiently differentiable function of L_2. Pansions of polynomial growth are defined as pansions of the form $\phi = [H]^n f_1$ where $n \geqq 0$, $f_1 \in L_2$. The class of these pansions contains all functions of at most polynomial growth and it is closed under Fourier transformation: $[T][H]^n f_1 = [H]^n g_1$ where $g_1 = [T]f_1$ is also in L_2. The class is also closed under the operations of global differentiation $[D]$, global multiplication by $[x]$ and translation $[e^{aD}]$, a real. Every pansion of polynomial growth is a finite order global derivative of a function of at most polynomial growth. Hence these pansions correspond to Laurent Schwartz's tempered distributions. It is finally noted that for a function of polynomial growth $[T]f$ is essentially equal to the global *kth* derivative of Bochner's "*k*-Transformierte" of f. (Received June 17, 1957.)

712. Jacob Korevaar: *Fourier transform theory for pansions of exponential growth.*

Using global properties of the solutions of the differential equation $(H+p)y = (x^2+p-D^2)y = f$ $(p \geqq 0)$ it is shown that every function of at most exponential growth on $(-\infty, \infty)$ can be represented in the form $E_0(H)f_0$. Here $E_0(s^2) = s^2\prod(1+s^2/p_j)$ is a special even entire function of exponential type and f_0 an infinitely differentiable function of L_2. Pansions of exponential growth are defined as pansions of the form $\phi = [E(H)]f_1$ where $E(s^2)$ is any even entire function of exponential type and $f_1 \in L_2$. The class of these pansions contains all functions of at most exponential growth and is closed under Fourier transformation as well as the operations $[G(D)]$ and $[G(x)]$ for every entire function G of exponential type. Every pansion of exponential growth is an infinite order global derivative of exponential type of a function of at most exponential growth. The proof depends on explicit solution of the equation $\phi = [\cos aD]f$. For sufficiently large $a > 0$ (depending on the type of ϕ) the equation possesses a function solution f of at most exponential growth. (Received June 17, 1957.)

713. Stephen Kulik: *A method for approximating the imaginary roots of analytic equations.*

Let $f(s) = 0$ be an analytic equation; $D_n = \sum_{i=0}^{n-2} [-f(s)]^i f(s)^{(1+i)} D_{n-1-i}/(1+i)! + [-f(s)]^{n-1} f(s)^{(n)}/(n-1)!$; $Q_n = \sum_{i=0}^{n-1} (i)^n (u-s)^{n-i} f^i(s) D_{n-i}$. Then, if u and s are given arbitrary values such that $|(u-a_1)/(s-a_1)| > |(u-a_i)/(s-a_i)|$, where a_1 and a_i, $i = 2, 3, \cdots$, are the roots of the equation, $(u-a_1)/(s-a_1) = \lim Q_n/f(s)Q_{n-1}$, as $n \to \infty$. If $s = x$ is real, the two real roots adjacent to x can be approximated by using Q_n with two different appropriately chosen real values of u. If $s = x$ and $u = x+it$, where x and $t > 0$ are real, and the second of two roots, $a+bi$ and $a-bi$, satisfies the above condition, then a and b are found by separating the real and imaginary parts in $Q_n = A_n + iB_n$, namely, $x - a = tf(x) \lim \{B_{n-1}[f(x)A_{n-1} - A_n] - A_{n-1}[f(x)B_{n-1} - B_n]\} \cdot \{[f(x)A_{n-1} - A_n]^2 + [f(x)B_{n-1} - B_n]^2\}^{-1}$ and $b = -tf(x) \lim \{A_{n-1}[f(x)A_{n-1} - A_n] + B_{n-1}[f(x)B_{n-1} - B_n]\} / \{[f(x)A_{n-1} - A_n]^2 + [f(x)B_{n-1} - B_n]^2\}$, as $n \to \infty$. The rule for the selection of u and t values in these two cases is simple. (Received June 19, 1957.)

714. R. A. Kunze: *An operator theoretic approach to generalized Fourier transforms.*

An intrinsic generalization of the Fourier transform is given, based upon the Plancherel theorem and the spectral theorem for normal operators, involving no special kernels or methods of summation, that reduces to the classical transform in the case of L_p $(1 \leqq p \leqq 2)$ and at the same time considerably extends the class of functions having transforms. The Fourier transform of a function is, in this approach, again a measurable function, not a distribution or generalized function, and is characterized by its properties as an unbounded operator in L_2. Specifically, it is shown that corresponding to any measurable function f such that a suitable restriction L_f' of the operation of convolution by f is essentially normal, there exists a unique measurable function F (the generalized Fourier transform of f) with the property that the closure L_f of L_f' is unitarily equivalent via the Plancherel transform to multiplication by F. A sufficient condition that L_f' be essentially normal is the existence of a spanning set of functions with compact supports in the domain of L_f'. Many of the functions such as the Heaviside unit function and others commonly used in electrical engineering are shown to have Fourier transforms in this sense. (Received July 11, 1957.)

715*t*. Joseph Lehner and G. M. Wing: *On Weyl's inequality for Eigenvalues.*

It is proved that if K and Q are compact symmetric operators on a Hilbert space to itself, and Q is positive definite, then $\lambda_n(K+Q) > \lambda_n(K)$, $n = 1, 2, \cdots$. On the assumption that Q is positive semi-definite, Weyl proved the same statement with "$>$" replaced by "$\geqq$" (Göttinger Nachrichten (1911) pp. 110–117). The proof uses the Fischer form of the minimax principle (Monatshefte fur Mathematik und Physik vol. 16 (1905) pp. 234–249). (Received July 8, 1957.)

716. H. G. Loomis (p) and I. M. Sheffer: *Solution of some linear partial differential equations.*

Let $a_j(x) = a_j(x_1, \cdots, x_n)$ $(j = 1, \cdots, n)$ be of class C^1 in some region R, with $a_1, \cdots, a_n$ never simultaneously zero there; and define the operator (1) $\theta f = \sum_1^n a_j(x)\partial f/\partial x_j$. Let $\phi(x)$ be a fixed solution of $\theta\phi = 1$. To each $f(x) \in C^1$ in R, associate the function $F(x, t) = F(x_1, \cdots, x_n, t)$ satisfying the Cauchy problem (2) $\theta F = 0$, $F(x_1, \cdots, x_n, \phi) = f(x_1, \cdots, x_n)$. Denote this correspondence by $F \leftrightarrow f$. If a_j, f belong to class $C^{(n)}$ then so does F. Let $d(x)$, $b_j(x) \in C^1$ $(j = 0, 1, \cdots, s-1)$. Every solution of (3) $\theta^s f + \sum_0^{s-1} b_j(x)\theta^j f = d(x)$ is the correspondent of some F satisfying the ordinary differential equation (4) $d^sF(x, t)/dt^s + \sum_0^{s-1} B_j(x, t)d^jF(x, t)/dt^j = D(x, t)$ where $D \leftrightarrow d$, $B_j \leftrightarrow b_j$; and the Cauchy problem for equation (3) with auxiliary conditions is solved by first solving (4) with suitable auxiliary conditions. The method is also applied to systems of equations in the operator θ. (Received July 11, 1957.)

717*t*. E. R. Lorch: *On integration theory.*

The Daniell program is extended, giving a theory of $\mathfrak{A}$-integrals (def. below) in structures that may be devoid of measure. Applications include rotation invariant integrals in infinite dimensional spaces. A projection theorem of the space of continuous functionals onto the manifold of $\mathfrak{A}$-integrals is given. Nec. and suff. conditions are given that a multiplicative functional F (extreme functional) be an $\mathfrak{A}$-integral.

These involve the relations of the topological structure of the neighborhoods of F in the space of maximal ideals and the set $\mathfrak{A}$. Definition of an $\mathfrak{A}$-integral: E is a space of points $x, y, \cdots$. $\mathfrak{B}$ is (for convenience) a Banach algebra of bounded functions $f(x)$ with uniform norm and containing a unit. $\mathfrak{A}$ is a family of sets $A \subset E$. A monotone sequence $\{f_n\}$ of positive functions converges *adequately* to zero, $f_n \downarrow 0$, if $f_n(x) \downarrow 0$ uniformly on each $A \in \mathfrak{A}$. If F is a continuous positive functional on $\mathfrak{B}$, then F is an $\mathfrak{A}$-integral if for every $\{f_n\}$ converging adequately to 0, $Ff_n \rightarrow 0$. The two extreme cases are: $\mathfrak{A}$ consists of all finite subsets of E (Lebesgue theory); $\mathfrak{A}$ consists of all subsets of E (Riemann theory). Intermediate cases give a spectrum of integrals. The $\mathfrak{A}$-integral F can be extended to the class of $\mathfrak{A}$-Baire functions and the standard Lebesgue convergence theorem can be established. Parts of this program have been extended by H. Gordon to topological Riesz spaces. (Received July 11, 1957.)

718t. E. H. Luchins: *On the concept of the strict radical.* Preliminary report.

Call the intersection of those maximal regular right ring (algebra) ideals of a ring (algebra) A which are also left ideals the *strict radical* of A. Call A *strictly semi-simple* (*sss*) if the strict radical, which contains the Brown-McCoy and Jacobson radicals, is the zero ideal. Examples: A strongly regular ring R (for every a in R, there exists x in R such that $a^2x = a$) is *sss*; Arens' BQ^*-algebras (Amer. J. Math. vol. 71 (1949) pp. 763–790) are *sss* Banach algebras. Let B be a real Banach algebra. Theorems: 1. The strict radical of B contains the set of topologically nilpotent elements of B and the Jacobson radical of any subalgebra of B. 2. B is *sss* if and only if it is algebraically isomorphic into some $C(X, Q)$, the Banach algebra of all continuous quaternion-valued functions vanishing at infinity on a locally compact Hausdorff space X. Corollary: Every subalgebra of a *sss* real Banach algebra is itself *sss*. (Received July 12, 1957.)

719t. E. H. Luchins: *On continuity of homomorphisms into Banach algebras.* Preliminary report.

Let B be a real Banach algebra. Call B *absolute* if an algebraic homomorphism into B of any real Banach algebra is necessarily continuous. An absolute Banach algebra has a unique Banach norm. Theorem 1: For B to be absolute it is necessary that zero be its only nilpotent element; it is sufficient that B be algebraically isomorphic into some $C(X, Q)$, the Banach algebra of all continuous quaternion-valued functions vanishing at infinity on a locally compact Hausdorff space X. An equivalent sufficient condition is that B be strictly semi-simple (see the preceding Abstract). Theorem 2: If B satisfies the descending chain condition on its right ring ideals, then B is absolute if and only if zero is its only nilpotent element. Rickart (Ann. of Math. vol. 51 (1950) pp. 615–628) has shown that if a Banach algebra is semi-simple and has a unique Banach norm, then any algebraic homomorphism of a Banach algebra *onto* B is necessarily continuous; the present report shows that the resulting proposition is false if the word *into* is substituted for *onto*. (Received July 12, 1957.)

720t. J. S. MacNerney: *Concerning quasi-harmonic operators.*

In a linear normed complete space S, let $QH(S)$ denote the class of *quasi-harmonic operators* over S (J. Elisha Mitchell Sci. Soc. vol. 71 (1955); Bull. Amer. Math. Soc. Abstract 61-1-99) and $Q\Phi(S)$ the corresponding class of *generators*, and let the established one-to-one correspondence between $QH(S)$ and $Q\Phi(S)$ be indicated by the functional notation $M = \mathcal{S}(F)$ for M in the former and F in the latter. THEOREM:

If each of F, G, and $F-G$ is in $Q\Phi(S)$ and $M=\mathcal{E}(F)$ and $N=\mathcal{E}(F-G)$, then the formula $K(u)=\int_0^u N(0, v)\cdot dG(v)\cdot N(v, 0)$ defines a member K of $Q\Phi(S)$ such that if $P=\mathcal{E}(K)$ then $M(s, t)=N(s, 0)P(s, t)N(0, t)$. Application is made to the problem of reducing systems of Stieltjes mean-integral equations to a canonical "second-order" form. Results will appear in J. Elisha Mitchell Sci. Soc. (Received June 3, 1957.)

721. Nathaniel Macon (p) and Abraham Spitzbart: *Inverses of Vandermonde matrices.*

Recently, the authors obtained explicit formulas for the derivatives of a polynomial $y=f(x)$ of degree n as a linear combination of its values $y_i=f(x_i)$ at $n+1$ points defined by $x_i=x_0+ih$, $(i=0, 1, \cdots, n)$, where the coefficients are expressed in terms of Stirling numbers (to appear in Amer. Math. Monthly). The results are applied here to obtain the elements of the inverse of the matrix whose element in the ith row and jth column is $(x_{j-1})^{i-1}$, $(i, j=1, 2, \cdots, n+1)$. The differentiation formulas mentioned above are then extended so as to apply to the case in which the x_i, $(i=0, 1, \cdots, n)$, are distinct but otherwise arbitrary, the coefficients of the y_i being expressed in terms of the elementary symmetric functions of certain subsets of the x_i. Finally, this result is used to obtain explicitly the inverse of the general Vandermonde matrix. This last derivation is an alternative to that one would obtain from the classical formula for the values of Vandermonde determinants with missing powers (cf. Polya and Szego, vol. 2, p. 99). (Received July 5, 1957.)

722. J. W. Neuberger: *Continuous products and nonlinear integral equations.*

This paper deals with the solution of integral equations in an abelian, normed, and complete space L, the points of which constitute S. Integral equations considered are of the form $Y(t)=A+\int_c^t dH\cdot Y$, where A is in S and H is a function from the numbers into the collection of all transformations from S into S. The continuous product from c to t of A with respect to H is defined as the limit of $\{\pi_R[I+\Delta H]\}A$ as the mesh of $R\rightarrow 0$. Under certain conditions on H, the continuous product from c to t exists for each t in some interval and defines the solution to the above integral equation. If L is Euclidean n-space, H can be chosen so that this integral equation is equivalent to a system of n first order nonlinear integral equations. A method of computation for such a system is inherent in the argument. (Received March 6, 1957.)

723*t*. J. C. C. Nitsche: *On the isolated singularities of solutions of* $\Delta u=e^u$.

Suppose that $u(x, y)$ is solution of an elliptic differential equation, defined and single valued in the neighbourhood of a point (x_0, y_0). Without further information one cannot, in general, predict the singular behaviour of $u(x, y)$ when approaching the point (x_0, y_0). There are, however, remarkable examples of nonlinear elliptic equations for which this is possible. In case of the minimal surface equation, for example, the singularity must be removable (see L. Bers, Ann. Math. vol. 53). Here the following theorem is proved: Let the function $u(x, y)$ be of class C^2, and single valued, and satisfy the equation $\Delta u=e^u$ in $0<x^2+y^2\leqq R^2$. Then either $u(x, y)$ is regular in $x^2+y^2\leqq R^2$ or it has one of the two forms (i) $u(x, y)=(k-1)\log(x^2+y^2)+\phi(x, y)$, $k>0$, (ii) $u(x, y)=-\log(x^2+y^2)-2\log|\log(x^2+y^2)|+\theta(x, y)$. The functions $\phi(x, y)$ and $\theta(x, y)$ are continuous in $x^2+y^2\leqq R^2$ and real analytic in $0<x^2+y^2\leqq R^2$

and $D\phi = O(r^{2k-1})$, $D^2\phi = O(r^{2k-2})$, $D\theta = O(r^{-1}|\log^{-2} r|)$, $D^2\theta = O(r^{-2}|\log^{-2} r|)$. Here $r = (x^2+y^2)^{1/2}$ and the symbols D and D^2 mean any first or second order derivatives, resp. (Received June 5, 1957.)

724*t*. J. C. C. Nitsche: *Discontinuities in the solutions of degenerate parabolic equations.*

Let $z(x, y)$ be solution of a second order hyperbolic equation. It is possible that $z(x, y)$ belongs to class C^{k-1} ($k \geqq 2$) but that its derivatives of order $\geqq k$ are only piecewise continuous (cf. Courant-Hilbert, Method. Math. Phys. II, p. 356ff). A curve across which jumps can occur must be a characteristic C. The magnitude θ of the discontinuities satisfies, along C, an ordinary differential equation of first order. This is, in general, linear. It may become nonlinear for $k = 2$ if the hyperbolic equation itself is nonlinear (cf. Nitsche, J. Rat. Mech. Anal. vol. 2). In the latter case discontinuities, in general, cannot remain bounded and the continuation of a solution across a characteristic, if discontinuities are admitted, may only be possible in a restricted domain. Author shows that this is a situation, also peculiar to certain degenerate parabolic equations (heat equation is not of this type). The equation $z_{xx} = F(x, y, z, p, q)$ is investigated under the assumption that the conic $\alpha^2 - \alpha \cdot F_p^0 - \beta \cdot F_q^0 = \text{const}$, taken for elements of the solution in points of the x-axis, degenerates into a couple of lines. The transport equation for the discontinuities has the form $\theta'' + a\theta' + b\theta + c\theta^2 = 0$. Here $c = 0$, however $c \neq 0$ for $k = 2$ if $F_{qq}^0 \neq 0$. (Received June 5, 1957.)

725*t*. R. R. Phelps: *Convex sets and closest points.*

If A is a subset of a normed linear space E and $x, y \in E$ we say that y is *point-wise closer* to A than is x provided $\|y-a\| < \|x-a\|$ for each $a \in A$. If x is such that no point of E is point-wise closer to A than is x, we call x a *closest point* to A. Fejér [Math. Ann. vol. 85 (1922) pp. 41–48] has noted that in the euclidean plane the set $C(A)$ of all closest points to A is precisely $K(A)$, the closed convex hull of A. This result is extended to complete inner product spaces. We say that a normed linear space E has *property* (F) if for each $A \subset E$, $C(A) \subset K(A)$. A normed linear space of dimension at least three has property (F) if and only if it is a complete inner product space. In two-dimensional normed linear spaces property (F) is equivalent to strict convexity. If A is a *bounded* subset of a strictly convex two-dimensional space then $C(A) = K(A)$. The chief tool is a slightly extended version of a theorem of R. C. James [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 559–566, Theorem 4]. (Received July 8, 1957.)

726. F. J. Polansky: *Completeness and approximation theorems for dilatations of entire functions.* Preliminary report.

(1) Let $a_n \neq 0$, $n = 0, 1, 2, \cdots$, and let $f(z) = \sum_0^\infty a_k z^k$ be any entire function. The problem investigated here is the completeness of the family $\{f(\lambda_n z)\}$. Here $\{\lambda_n\}$ is a discrete sequence of complex numbers, the completeness is with respect to the closed linear extension of the set $\{z^n\}_0^\infty$, the domain of completeness being circles in the z plane. It is shown that any sequence $\{\lambda_n\}$ with a single finite limit point is complete on every circle for any entire function. For the case $\lambda_n \to \infty$, sufficient conditions for completeness are obtained which can be characterized in terms of the MacLaurin series remainder of the function together with products involving the λ_k. (2) The method of (1) depends essentially on approximation of z^n by linear combinations of

the $\{f(\lambda_{ks})\}$. Approximation theorems for functions which are polynomial approximable on arbitrary point sets can be deduced from the method of (1). (Received August 9, 1957.)

727. Pasquale Porcelli: *Uniform completeness and closure of powers of reciprocals of linear functions.*

Let $\{k_p\}_{p=1}^{\infty}$ be set of complex numbers such that $k_p \neq 0$, $k_p \notin [-\infty, -1]$, $k_p \neq k_q$, $p \neq q$, $p, q = 1, 2, \cdots$, n a positive integer, K_n the set of functions $\{(1+k_pt)^{-n}\}_{p=1}^{\infty}$, $0 \leq t \leq 1$, $M(K_n)$ the closed linear manifold generated by K_n in $F[0, 1]$($F[0, 1]$ denotes the collection of all continuous functions on $[0, 1]$ to the complex numbers), and $C_q(K_n)$ the closed linear manifold generated by K_n in the Lebesgue space $L^q[0,1]$. Theorem: If each of m and n is a positive integer and $q \geq 1$, then each two of the following statements are equivalent: (1) $M(K_n) = F[0, 1]$, (2) $M(K_m) = F[0, 1]$, (3) $C_q(K_n) = L^q[0, 1]$, (4) $\sum_{p=1}^{\infty} (1 - |[(1+k_p)^{1/2}-1]/[(1+k_p)^{1/2}+1]|) = \infty$, (5) $1 \in M(K_n)$, (6) $1 \in C_q(K_n)$, (7) there exists a function h in $F[0, 1]$ such that $M(K_n)$ is dense in some neighborhood of h, and (8) there exists a function f in $L^q[0, 1]$ such that $C_q(K_n)$ is dense in some neighborhood of f. (Received July 12, 1957.)

728. R. T. Prosser: *On the general Schrodinger equation.*

This paper considers the General Schrodinger Equation $dU(t)/dt = i\lambda H(t)U(t)$, where $\{H(t)\}$ is a prescribed strongly continuous one-parameter family of operators defined on a suitable Hilbert space, and λ a positive constant. We assume (1) For each t, $H(t)$ is defined on a common dense domain D, (2) For each t, $H(t)$ admits a self-adjoint extension, (3) For each $\phi \in D$ there is a constant K such that $\|H(t_1)H(t_2) \cdots H(t_n)\phi\| \leq K^n\|\phi\|$, $t_1 \geq t_2 \geq \cdots \geq t_n$. Under these conditions we show that there exists an essentially unique solution U which is unitary and satisfies the usual composition relations. U is given by a power series in λ which strongly converges for all values of λ. Conditions (1) and (2) are necessary; condition (3) is automatic if the $H(t)$ are uniformly bounded, or mutually commute, but some form of (3) is necessary in general. (Received July 12, 1957.)

729t. R. T. Prosser: *On the Schrodinger equation of quantum electrodynamics.*

This paper applies the results of the preceding abstract to the Schrodinger equation which arises in quantum electrodynamics, where $H(t)$ is the total interaction Hamiltonian in the interaction representation and λ the charge. Using the canonical particle representations for the electron and photon field operators, we show that this $H(t)$ fails to satisfy condition (2) of the preceding abstract, and hence that the Schrodinger equation admits no proper solution in this case. All of the nontrivial divergences of quantum electrodynamics may be traced to this fact. We show further that if a suitable form factor is introduced into $H(t)$, describing a finite space extension for all electrons, then the conditions of the preceding abstract are all satisfied, and the Schrodinger equation admits a proper solution. *We conclude that the divergence difficulties of the present theories are due not to the mathematical methods employed but rather to the physical assumptions made involving the notion of a point particle.* (Received July 12, 1957.)

730. Walter Rudin: *On lacunary trigonometric series.*

Define $c_n(f) = (1/2\pi)\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$. If $1 \leq r < s < \infty$, let (r, s) denote the class of all sets E whose elements are integers and which have the following property: if $f \in L^r$

and if $c_n(f)=0$ for all integers n not in E, then $f\in L^s$. The following results are obtained: I. $(r, s)=(1, s)$. II. Suppose $1<p<\infty$, $1/p+1/q=1$. Then $E\in(1, p)$ if and only if there corresponds to every $f\in L^q$ a continuous function g such that $c_n(g)=c_n(f)$ for all $n\in E$. III. If $E\in(1, p)$ for some $p>1$, then E does not contain arbitrarily long arithmetic progressions. This disproves a conjecture of Sidon (Acta Szeged. vol. 10 (1943) p. 230) to the effect that $E\in(1, 2)$ if every term of E is positive and if no term of E is the sum of two terms of E. IV. If $E\in(1, q)$ for some $q>2$, then there is a constant A such that no arithmetic progression of n terms contains more than $A\cdot n^{2/q}$ terms of E. V. From IV it follows that there is an $E\in(1, 4)$ which is not in $(1, q)$ for any $q>4$. It is conjectured that $(1, r)$ is a proper subset of $(1, q)$ whenever $2<q<r$. V. If $2<q<r$ and if $(1, r)$ is a proper subset of $(1, q)$, then the automorphism group $\Gamma(L^r)$ of the convolution algebra L^r is a proper subgroup of $\Gamma(L^q)$. (Received July 3, 1957.)

731t. Walter Rudin: *Factorization in group algebras.*

The group algebra $L^1(G)$ of the locally compact abelian group G is the algebra of all complex-valued functions on G which are Haar integrable, with convolution as multiplication. Theorem: If G is abelian and locally euclidean, then every $f\in L^1(G)$ is the convolution of two members of $L^1(G)$. That is to say, unrestricted factorization is possible in $L^1(G)$; there are no primes. The following lemma is used to established the theorem if $G=R^n$ (n-dimensional euclidean space). Lemma: Choose an integer $p>(n+1)/2$. Suppose g is continuous on $0\leqq t<\infty$, $g(t)\to 0$ as $t\to\infty$, and $(-1)^k\Delta^k g(t)>0$ $(k=0, \cdots, p;\ 0<t<\infty)$, where $\Delta g=t^{-1}(dg/dt)$, $\Delta^k g=\Delta(\Delta^{k-1}g)$. Define h on R^n by: $h(x)=g(|x|)$, where $|x|$ is the distance from x to the origin. Then h is the Fourier transform of some $f\in L^1(R^n)$. Once the theorem is proved for $G=R^n$, the general case follows from known facts about the structure of locally euclidean abelian groups. (Received July 3, 1957.)

732t. Walter Rudin: *Discrete flows on compact spaces.*

Let $C(X)$ be the set of all continuous real functions on X. The following is a known consequence of the individual ergodic theorem: If X is a compact metric space and if h is a homeomorphism of X onto X, then there exists a point $p\in X$ at which the sequence $(1/n)\sum_1^n f(h^i(p))$ converges for every $f\in C(X)$, as $n\to\infty$. Jerison (Canadian J. Math. vol. 9 (1957) p. 83) conjectures that the same is true if X is the (nonmetric) compact space $\beta N-N$ (where N is the set of all positive integers, with the discrete topology, and βN is the Čech compactification of N) and if h is the homeomorphism of X induced by the mapping $k\to k+1$ of N into N. Since there are no finite orbits under this homeomorphism, the conjecture is disproved by the following theorem, which is the main result of the present paper: If $\{p_i\}$ is a simple countable sequence of points in βN, such that $p_i\neq p_j$ if $i\neq j$, then the sequence $(1/n)\sum_1^n f(p_i)$ fails to converge, as $n\to\infty$, for some $f\in C(\beta N)$. (Received July 3, 1957.)

733t. Diran Sarafyan: *Determination of singular points of first order ordinary differential equations through the use of integrating factors.*

In a previous work (Bull. Amer. Math. Soc. Abstract 63-4-529) (534th meeting of AMS) the author established results which can be crystallized in the following Theorem I: The exact differential equation $Mdx+Ndy=0$ has no singular points in a region R where M, N and $f(x, y)=-M/N$ are continuous. Outside of R they are determined by $1/M=0$ and/or $1/N=0$ on condition that at such points M and N

are continuous. Although this paper creates a new method for the determination of the singular points of $y' = f(x, y)$ through the use of its integrating factors, it may be considered as the extension and enlargement of the above mentioned work, and concludes with a generalized theorem which covers exact as well as non exact differential equations. Let $\lambda = \lambda'(x, y)$ be an integrating factor for $Mdx + Ndy = 0$. Since $\lambda Mdx + \lambda Ndy = 0$ is an exact differential equation, the application of the above Theorem I will give the following generalized Theorem II: The differential equation $Mdx + Ndy = 0$ has no singular points in a region R where λM, λN and $f(x, y) = -M/N$ are continuous, λ being an integrating factor. Outside of the region R the singular points are determined by $1/(\lambda M) = 0$ and/or $1/(\lambda N) = 0$, on condition that at such points λM and λN are continuous. (Received July 15, 1957.)

734t. J. C. Scanlon: *A note to Poincaré's perturbation method.*

Following K. O. Friedrichs' treatment [*Advanced ordinary differential equations* (mimeographed notes)], a study is made of the periodic solutions of a nonautonomous system of differential equations $dx/dt = F(x, t, \epsilon)$, where x and F are n-vectors and F satisfies appropriate differentiability and periodicity conditions, by using Poincaré's perturbation method. The results are then extended by using topological degree. The technique yields new existence theorems if the variational equation has at least two solutions, i.e., if the degree of degeneracy exceeds one. This is usually the case in problems of mechanical or electrical oscillations. (Received May 13, 1957.)

735. J. A. Schatz and Albert Wilansky (p): *Depressing subspaces of a conjugate Banach space.*

For a Banach space X, let L be a norm-closed proper linear subspace of X^* which is total over X i.e. $f(x) = 0$ for $f \in L$ implies $x = 0$. (Then X cannot be reflexive). Let $\|x\|_L$ be the norm of $x \in X^{**}$ as a function on L. (a) It can happen that $\|x\|_L < \|x\|$ for some $x \in X$. Then although L is weak * dense in X^*, $L \cap B$ is not weak * dense in B ($B =$ unit ball). (b) For some $y \in X^{**} \sim X$, $y^{\perp}$ has property (a); for others it does not. Call L depressing if $\|x\|_L$ is not equivalent to the norm of X i.e. given $\epsilon > 0$, $\|x\|_L < \epsilon \|x\|$ for some x. For $y \in X^{**}$, $y^{\perp}$ is never depressing. (Compare (b)). THEOREM. *L is depressing* iff *$L^{\perp} + X$ is not norm-closed in X^{**}*. Since $L^{\perp}$ and X are disjoint this means $L^{\perp}$ and X are "far from orthogonal" (Kober, Lorch). Examples of nonclosed $A + B$ are known. We give one in which $A = L^{\perp}$, $B = X$. Hence THEOREM. *There exists a depressing subspace.* (Received July 10, 1957.)

736. V. L. Shapiro: *Upper and lower divergence of vector fields.*

Using the notation $X = (x, y)$ and $C(X, t) =$ circumference of the circle with radius t and center X_0, and letting $V(X) = [A(X), B(X)]$ be a continuous vector field in the neighborhood of the point X_0, define $\text{div}^* V(X_0) = \limsup_{t \to 0} (\pi t^2)^{-1} \cdot \int_{C(X_0, t)} Ady - Bdx$. Define $\text{div}_* V(X_0)$ similarly using lim inf. If div $V_*(X_0) = \text{div}^* V(X_0)$ is finite, call this common value div $V(X_0)$. In this paper, the following theorem is proved: *Theorem.* Let D be a bounded domain in the plane and let E be a closed set of capacity zero contained in D. Suppose (i) $V(X) = [A(X), B(X)]$ is a continuous vector field defined in D. (ii) $\text{div}^* V(X)$ and $\text{div}_* V(X)$ are finite in $D - E$. (iii) $\text{div}_* V(X) \geq F(X)$ for X in D where $F(X)$ is Lebesgue integrable on D. Then div $V(X)$ is defined almost everywhere in D and is Lebesgue integrable on every closed subdomain of D. Furthermore if R is closed subdomain of D bounded by a simple closed rectifiable curve C, then $\int_C Ady - Bdx = \int_R \text{div } V(X) dX$. This theorem can be extended to n-

dimensional Euclidean space, $n \geqq 2$, if due consideration is paid to the boundary surface of R. (Received April 17, 1957.)

737t. V. L. Shapiro: *Negligible sets for the divergence theorem.*

Let D be a bounded domain in the plane with a simple closed rectifiable curve C as its boundary and let Z be a closed set of Lebesgue measure zero contained in D. With respect to this set Z, define a class of vector fields C_Z as follows: $V(X)$ is in C_Z if (1) $V(X)$ is continuous in $D+C-Z$. (2) $V(X)$ is in L^2 on D. (3) $\operatorname{div}_* V(X)$ and $\operatorname{div}^* V(X)$ are finite in $D-Z$. (4) div $V(X)$ exists almost everywhere in D and is in L^1 on D. (The notation of the preceding abstract *Upper and lower divergence of vector fields* is being used.) Z is then said to be a negligible set for the divergence theorem if V is in C_Z implies that the divergence theorem holds for V with respect to D, i.e. $\int_C A(X)dy-B(X)dx=\int_D \operatorname{div} V(X)dX$. The following theorem is then proved: A necessary and sufficient condition that Z be a negligible set for the divergence theorem is that Z be of logarithmic capacity zero. Condition (2) in the definition of class C_Z is necessary if one wants Z to be different from the empty set as the vector field $V(X)=\operatorname{grad}\log|X|$ illustrates. The above theorem can be extended to n-dimensional Euclidean space, $n \geqq 2$, if due consideration is paid to the boundary of the domain D. (Received July 5, 1957.)

738. Harold Shniad: *Bieberbach's conjecture for schlicht functions.*

For functions which are analytic and schlicht in the unit circle and which have the usual normalization, Bieberbach has conjectured that the nth coefficient is at most n. In the present paper this conjecture is verified up to and including $n=6$. The basic tools used are a combination of various types of variations on Loewner's function k and a representation theorem which characterizes functions whose real part is non-positive in the unit circle. The method can be used for higher values of n, but the calculations required would be of considerable length. (Received July 10, 1957.)

739. W. F. Stinespring: *An element of the enveloping algebra of a Lie group.*

Let G be a Lie group which is unimodular. Let $X_1, \cdots, X_n$ be any basis for the right invariant Lie algebra of G. Let $p=[n/2]+1$. Then $(1-X_1^2-\cdots-X_n^2)^p$ acting on $C_0^\infty(G)$, the infinitely differentiable functions with compact support on G, has an extension D to a self-adjoint operator on $L_2(G)$ that commutes with right translations such that D^{-1} is left convolution by a positive, positive-definite function in $L_1(G)$. The operator D^{-1} is integrable with respect to the dual gage space of G. Let P be any subrepresentation of the regular representation of G, and let t be the gage on the algebra of P induced by the dual gage of G. If f is in $C_0^\infty(G)$, then $P_f=\int f(x)P(x)dx$ is such $|t(P_f)| \leqq \|D^{-1}\|_1 \|Df\|_1$. Therefore, $\chi_P(f)=t(P_f)=\int (Df)(x)k(x)dx$ for some k in $L_\infty(G)$. In other words, the "character" χ_P of P is a $2p$-fold distribution derivative of a bounded measurable function. (Received July 12, 1957.)

740. I. H. Sublette: *On a question raised by J. Schur.* Preliminary report.

J. Schur studied the properties of power series $f_n(z)$ which are analytic and have a modulus not greater than unity in $|z|<1$ [*Journal fur Reine und Angewandte Mathematik*, vol. 147 (1917) pp. 205–232; also vol. 148 (1918) pp. 122–145]. By means of a special algorithm he generated an infinite sequence of bounded functions $\{f_n\}$ ($n=0, 1, 2, \cdots$). He showed that if the series $\sum f_n(0)$ converges absolutely, f_0 is continuous in the set $|z| \leqq 1$ and its upper bound there is less than unity. He raised, but did not answer, the following question: do the above conditions on f_0 imply the absolute convergence of the series $\sum f_n(0)$? A recent investigation into electrical network theory by the present author has produced the following results for the special case of rational f_0: (a) if $|f_0|$ is not equal to unity everywhere on $|z|=1$, $\lim_{n\to\infty} f_n(0)=0$, (b) if $|f_0|<1$ everywhere on $|z|=1$, the series $\sum f_n(0)$ converges absolutely. (Received June 20, 1957.)

741. A. H. Van Tuyl: *A property of some nonlinear sequence transformations.*

Let A_n be a convergent complex sequence such that $\Delta A_{n+1}/\Delta A_n = \sum_{i=0}^{\infty} a_i n^{-i}$, where $\Delta A_n = A_{n+1} - A_n$, $|a_0| < 1$, and the a_i are independent of n and not all zero for $i \geqq 1$. Let $e_k(A_n)$ be the kth order transformation of A_n defined by D. Shanks (Journ. Math. Phys. vol. 34 (1955) pp. 1–42), let $e_k^m(A_n) = e_k(e_k^{m-1}(A_n))$, $m=1, 2, \cdots$, with $e_k^0(A_n) = A_n$, and let $e_{k_1}^{m_1} e_{k_2}^{m_2}(A_n) = e_{k_1}^{m_1}(e_{k_2}^{m_2}(A_n))$. Then if A_n and B_n have the same limit, and $\Delta B_n/\Delta A_n = n^{-l} \sum_{i=0}^{\infty} c_i n^{-i}$, where l is an integer $\geqq 1$ and $c_0 \neq 0$, it follows that $B_n = e_{k_1}^{m_1} \cdots e_{k_r}^{m_r}(A_n) + \alpha_n(k_1, \cdots, k_r; m_1, \cdots, m_r) \Delta e_{k_1}^{m_1} \cdots e_{k_r}^{m_r}(A_n)$, where the k_i and m_i are any positive integers such that $m_1 k_1 + \cdots + m_r k_r = l$, and $\alpha_n(k_1, \cdots, k_r; m_1, \cdots, m_r)$ tends to a finite limit as $n \to \infty$. In general, two transformations of the form $e_{k_1}^{m_1} \cdots e_{k_r}^{m_r}(A_n)$, where $m_1 k_1 + \cdots + m_r k_r = l$, converge with the same order of rapidity when A_n is of the class considered. However, for special values of a_0, a_1, and k, the rates of convergence are different. Further transformations $e_{j,k}(A_n)$ are defined for $j=1, 2, \cdots$ which accelerate the convergence of certain sequences A_n for which $\Delta A_{n+1}/\Delta A_n = 1 - 1/n - 1/(n \log n) - \cdots - s/(n \log n \log_2 n \cdots \log_{j-1} n) + \sigma_n/n^s$, where $\operatorname{Re} s > 1$, $\log_i n$ is the i-times iterated logarithm of n, and σ_n tends to a finite limit as $n \to \infty$. Results similar to the preceding hold for a given $j \geqq 1$. (Received July 15, 1957.)

742*t*. John Wermer: *Rings of analytic functions.*

Let D be a finite region on a Riemann surface whose boundary is a simple closed analytic curve Γ. Denote by $R(D)$ the ring of all functions analytic on $D \cup \Gamma$. Let A be a subring of $R(D)$ which contains the constants. In order that A be uniformly dense in $R(D)$ on $D \cup \Gamma$ it is clearly necessary that: (i) A separates points on $D \cup \Gamma$, and (ii) If $p \in D$, there is some f_p in A which is locally schlicht at p. *Theorem*: Assume (i) and (ii) and also assume that there is some ϕ in A which is locally schlicht at each point of Γ. Then A is uniformly dense in $R(D)$ on $D \cup \Gamma$. (Received July 11, 1957.)

743*t*. John Wermer: *Polynomial approximation on curves.*

Let Γ be a simple closed Jordan curve in the space of n complex variables $Z_1 \cdots Z_n$. By the *hull* of Γ, $h(\Gamma)$, we mean the set of all points $(Z_1^0, \cdots, Z_n^0)$ such that for all polynomials P in n variables: $|P(Z_1^0, \cdots, Z_n^0)| \leqq \max |P(Z_1, \cdots Z_n)|$ taken over Γ. It is trivial that $h(\Gamma)$ is a compact set containing Γ. Also, it is easy to give examples of curves Γ with $h(\Gamma) = \Gamma$. *Theorem* 1: Assume Γ admits an analytic parametrisation $Z_i = \phi_i(u)$, $|u| = 1$, $i=1, \cdots n$, where each ϕ_i is analytic on the circle $|u|=1$. Assume also $\phi_i^1(u) \neq 0$ for $|u|=1$. Then $h(\Gamma) - \Gamma$ is either empty or is a complex analytic variety of one complex dimension with at most finitely many singularities. *Theorem* 2: Let T be a Jordan *arc* in the space of n variables and let T admit an analytic parametrisation $Z_i = \psi_i(t)$, $0 \leqq t \leqq 1$, $i=1, \cdots n$, and $\psi_1^1(t) \neq 0$, $0 \leqq t \leqq 1$. Then every continuous function on T is uniformly approximable on T by polynomials in $Z_1, \cdots, Z_n$. (Received July 11, 1957.)

744. Harold Widom: *Equations of Wiener-Hopf type.*

A general procedure is given for solving homogeneous integral equations of Wiener-Hopf type $(^*)\int_0^\infty k(x-y)f(y)dy=f(x)$, where the only growth restriction on the kernel is $k(x)\in L_1(-\infty, \infty)$. It is seen that the number and general nature of the solutions of $(^*)$ are determined by the behavior of $K(\xi)=\int_{-\infty}^{\infty}e^{i\xi x}k(x)dx$ near the zeros of $1-K(\xi)$. For example, if $1-K(\xi)$ has finitely many zeros α possessing multiplicities m_α, the solutions of $(^*)$ (under certain additional assumptions) are all of the form $\sum_\alpha Q_\alpha(x)e^{i\alpha x}+g(x)$ where each $Q_\alpha(x)$ is a polynomial of degree $<m_\alpha$ and $g(x)\in L_1(0, \infty)$; if $K(\xi)$ is real, there are exactly $\sum m_\alpha/2$ linearly independent solutions of this form. Again, if $K(\xi)$ is real, $1-K(\xi)=0$ at $\xi=0$ and nowhere else and $|\xi|^{-\sigma}(1-K(\xi))$ is bounded and bounded away from zero near $\xi=0$, then (again under certain additional assumptions) r independent solutions of $(^*)$ are obtained, where $r=\sigma/2$ if this is an integer and $r=[\sigma/2]+1$ otherwise. (Received July 2, 1957.)

745. Hidehiko Yamabe: *A kernel function of a parabolic differential equation.*

D denotes an open domain with smooth boundary in a d-dimensional euclidean R. Given two integrable functions ϕ and ψ over $R\times R$, $(\phi*\psi)(x, y)=\int_D\phi(x, s)\psi(s, y)ds$. $E_t(x, y)$ denotes the kernel function $(2(\pi t)^{1/2})^{-d}\exp(-|x-y|/4t)$ of an equation $\partial U/\partial t=\Delta U$. Set $(E_{t/n}*)^n(x, y)=(E_{t/n}*(n)*E_{t/n}(x, y)$. Then $\lim_n (E_{t/n}*)^n(x, y)=K(x, y; t)$ exists and coincides with the kernel function of an equation $\partial U/\partial t=\Delta U$ with 0 boundary. Here Δ denotes the ordinary Laplacian. (Received June 13, 1957.)

APPLIED MATHEMATICS

746t. R. W. Bass: *Stability of nonlinear control systems* (I).

Notations: Bull. Amer. Math. Soc. Abstracts 63-3-359 through 63-3-361. Let S represent a control system, which is "stable" when the rest point x_0 is globally asymptotically stable. A NASC for this is criterion (iv), Abstract 361. By strengthening (iv) we attain a sufficient criterion, easier to apply in specific cases than Lyapunov's Direct Method, and of wide usefulness. Let P' denote P transposed, and curl g, the skew-symmetrized Jacobian matrix. *Theorem.* S is *stable* if there exists a continuous matrix $P(x)$ such that in E^n (i) the tensor curl $P(x)f(x)=0$; (ii) $P(x)$ is positive definite for all $x\neq 0$; (iii) $P'(x)x\cdot f(x)\leq -\epsilon\|x\|$ for some $\epsilon>0$ and all $\|x\|\geq 1$. *Corollaries.* Routh-Hurwitz linear stability criteria; (b) criteria of Ayzerman, Erugin, Malkin; Barbasin, Simanov; and Cartwright for nonlinear systems of second, third, and fourth order; (c) general criteria (nth order, nonlinear) of Hahn, Zubov, Krasovskii, and Kalman; (d) all general criteria for controls given in the books of Lur'e and Letov; (e) (after modifications) stability criteria for "pre-endpoint motion" of discontinuous systems, such as studied by e.g. Flügge-Lotz, and André and Seibert. (Received July 12, 1957.)

747. E. K. Blum: *On the Runge-Kutta fourth-order method.*

Consider the system of differential equations, $y'=f(x, y)$, with initial value y_0 at x_0. The numerical solution of this initial value problem by the Runge-Kutta fourth order method takes the form, $y_{n+1}=y_n+(K_1+2K_2+2K_3+K_4)/6$, where $K_1=hf(x_n, y_n)$,

$K_2=hf(x_n+h/2,\ y_n+K_1/2)$, $K_3=hf(x_n+h/2,\ y_n+K_2/2)$, $K_4=hf(x_n+h,\ y_n+K_3)$. A variation of this method was proposed by S. Gill [Proc. Cambridge Philos. Soc. vol. 47 (1950) pp. 96–108]. The two advantages claimed for the Gill modification are: (1) In automatic computers, it requires $3K+B$ storage registers whereas the Runge-Kutta formulas require $4K+B$, where K is the order of the differential system; (2) The computation can be arranged in such a manner that rounding errors are reduced appreciably. In the present paper, it is shown that (1) the Runge-Kutta formulas can also be arranged so that the process requires $3K+B$ storage registers, and (2) the ingenious techniques employed by Gill to reduce rounding error can be applied to the rearranged Runge-Kutta formulas with equal effectiveness. All constants in this modification are rational, whereas in the Gill version there are irrational constants. (Received July 12, 1957.)

748. Ward Cheney and A. A. Goldstein (p): *A finite algorithm for the linear inequality and Tchebycheff problems.*

The problem of solving a system of linear inequalities and the Tchebycheff problem for an inconsistent system of linear equations reduce to the problem of minimizing (or making negative) the function $F(x)=\max_i\ [(A^i,\ x)-b_i]$ where $A^i\in E_n$, $x\in E_n$, $b\in E_m$, $i=1,\cdots,m$. An algorithm is given for accomplishing this, but a simpler version adequate for the usual case in practice will be outlined here. Define residuals $R^i(x)=(A^i,\ x)-b_i$. Let x^0 be a first approximation. Suppose that $F(x^0)=R^1(x^0)>R^i(x^0)$, $i\neq 1$. The minimum of F on the ray of steepest descent $x^0-\lambda A^1$ is attained at a point x^1 which induces two maximum residuals, say $R^1(x^1)$ and $R^2(x^1)$. The relation $R^1=R^2$ serves to eliminate a variable from the original system and the process may be repeated in the reduced system. After $n-1$ eliminations, $F(x)$ becomes a function of one variable in the reduced system. At a minimum point in this system the eliminated variables are restored and this results in a vector inducing $n+1$ equal maximum residuals in the original system. A method is given for determining a ray D of descent by solving an $n\times(n+1)$ system of inequalities. $F(x)$ is minimized on D and the entire process may begin anew. The set of solution vectors is described. (Received June 24, 1957.)

749*t*. Nathanael Coburn: *A class of steady supersonic rotational axial symmetric flows.*

In this paper, we study a class of rotational axial symmetric supersonic flows which possess a family of ∞^1 parallel planes as characteristic surfaces. By use of the theory of a previous paper [*Intrinsic form of the characteristic relations in the steady supersonic flow of a compressible fluid*, Quarterly of Applied Math., forthcoming], the basic equations are obtained for the case, $a_{33}=0$, of that paper. This case leads to a set of three partial differential equations for M, the Mach number, h_0, the stagnation enthalpy, and r, the radial distance, in terms of z, the distance along the axis of the flow, and another variable. Two of these equations are linear. For any proper choice of M as a function of h_0, r, the linear equations can be solved to determine h_0 as a function of z and r. The case $M=M(r)$ is treated in detail. (Received July 5, 1957.)

750*t*. Nathanael Coburn: *The characteristic relations in the non-steady flow of a compressible gas*. Preliminary report.

Since the Newtonian mechanics of continuous media is based on three space variables, with time playing the role of an independent parameter, it is very difficult to find the geometric relations between the velocity vector, the bicharacteristic and normal cones for the nonsteady flow of a compressible gas. Hence, the space-time metrics of special and general relativity are introduced and the theory of characteristic systems [for the steady case, see *Characteristic relations for the steady flow of a compressible fluid*, C. L. Dolph and N. Coburn, Proc. of the First Symposium of Applied Math., McGraw-Hill, New York, 1949, pp. 55–66] is studied. The partial differential equation satisfied by the characteristic manifolds is determined and it is shown that the velocity four-vector, a normal to a characteristic manifold, and the corresponding bicharacteristic are related in a similar manner to the steady case. The return to Newtonian mechanics is made via the usual first approximation. A complete characteristic system is yet to be found. (Received July 5, 1957.)

751*t*. R. T. Dames: *Stability and convergence for a numerical solution of the Goursat problem.*

This paper considers the notions of stability and convergence as applied to the numerical solution of systems of second order hyperbolic equations expressed in characteristic coordinates. Emphasis is placed on linear equations in two independent variables and boundary conditions that lead to the Goursat problem. The same methods of analysis however, are extended to the Cauchy problem and the initial-boundary value problem. Stability and convergence are defined for a class of finite difference problems and an equivalence theorem is derived which relates the two concepts. Sufficient conditions for stability and convergence are obtained for a particular explicit difference scheme. By a method similar to that used by Fritz John for parabolic equations (Comm. on Pure and Appl. Math. vol. 5 (1952) pp. 155–211), existence and uniqueness theorems are derived. This provides a systematic method for bounding the truncation error. Conditions for stability and convergence for quasi-linear systems are also given. (Received April 15, 1957.)

752. W. K. Ergen and J. A. Nohel (p): *Stability of the continuous medium reactor.*

Criteria have been obtained recently for the stability of various types homogeneous and heterogeneous reactors by giving sufficient conditions for the stability of equilibrium solutions of the system of nonlinear ordinary differential equations which describe, under suitable restrictions, the dynamics of the reactor (W. K. Ergen, H. J. Lipkin, J. A. Nohel, *Applications of Liapounov's second method in reactor dynamics*, to appear, J. M. P., 57). In the continuous medium reactor—a limiting case of the heterogeneous reactor—the dynamic equations consist of a nonlinear differential-integral equation and a diffusion type partial differential equation; here the methods developed previously cannot be applied directly. However, a generalization of these together with Welton's Criterion for stability (see W. K. Ergen, *Kinetics of the circulating-fuel nuclear reactor*, Journal of Applied Physics, June, 1954), does yield sufficient conditions for stability in this case. Unfortunately Welton's Criterion hinges on an assumption which though physically reasonable is yet to be rigorously justified. A proof of Welton's Criterion using this assumption is also included. (Received June 10, 1957.)

753. Abolghassem Ghaffari: *On some mathematical properties of wedge solutions.*

The notation and references are the same as in Bull. Amer. Math. Soc. Abstract 63-4-467. It was found that the solution for the stream function ψ of a steady, irrotational, subsonic compressible flow past a wedge is either: (I) for $m=(n+1)/a$ and $\tau<\tau_1$, $\psi=lq_1\sum_{n=0}^{\infty}(\psi_m(\tau)/\psi_m(\tau_1))\sin((n+1)/a)\theta$ or (II) for $m=n/a$ and $\tau>\tau_1$, $\psi=lq_1\sum_{n=0}^{\infty}(\psi_{-m}(\tau)/\psi_{-m}(\tau_1))\sin(n/a)\theta$, where $\psi_m(\tau)=\tau^{m/2}F(a_m, b_m, c_m; \tau)$, $\psi_{-m}(\tau)=\tau^{-m/2}F(a_m+1-c_m, b_m+1-c_m, 2-c_m; \tau)$ and $0\leqq\tau\leqq 1/6$. It is proved that in the subsonic interval, $0\leqq\tau\leqq 1/6$, $\psi_m(\tau)$ is a positive monotonically increasing function of τ, and $\psi_{-m}(\tau)$ is a positive decreasing function of τ. When τ exceeds 1/6, $\psi_m(\tau)$ is an oscillatory function of τ and its period of oscillation decreases as τ increases. For large m, Shen Yun (F.M. 973–9873) obtained an asymptotic expansion of $\psi_m(\tau)$ and also showed that $\psi_{-m}(\tau)$ is numerically smaller than its value at $\tau=1/6$. One can show that in the supersonic interval, $1/6\leqq\tau\leqq 1$, only the solution (II) is valid. (Received July 10, 1957.)

754. Seymour Ginsburg: *On the length of the smallest uniform experiment which distinguishes the terminal states of a machine.*

The terminology used is the same as in *Gedanken-experiments on sequential machines*, by E. F. Moore, appearing in the Annals of Mathematical Studies, No. 34; *Automata studies*. The problem considered here is to obtain estimates of the length of the smallest experiment (on a machine) which is independent of the unknown initial state and which allows us, by observing the outputs, to distinguish the terminal state. The estimates obtained depend, of course, on the assumptions placed on the machine. In general, the bounds derived are less than nk, n being the number of distinguishable states in the machine and k, the number of admissible initial states. For no really general class of machines is a best bound known. (Received June 13, 1957.)

755t. Peter Henrici: *Symmetric Runge-Kutta methods for integrating ordinary differential equations of the first order.*

The ordinary Runge-Kutta methods for integrating $y'=f(x, y)$ are of the form $y_{n+1}-y_n=h\Phi_n$, where Φ_n is a combination of explicitly calculable values of $f(x, y)$ which differs from $y_n'+(1/2!)hy_n''+\cdots$ by $O(h^p)$, p being the order of the method. The paper exhibits "symmetric" Runge-Kutta methods of the form $y_{n+1}-y_{n-1}=2h\psi_n$, where ψ_n approximates $y_n'+(1/3!)h^2y_n^{(3)}+\cdots$. For $p=2$ there results the midpoint rule, $\psi_n=f(x_n, y_n)$. All possible forms of ψ_n requiring four substitutions are given for $p=4$, and a two-parameter family of expressions ψ_n, requiring 12 substitutions, is constructed for $p=6$. Symmetric Runge-Kutta methods are only conditionally stable and are not self-starting, but there is the following advantage: If the starting values are judiciously chosen, the overall error (x fixed, $h\to 0$) satisfies $e(x)=\eta(x)h^p+O(h^{p+2})$, $\eta(x)$ being the solution of a well-defined differential equation. Hence Richardson's deferred approach to the limit improves the accuracy by two orders of magnitude. Extensions to systems of equations are also given. (Sponsored by the Office of Ordnance Research.) (Received July 5, 1957.)

756t. Peter Henrici: *On the error of a method of Hammer and Hollingsworth for integrating ordinary differential equations.*

P. C. Hammer and J. W. Hollingsworth (MTAC vol. 9 (1955) pp. 92–96) proposed a step-by-step method for integrating $y'=f(x, y)$ which is based on the two-point (fourth order) Gaussian quadrature formula. The error of the method after one step was implied to be (1) $h^5y^{(5)}\xi/4320$. It is now shown that the overall error (x fixed, $h\to 0$) is of the form (2) $e(x)=h^4\eta(x)+O(h^5)$. Here $\eta(x)$ is the initially vanishing solution of (3) $\eta'(x)=g(x)\eta(x)+\phi(x)$, where $g(x)=f_y(x, y(x))$ and $\phi(x)$ is defined in terms of derivatives of $f(x, y(x))$, $y(x)$ being the true solution. For $y' = y$, $y(0) = 1$ there results the overall error $\epsilon(x) = -(1/720)xe^x h^4 + O(h^5)$, that is, about six times the error that would be present if (1) were correct. [For the ordinary fourth-order Runge-Kutta method the error is $-(1/120)xe^x h^4 + O(h^5)$; for Simpson's rule, if properly started, $+(1/180)xe^x h^4 + O(h^5)$.] The full accuracy of the Gaussian formula is not realized because the values of $y(x)$ at the irrational abscissas are predicted with insufficient accuracy. Analogous results are proved for the application of the Hammer-Hollingsworth method to systems of equations. (Sponsored by the Office of Ordnance Research.) (Received July 5, 1957.)

757t. Peter Henrici: *Methods for integrating ordinary differential equations based on Gaussian quadrature.*

If a Gaussian quadrature formula involving N abscissas is used for evaluating $\int_{x_n}^{x_{n+1}} f(x, y(x))dx$ in a step-by-step procedure for solving $y' = f(x, y)$, and if the high accuracy of the formula is not to be spoiled, the values of $y(x)$ at the abscissas of integration must be predicted with an error of at most $O(h^{2N+1})$. For the one-point Gaussian formula (midpoint rule) a simplified Runge-Kutta formula can be used as a predictor. There results the formula $y_{n+1} - y_n = hf(x_n + h/2,\ y_n + (h/2)f(x_n + h/4,\ y_n + (h/4)f(x_n, y_n)))$. This formula is explicit, self-starting, and strongly stable; it requires 3 substitutions per step. The overall error is $\epsilon(x) = h^2\eta(x) + O(h^3)$, where $\eta(x)$ is given by (3) of preceding abstract with $\phi(x) = -(1/24)y^{(3)}(x)$ [as contrasted to $-(1/12)y^{(3)}$ for the trapezoidal rule and $-(1/6)y^{(3)}$ for $y_{n+1} - y_{n-1} = 2hf_n$]. Similar results are obtained for the two-point Gaussian rule used with the ordinary fourth order Runge-Kutta formula as a predictor. The error then satisfies (2) and (3) of preceding abstract with $\phi(x) = -(1/4320)y^{(5)}(x)$, at the expense of 9 substitutions per step. Example: For $y' = y$, $y(0) = 1$ the error is $\epsilon(x) = -(1/4320)xe^x h^4 + O(h^5)$. Methods and results concerning error generalize to systems of equations. (Sponsored by the Office of Ordnance Research.) (Received July 5, 1957.)

758t. Erwin Kreyszig: *On relaxation oscillations.*

The van der Pol equation can be generalized in different ways. Of particular interest are those generalized equations whose solutions can be investigated by special methods which yield more detailed knowledge than the general theory of relaxation oscillation does. The equation (1) $\ddot{q} - (\operatorname{sgn} \dot{q})\frac{1}{2}(a - b\dot{q}^2) + 2^{-1}cf(q) = 0$ is of this type; here $q(t)$ is the elongation, a, b, and c are positive constants. (1) can be transformed into (2) $dV/dq - (\operatorname{sgn} \dot{q})(a - bV) + cf(q) = 0$ where $V = \dot{q}^2$. Let $f(q)$ be non-negative for positive values of q and nonpositive for negative values of q. Then the oscillations tend to a limit cycle. If $f(q) = \sum_{n=0}^{m} c_n q^n$, the solution $V(q)$ of (2) can be represented in terms of elementary functions and the sequence of maximum amplitudes of the oscillation can be determined. In particular, if $f(q) = q^{2m+1}$, $m = 0, 1, \cdots$, the maximum amplitude of the limit cycle is the root of the equation $(\sum_{s=0}^{m}((2s+1)!)^{-1}x^{2s+1}) \cdot (k_m + \sum_{s=1}^{m}((2s)!)^{-1}x^{2s})^{-1} = \tanh x$, where $x = bq$ and k_m is a constant, depending on a, b, c, and m. If $f(q)$ satisfies the above condition and is not an odd function, then the positive and negative maximum amplitude of the limit cycle are different from each other. (Received March 28, 1957.)

759t. H. W. Kuhn: *A matrix game equivalent to the Transportation Problem.*

The Hitchcock-Koopmans Transportation Problem asks for the minimum of the linear form $\sum c_{ij}x_{ij}$ subject to $x_{ij} \geq 0$, $\sum_j x_{ij} = a_i$, $\sum_i x_{ij} = b_j$, where $i = 1, \cdots, m$,

$j=1, \cdots, n$ and we may assume $a_i>0$, $b_j>0$, and $\sum_i a_i=\sum_j b_j$. The following matrix game Γ is based on the same data: Player A chooses an index i from $1, \cdots, m$. Independently, Player B chooses, for each $j=1, \cdots, n$, an index $k(j)$ from $1, \cdots, m$. Player B pays Player A the amount $Ca_i+\sum_j b_j(c_{k(j),j}-(C\delta_{k(j),i})$ where C is a constant greater than $(m-1)\max_j \max_{k,i} (c_{kj}-c_{ij})$, $\delta_{ki}=1$ if $k=i$, and $\delta_{ki}=0$ if $k\neq i$. Theorem: The matrix game Γ is equivalent to the Transportation Problem. The n distributions $q_{ij}\geqq 0$, $\sum_i q_{ij}=1$ constitute an optimal (behavior) strategy for B if and only if $x_{ij}=b_jq_{ij}$ solve the Transportation Problem. The distribution $p_i\geqq 0$, $\sum_i p_i=1$ is an optimal (mixed) strategy for A if and only if $u_i=Cp_i$ and $v_j=\min_k (c_{kj}-u_k)$ solve a program dual to the Transportation Problem, namely: Maximize $\sum_i a_iu_i+\sum_j b_jv_j$ subject to $u_i+v_j\leqq c_{ij}$, $u_i\geqq 0$, and $\sum_i u_i=C$. Preliminary computational experience using fictitious play favors this game over other matrix game versions of the problem. (Received July 16, 1957.)

760. H. M. Lieberstein: *On a mixed problem for the Euler-Poisson-Darboux equation.*

Solution of the following boundary value problem is given here. Find a regular solution $u^{(k)}$, $-\infty<k<1$, of the two dimensional EPD equation $u^{(k)}_{xx}=u^{(k)}_{yy}+(k/y)u^{(k)}_y$, such that $u^{(k)}(x, 0)=f(x)$ and $u^{(k)}(x, x)=g(x)$. Here $f(x)$ and $g(x)$ are defined on $[0, a]$, $[0, b]$, respectively, have a specified number of continuous derivatives there and another specified number of vanishing derivatives at the origin, both numbers depending on k. This problem was solved earlier for $g=0$ and for somewhat more restrictive conditions on f by A. Weinstein [Summa Brasiliensis Mathematicae vol. 3, September, 1955]. Uniqueness considerations given by the author in another paper cover this problem. The general solutions of E. K. Blum [Duke Math. J. vol. 21, pp. 257–274] and certain Riemann-Liouville integrals are used to obtain solutions for $-2<k<1$, and solutions for all negative k are generated using the recursion, $u^{(k)}=(y/(k+1))u_y^{(k+2)}+u^{(k+2)}$, due to L. Payne. It is found that polynomial data yield polynomial solutions if and only if k is a negative even integer though polynomial solutions do exist for other values of k. A generalized Huyghens' principle is noted for $k=0$ only. (Received February 18, 1957.)

761. Walter Littman: *On Struik's gravity waves.*

Struik's proof for the existence of two dimensional periodic gravity waves in a canal of finite depth (Math. Ann. vol. 95 (1926) pp. 595–634) does not offer a convenient means of studying its own range of validity in terms of the physical variables involved. In particular, it is not clear that the interval of convergence of the series expansion (on which the proof is based) remains bounded away from zero as $k\rightarrow\infty$ (k = depth) thus making it impossible to obtain the case of infinite depth (treated earlier by Levi Civita) in the limit as $k\rightarrow\infty$. Replacing the power series expansion of Struik's proof by an iteration scheme (cf. Stoker, *Water waves*, Interscience Publishers, to appear, section 12.2) results in a somewhat simplified proof whose range of validity can be studied more easily, in terms of the physical variables. In particular the case of infinite depth is obtained in the limit as $k\rightarrow\infty$. Approximate formulas for various physical quantities are derived. This proof also extends to flows over certain periodic bottoms which are "approximately horizontal." (Received July 12, 1957.)

762. Mark Lotkin: *Integration of heat conduction equations for composite media.*

The quasi-linear parabolic differential equation $c^{(m)}\rho^{(m)}(\partial u/\partial t)=\partial/\partial x(k^{(m)}\partial u/\partial x)$,

$m = 1, 2, \cdots$, together with the accessory boundary, initial, and interface conditions governing the conduction of heat through composite media are approximated by a tri-diagonal system of finite difference equations. By considering the propagation of the local truncation error the recurrence relations for this error are established. Assuming the boundedness in the basic region of definition of certain derivatives of the heat function u up to the fourth order, the boundedness of the overall truncation error is assured, thus proving the convergence of the solution of the difference equation to that of the differential equation. The treatment is in some respects similar to that discussed by Douglas, Pacific J. Math. vol. 6 (1956). (Received May 13, 1957.)

763. Walter Noll: *Relativistic aspects in classical mechanics.*

Classical mechanics must nowadays be regarded as essentially identical with the mechanics of continuous media. A satisfactory axiomatic foundation of continuum mechanics has never been given, and an attempt in this direction is made here. Precise definitions in terms of vector measures of such notions as force, mutual force, contact force, body force, and external force are given. It turns out that a basic axiom is a principle of relativity, which states that the mutual action between bodies is independent of the frame of reference. The particular role of Galilean frames of reference disappears in an appropriate formulation of the theory. The principle of relativity has far reaching consequences in the form of invariance requirements for general constitutive equations, i.e. equations describing the mechanical properties of general classes of ideal materials. These general materials have been the object of much research recently. (Received July 5, 1957.)

764. R. W. Preisendorfer: *Radiative transfer and measure theory.*

It is shown that four *structure axioms* C_1, M_1, R_1, T_1, formulated in terms of modern measure theory can serve as a mathematical foundation for radiative transfer theory. C_1 defines a *carrier space*: a measure space with totally sigma-finite measure on a sigma-algebra: (X, S, ν). M_1 deals with a non-negative cone $\mathfrak{M}_\nu$ of ν-continuous measures on S (the *radiative measures*). R_1 and T_1 employ abstract variants of the major premise of Huygens' Principle: R_1 defines a family $\mathcal{R}_\phi$ (the *radiative process*) of linear transformations $R_{t,t'}$ on $\mathfrak{M}_\nu$ into $\mathfrak{M}_\nu$; T_1 defines a family $\mathcal{T}_\phi$ (the *transfer process*) of measure preserving transformations $T_{t,t'}$ on (X, S, ν) onto (X, S, ν). The pair $(\mathcal{R}_\phi, \mathcal{T}_\phi)$ is the *radiative transfer process*. Polarization is included in the formulations. Results: solution of the long-standing problem of Hopf; connections between r.t. and electromagnetic theory via generalized harmonic analysis; connections between r.t. and Mueller phenomenological optics algebra; subsumption of classical r.t. and neutron transport theories. Reference: R. Preisendorfer, *A mathematical foundation for radiative transfer theory*. Dissertation, University of California at Los Angeles, May, 1956. (Received July 2, 1957.)

765*t*. I. F. Ritter: *The optimum linear program.*

The theory of the Linear Programming problem—to maximize a linear form of many real variables constrained by linear inequalities—together with its dual minimum problem, is derived by most elementary algebra. On this is based a practical method of computing simultaneously the solutions of the dual problems. Running parallel with the well-known Simplex Method, the present algorithm employs a rectangular array that differs essentially from the Simplex Tableau, particularly by the explicit appearance of the edge vectors leading away from any vertex of the polyhedron of admissible vectors. Besides some simplification of the computational routine, advantages derived from using the new array are: The difficulties of the Simplex Method in securing an initial "basic feasible solution" and in dealing with "degeneracy" are avoided. A correction method is readily available for the reduction of rounding errors. If the problem is not too large, it can be solved under the additional constraint that the solution must consist of integers. (Received July 1, 1957.)

766t. H. E. Salzer: *Formulas for hyperosculatory interpolation, direct and inverse.*

A convenient new procedure for direct "hyperosculatory" interpolation for $f(x)=f(x_0+ph)=f$, when given $f(x_i)=f_i$, $f'(x_i)=f_i'$ and $f''(x_i)=f_i''$ at n equally spaced points $x_i=x_0+ih$, $i=-[(n-1)/2]$ to $[n/2]$, is derived from Hermite's $(3n-1)$th degree osculatory interpolation formula, for $n=2(1)7$, (i.e. up to 20th degree accuracy). Certain fixed auxiliary quantities a_i, b_i and c_i, which are independent of p, f_i, f_i' and f_i'', are tabulated exactly. The method is an extension of the author's earlier adaptation of the simple osculatory interpolation formula, employing both a decomposition and uniqueness property of Hermite's general formula. The remainder term indicates the vast increase in accuracy and permissible size of h, when compared with simple osculatory interpolation formulas which are the next most accurate. For inverse interpolation the coefficients of the first ten powers of $r=(f-f_0)/hf_0'$, for $n=2(1)5$, (i.e. up to 14th degree accuracy in the direct function) are given in terms of f_i, f_i' and f_i''. Hyperosculatory interpolation is specially suitable in (1) practical problems in astronautics involving rocket or missile flight, where the acceleration data, or f_i'' are available as well as position and velocity data, or f_i and f_i', and where (2) higher mathematical functions that are tabulated with their first derivatives, are solutions of simple second order differential equations, so that f_i'' is readily obtained. (Received June 20, 1957.)

767t. H. E. Salzer: *Toward a Gibbsian approach to the problems of growth and cancer*, I: *Growth potentials.*

Certain sections of Josiah Willard Gibbs' thermodynamics papers might be applicable to biological equilibrium and growth, normal or abnormal. Gibbs added terms $\sum \mu_i dm_i$ to the differential of the internal energy $d\epsilon = t\,d\eta - p\,dv$, ($t$ = temperature, p = pressure, η = entropy, v = volume) where $\mu_i = \partial\epsilon/\partial m_i$ is the potential of substance m_i, to provide for chemical as well as thermal and mechanical equilibrium. In this article a further generalization is suggested, to include biological equilibrium by adding to $d\epsilon$ terms of the form GdN, the variable N being the number of cells, where $G=\partial\epsilon/\partial N$ is a "growth potential" that measures exactly the resistance toward spontaneous growth. The function G, like μ_i, is intensive in nature (i.e. depends on intensive variables only) except for a conversion factor dM/dN, $M=\sum m_i$, affording possible insight into why incipient abnormal growth is often independent of the number of cells. Useful applications might follow from identities between $\partial G/\partial\eta$, $\partial G/\partial v$ or $\partial G/\partial m_i$ and $\partial t/\partial N$, $-\partial p/\partial N$ or $\partial\mu_i/\partial N$ respectively. The following new function is studied, $\bar{\zeta}=\zeta-GN$, a natural generalization of the Gibbs free energy function ζ, the possibility of measuring it electrically, and comparison of its role with that of ζ for the possible experimental determination of G. (Received July 15, 1957.)

768*t*. H. E. Salzer: *Toward a Gibbsian approach to the problems of growth and cancer*, II: *Equilibrium, stability, pressure, surface tension.*

Gibbs' necessary and sufficient conditions for heterogeneous equilibrium of n components in m phases are generalized and also modified to include broader restraining conditions like $\sum_{i=1}^{m} \delta N_j^{(i)} \geq 0$, $j=1, \cdots, n$, the $>$ being characteristic of only living cellular phases. Careful appraisal of the term "biological stability" is followed by new criteria for stability, instability, and limits of stability (neutral equilibrium) in terms of derivatives of G, with possible medical applications. Three different sections of Gibbs' works tend to indicate that, for a biological phase, lower pressure usually increases its stability. The equation $p''-p'=\sigma(1/r+1/r')$, where $\sigma=$surface tension, p'', $p'=$pressures, r, $r'=$radii of curvature, is applied to possible control of tissue growth at interfaces. Methods of altering the equilibrium between three phases A, B, C by varying the interfacial tensions σ_{AB}, σ_{BC}, σ_{AC}, using relations like $\sigma_{AB}<\sigma_{AC}+\sigma_{BC}$ for stability of the A, B interface, suggest different means for shifting biological equilibrium between normal and abnormal cells through the introduction of new third phases at the interface. Various devices are mentioned for possible control of growth through proper channeling of surface or other equivalent forms of energy. (Received July 15, 1957.)

769. D. E. Spencer: *Exact and approximate solutions of integral equations.*

Approximate solutions of the Fredholm-type integral equations of interflection theory have previously been obtained (P. Moon and D. E. Spencer, J. Franklin Inst., vol. 242 (1946) p. 111; vol. 250 (1950) p. 151. Exponential approximations to the kernels were employed and exact solutions of the resulting approximate integral equations were obtained. A new method permits an exact solution to be obtained, by solving an auxiliary integral equation. The method is applied to the infinitely long cylindrical black body and to finite cylinders of various cross sections. A theoretical basis is thus provided for determining the accuracy of the earlier approximate solutions. (Received July 15, 1957.)

770*t*. R. L. Sternberg, J. S. Shipman, and L. S. Rose: *Multiple Fourier analysis in rectifier problems.*

The nonlinear problem of the multiple Fourier analysis of the output from a cut-off power law rectifier responding to a several-frequency input is reviewed for the one- and two-frequency problems and is briefly investigated for the three-frequency problem. The solutions for the modulation product amplitudes or multiple Fourier coefficients are obtained in exact although transcendental form. An account of the mathematical properties of these multiple Fourier coefficients or Bennett functions, including hypergeometric representations and power series expansions for them as well as recurrence relations satisfied by them, is given in the paper together with numerical tables of the first ten basic functions for the one-frequency problem and of the first fifteen basic functions for the two-frequency problem. Further applications of the theory are also given to the computation of average output power with the aid of the multiple Fourier coefficients or Bennett functions tabulated in the paper, and the work is concluded with some brief remarks concerning the interpretation of the results in terms of the theory of almost periodic functions and the generalized Fourier series of Bohr under appropriate conditions. The entire theory is based on the original method of the expansion of the rectifier output in multiple Fourier series introduced by Bennett in 1933 and 1947. (Received June 14, 1957.)

771. R. S. Varga: *The p-color problem: A generalization of the Young-Frankel successive overrelaxation scheme.*

The Young-Frankel method of successive overrelaxation, which has been shown by Young [Trans. Amer. Math. Soc. vol. 76, pp. 92–111] to correspond to a two color problem, is generalized to a method of successive overrelaxation for a p color problem, $p \geq 2$, the results holding for either point or line relaxation. The generalization, based on works of Romanovsky and Frobenius concerning non-negative matrices, makes use of the properties of cyclic matrices of index p. The fundamental equation relating eigenvalues of iteration matrices becomes $(\lambda+\omega-1)^p = \lambda^{p-1}\omega^p\mu^p$. An analogous formula for the optimum overrelaxation factor ω is derived, and an application is given to the solution of elliptic difference equations on a triangular mesh. (Received June 10, 1957.)

772. Paul Weiss: *On the evaluation of the general bialternant.*

Consider n functions $f_1(x), \cdots, f_n(x)$ of one independent variable x. Let x take on n distinct values $x_1, \cdots, x_n$. Form the general alternant of the $f_i(x_k)$, i.e. the determinant $|f_i(x_k)|$. Divide this determinant by the simple alternant $|x_k^{(i-1)}|$. The resulting bialternant can be expressed as a $(n \times n)$ determinant whose element in the ith row and kth column is the $(n-1)$st divided difference of the function $x^{n-k}f_i(x)$. This result reduces (1) to Jacobi's classical bialternant theorem if the functions $f_i(x)$ are monomials x^{r_i}, (2) to the expression of the $(n-1)$st divided difference as a quotient of two $(n \times n)$ determinants (see e.g. W. E. Milne: *Numerical calculus*, Princeton 1949, p. 206) if $f_i(x) = x^{(i-1)}$, $i = 1, 2, \cdots, n-1$; $f_n(x) = F(x)$; (3) by a limiting operation it leads to a general mean-value theorem due to Schwarz and Stieltjes (T. J. Stieltjes: Oeuvres, vol. 2, p. 110; G. Pólya und G. Szegö: Aufgaben und Lehrsatze II, ex. V, 95, p. 54). (Received July 11, 1957.)

GEOMETRY

773. D. C. Benson and L. E. Dubins (p): *On subsets of the unit sphere, which are sufficiently small to be subsets of some hemisphere.*

The authors show that if an arcwise connected subset of the surface of the unit sphere, S^2, has a diameter less than $2\pi/3$, then it is a subset of some open hemisphere of S^2. Furthermore $2\pi/3$ is best possible. (By the diameter of a subset A of S^2 is meant, as usual, the least upper bound of the distances between pairs of points of A where the distance is the length of the smaller great circular arc joining the pair of points.) (Received July 12, 1957.)

774*t*. D. W. Crowe: *On the geometry of quaternions.*

It is known that the plane $\pi(A)$ in real euclidean 4-space E^4 defined by the complex equation $w = Az$, $(w = u+iv$, $A = a+ib$, $z = x+iy)$ (i.e. by the two real equations $u = ax-by$, $v = bx+ay$) is isocline to the plane $\pi(0)$ at the origin. We consider the analogous properties of the 4-dimensional subspace $P(R)$ of E^8 defined by the quaternion equation $p = Rq$, p, R, q quaternions (i.e. by four real equations). It is shown that $P(R)$ is isocline to $P(0)$ at the origin and that the angle α between two such 4-spaces $P(R)$, $P(S)$ is given by $\cos^2 \alpha = |1+RS^*|^2/(1+RR^*)(1+SS^*)$ where R^*,

S^* denote the quaternion conjugates of R, S. This is the same formula as that for the angle between two planes in E^4 (with "quaternion conjugate" replaced by "complex conjugate"). This implies that the sphere model and stereographic projection property of isocline planes in E^4 (Bull. Amer. Math. Soc. Abstract 63-1-53) can be extended to the case of isocline 4-spaces in E^8. Similar results hold for 4-spaces defined by $p=Rq^*$. (Received July 1, 1957.)

775. Michael Goldberg: *Intermittent rotors.*

Rotors include noncircular shapes which may be rotated through all orientations in the plane while keeping contact with a number of fixed elements such as points, lines or circles. Since only three contacts are sufficient to constrain the motion, the rotors may be modified to remove some of the excess constraints. The rotor then makes contact, not continuously with all the fixed elements, but intermittently with different sets of the elements. Intermittent rotors derived in other ways are also shown. Of particular interest is the shape of least area which, when placed at random on a square lattice of points, always includes at least one of the points. This was shown by J. J. Schäffer [Math. Ann. vol. 129 (1955) pp. 265–273] and D. B. Sawyer [Quart. J. Math. vol. 4 (1953) pp. 284–292] to be a square to which has been added the areas enclosed by two parabolic arcs. This shape is an intermittent rotor which may be rotated while passing through at least three vertices of a fixed square. (Received July 3, 1957.)

776. V. G. Grove: *Closed convex surfaces.*

Let S be a closed, convex, orientable surface of class C'''. Let the second fundamental form of S be chosen as positive definite. Let F be a positive definite form on S whose coefficients $b_{\alpha\beta}$ are such that $b_{\alpha\beta,\gamma}-b_{\alpha\gamma,\beta}=0$. Such a form will be called trisymmetric on S. Let the discriminants of F and of the first and second fundamental forms be b, g and d respectively, and let $J=b/g$, $K=d/g$. Let $\overline{S}$ be a second closed, convex orientable surface of class C''', whose second fundamental form is positive definite. Let h be a differentiable homeomorphism of S into $\overline{S}$. Let $\overline{F}$ be the image of F under h, $\overline{F}$ being assumed trisymmetric on $\overline{S}$. Then if $\overline{J}=J$, $\overline{K}=K$ at corresponding points under h, and if h is orientation preserving then h is a rigid motion. (Received July 1, 1957.)

777. David Hertzig: *On simple algebraic groups.* Preliminary report.

Let G be a connected linear algebraic group defined over a field k; an algebraic group G' defined over k is a k-form of G split by K if there is an isomorphism between G and G' defined over an algebraic extension K of k. When K is a Galois extension of k with Galois group Γ the distinct k-forms of G split by K are in one-to-one correspondence with the 1-dimensional cohomology set $H^1(\Gamma, \mathrm{Aut}_K\, G)$ where $\mathrm{Aut}_K\, G$ is the group of automorphisms of G defined over K (particular case of a theorem of Weil). The following is derived from a theorem of Chevalley: Let G' be a simple linear algebraic group defined over k. Then there exists a group G defined over the prime field such that G' is a k-form of G; G is separably isogenous to a pseudo-classical or pseudo-exceptional group. The pseudo-classical and pseudo-exceptional groups are certain groups described by Dieudonné which are inseparably isogenous to classical and exceptional groups. Their Lie algebras are constructed, following Chevalley; the k-forms are then obtained, with a few exceptions. Due to the principle of triality new simple Lie algebras, hence new simple algebraic groups, are constructed as forms of the classical Lie algebra of type D_4. (Received July 15, 1957.)

778. C. C. Hsiung: *Curvature and Betti numbers of compact Riemannian manifolds with boundary.*

The purpose of this paper is to study harmonic and Killing tensor fields on compact orientable Riemannian manifolds M^n of dimension n ($\geqq 2$) and class C^3 with connected boundary B^{n-1} of dimension $n-1$. By using a duality theorem of Lefschetz and a theorem of Duff and Spencer (Ann. of Math. vol. 56 (1952), pp. 128–156), a typical result can be stated as follows: *On a M^n not homeomorphic to an n-dimensional submanifold, with an $(n-1)$-dimensional boundary, of an n-dimensional Euclidean space, if the Ricci curvature is positive definite everywhere; then there exists no harmonic vector field, other than the zero vector field, with zero tangential or normal component on B^{n-1}, and the absolute first Betti number of M^n and the relative first Betti number of M^n mod B^{n-1} are zero.* For the case of empty B^{n-1}, this result was proved first by S.B. Myers (Duke Math. J. vol. 8 (1941) pp. 401–404) by calculus of variations, and again by S. Bochner (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 776–797) by an entirely different method with the use of a result of E. Hopf on linear elliptic differential equations of the second order. In this paper, Bochner's method is modified by replacing the use of Hopf's result by constructing some differential forms and applying Stokes' theorem. (Received June 18, 1957.)

779. W. R. Hutcherson: *Oscillating inequality, Fibonacci sequence, and quadratic transformations.*

This paper gives a general proof of an inequality [J. C. Morelock, N. C. Perry, and W. R. Hutcherson, *Fibonacci sequence applied to quadratic transformations*, Bull. Amer. Math. Soc. Abstract 62-6-596] which oscillates in sense, as one uses the Fibonacci sequence related to quadratic transformations. (May 23, 1957.) (Received May 27, 1957.)

780. T. K. Pan: *Existence of surfaces in a certain conformal correspondence.*

Let v and $\bar{v}$ be corresponding unit vector fields associated with two conformal surfaces S and $\bar{S}$, respectively. Let the indicatric torsions of v and $\bar{v}$ along two corresponding curves C and $\bar{C}$ be equal (T. K. Pan, *Torsion of a vector field*, Proc. Amer. Math. Soc. vol. 7 (1956) p. 453). The existence of such surfaces is investigated. Theory of exterior differential system is applied for the study. It is found that the general solution depends on one arbitrary function of two variables. Singular solution, characteristic and Cauchy problem are discussed. (Received July 11, 1957.)

LOGIC AND FOUNDATIONS

781t. J. W. Addison: *Separation principles in the hierarchies of classical and effective descriptive set theory.*

Let $\mathfrak{N}$ be the product of n copies of the set N of natural numbers with the discrete topology and f copies of N^N under the product topology, where $n, f \in N$ and $n+f>0$. If $C \subseteq N^N$ the C-arithmetical (C-analytical) hierarchies on $\mathfrak{N}$ are those obtained by quantifying over N (over N^N) predicates on $\mathfrak{N}$ recursive in functions of C. Thus in particular the ϕ-arithmetical (ϕ-analytical) hierarchy is the arithmetical (analytic) hierarchy of recursive function theory (cf. Kleene, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 193–213) and the N^N-arithmetical (N^N-analytical) hierarchy is the finite Borel (projective) hierarchy of classical descriptive set theory. By a uniform method the reduction (1st and 2nd separation) principle(s) is (are) shown to hold (fail) on the $\exists$ and fail (hold) on the $\forall$ side of the C-arithmetical and C-analytical hierarchies with the exception of the 1st level of the C-analytical hierarchies where the situation is exactly reversed, a consequence of the axiom of constructibility being assumed in the proof for the 3rd and higher levels of the C-analytical hierarchies. Thus in particular we obtain the result announced without proof by Novikov (cf. Trudy Mat. Inst. Steklova vol. 38 (1951) pp. 279–316) that it is consistent with the axioms of set theory that for some k at the kth and higher levels of the projective hierarchy the separation principles behave as they do at the 2nd level (our result yielding further that k can be chosen as 3). (Received July 3, 1957.)

782. Ronald Harrop: *On the existence of finite models and decision procedures for propositional calculi.*

In this paper certain general properties of propositional calculi are considered. Two forms of the definition of a finite model of a propositional calculus with a finite number of axiom schemes and of rule schemes are discussed, these forms differing in the manner prescribed for the satisfaction of the rule schemes. In the first case it is insisted that when the premises of an application of such a scheme are valid then the conclusion of that application of the scheme must also be valid. In the second case the stronger condition is used that when the premises of a rule scheme take designated values for a certain substitution of values for the variables for arbitrary formulae in the scheme, then, for that substitution of values for the variables, the conclusion also takes a designated value. It is in general much easier to test for satisfaction of the second condition than of the first condition. The methods of definition are shown to be equivalent for the application of the finite model method for the demonstration of the unprovability of formulae in the calculus. It is further proved that, although the decidability of a calculus follows from the existence of a finite model counterexample for each unprovable formula of the calculus, the converse result is not true. (Received June 27, 1957.)

783t. S. C. Kleene: *Recursive functionals and quantifiers of finite types.* I.

Natural numbers are of *type* 0, functions $\alpha^{j+1}(\alpha^j)$ from type-j objects α^j to natural numbers of *type* $j+1$. The *primitive recursive* functions $\phi(\mathfrak{A})$, for $\mathfrak{A}$ a list of variables of these types, are those definable by eight schemata S1–S8, e.g.: S8 $\phi(\alpha^j, \mathfrak{a}) = \alpha^j(\lambda\alpha^{j-2}\chi(\alpha^j, \alpha^{j-2}, \mathfrak{a}))$. They have as *indices* numbers codifying the schema applications. *Partial recursive* functions $\phi(\mathfrak{A})$ are those definable by S1–S8 and: S9 $\phi(a, \mathfrak{B}, \mathfrak{C}) \cong \{a\}(\mathfrak{B})$ where $\{a\}(\mathfrak{B})$ is the value if defined of the function with index a for arguments $\mathfrak{B}$ (and S9 is included in the indexing). A completely defined partial recursive function is *general recursive*. *Partial* μ-, and μ-, *recursive* functions are obtained when S9 is replaced by: S10 $\phi(\mathfrak{A}) \cong \mu y[\chi(\mathfrak{A}, y) = 0]$. For $\mathfrak{A}$ of types $\leqq 1$, the notions coincide with the former ones. Otherwise, μ-recursiveness is narrower than general recursiveness. Indeed, let $\mathsf{E}(\alpha^1) = 0$ if $(Ea)[\alpha^1(a) = 0]$, $= 1$ otherwise; then for number variables $\mathfrak{a}$, the functions $\lambda\mathfrak{a}\, \phi(\mathfrak{a}, \mathsf{E})$ for $\phi(\mathfrak{a}, \alpha^2)$ μ- (general) recursive are exactly the arithmetical (hyperarithmetical) functions. The general recursive functions are closed under substitution of functionals $\lambda\alpha^j\theta(\mathfrak{A}, \alpha^j)$ (θ general recursive). For $\mathfrak{A}$ of types $\leqq r$, if ϕ is partial recursive with index z, $\phi(\mathfrak{A}) \cong w$ is expressible by either $(\beta^{r-1})(E\xi^{r-2})$ or $(E\beta^{r-1})(\xi^{r-2})$ prefixed to a primitive recursive predicate of z, etc., if $r > 2$, (but if $r = 2$, by only the first). (Received June 13, 1957.)

784*t*. Richard Montague: *An extension of Gödel's incompleteness theorem.*

For terminology see Tarski, Mostowski, Robinson, *Undecidable Theories*. If S, T are theories without common (nonlogical) constants and there is a one-place predicate belonging to neither, then A is said to be an *interpretation* of S in T if A is a set of possible definitions in T such that $S^{(\pi)} \subseteq T+A$, for some one-place predicate π which belongs to neither S nor T, and A contains at most one possible definition for each constant. The extension of this notion to arbitrary theories is obvious. Let P be Peano's arithmetic. Rosser's version of Gödel's theorem may be stated as follows: For any recursively axiomatizable and consistent theory T and any interpretation A of P in T, there is a sentence ϕ of P such that neither $\vdash_{T+A}\phi$ nor $\vdash_{T+A}\sim\phi$. Extension: For any recursively axiomatizable and consistent theory T, there is a sentence ϕ of P such that, for any interpretation A of P in T, neither $\vdash_{T+A}\phi$ nor $\vdash_{T+A}\sim\phi$; further, P may be replaced here by a certain finitely axiomatizable subtheory of P. Thus the choice of an undecidable sentence need not depend on the choice of definitions for the constants of arithmetic. (Received July 12, 1957.)

STATISTICS AND PROBABILITY

785. D. G. Austin and R. V. Chacon (p): *The continuity postulate for Markov processes—Abstract space case.* Preliminary report.

We show that a set of necessary and sufficient conditions for the continuity of the transition probabilities of an abstract space, continuous parameter Markov process (i.e. the transition probabilities $P_t(x, A)$ are defined for $t \in (0, +\infty)$, $x \in X$ and $A \in \mathcal{B}$ where $\mathcal{B}$ is a Borel field of subsets of the abstract space X) is: (i) for all x and $AP_\cdot(x, A)=0$ a.e.$\Rightarrow P_\cdot(x, A)=0$, (ii) for all x and $AP_\cdot(\cdot, A)$ is measurable $(0, +\infty)\times X$ and $P_\cdot(x, A)$ is measurable $(0, +\infty)$. We then use this result to show that the process is continuous under several modifications of Doob's continuity postulate, for all x, $\lim_{t\to 0} P_t(x, \{x\})=1$. (Received August 5, 1957.)

786. G. E. Baxter: *An operator identity.*

For any matrix N let N^+ be the matrix formed from N by setting equal to zero all elements of N which lie on or below the diagonal, and let $N^-=N-N^+$. Define sequences $\{P_k\}$ and $\{Q_k\}$ by $P_0=Q_0=I$, $P_{n+1}=(MP_n)^+$, and $Q_{n+1}=(Q_nM)^-$. If P and Q denote the generating functions (in s) of $\{P_k\}$ and $\{Q_k\}$, then (i) $P=I+s(MP)^+$, (ii) $Q=I+s(QM)^-$, (iii) $PQ=(I-sM)^{-1}$, and (iv) if $(M^k)^+M=M(M^k)^+$ for all $k=1, 2, \cdots$, then $P=\exp\left(\sum_1^\infty s^k(M^k)^+/k\right)$ and $Q=\exp\left(\sum_1^\infty s^k(M^k)^-/k\right)$. These results have application to probability theory. If M is the transition probability matrix of a Markov chain $\{X_k, k\geq 0\}$, with states $\alpha_1< \cdots <\alpha_K$, and if $({}_np_{ij})=P_n$ and $({}_nq_{ij})=Q_n$, then ${}_np_{ij}=P\{\max(X_0, \cdots, X_{n-1})<X_n, X_n=\alpha_j \mid X_0=\alpha_i\}$ and ${}_nq_{i,j}=P\{\max(X_0, \cdots, X_n)=X_0, X_n=\alpha_j \mid X_0=\alpha_i\}$. If (iv) holds, the P and Q form an operator analogue of Spitzer's identity (Trans. Amer. Math. Soc. vol. 82 (1956) pp. 323–339). These results have been extended to hold for certain integral operators of the form $Mf=\int_{-\infty}^{\infty} f(y)m(x; dy)$. (Received July 8, 1957.)

787*t*. A. T. Bharucha-Reid and L. W. Anderson: *Ergodic projections for periodic Markov chains and processes.*

This paper is concerned with the study of the ergodic properties of periodic Markov chains and processes. Let $P=(p_{ij})$ denote the matrix of transition probabilities associated with a temporally homogeneous Markov chain with a denumerable set of states. The Markov matrices satisfy the relation $P^{m+n}=P^mP^n$, and form a semi-group of operators $\mathfrak{S}=\{P_n: n=0, 1, \cdots\}$ on the Banach space l. We assume the Markov chain is periodic with period ω; hence $P^{n+\omega}=P^n$. Let $\pi^n=n^{-1}\sum_{\nu=1}^{n} P^{\nu-1}$. As $n\to\infty$ π^n converges to the operator $\pi(\omega)=\omega^{-1}(I+P+\cdots+P^{\omega-1})$, with $\pi P=P\pi=\pi$, and $\pi=\pi^2$. Now let $P(t)=(p_{ij}(t))$ denote the matrix of transition probabilities associated with a temporally homogeneous Markov process with a denumerable set of states. The Markov matrices satisfy the relations $P(0)=I$, $P(s+t)P(s)P(t)$, and form a semi-group of operators $\mathfrak{S}=\{P(t): t\geqq 0\}$ on l which has the properties $P(t)x\geqq 0$, and $\|P(t)x\|=\|x\|$ whenever $0\leqq x\in l$, and $\|P(t)x-x\|\to 0$ as $t\downarrow 0$, for each $x\in l$. For periodic processes we have $P(t+\omega)=P(t)$. Let $\pi(T)=T^{-1}\int_0^T P(t)dt$. As $T\to\infty$, $\pi(T)$ converges to the integral operator $\pi(\omega)=\omega^{-1}\int_0^\omega P(t)dt$. Fixed point theorems are also obtained. Examples, and applications to some problems in applied stochastic processes are given. (Received July 11, 1957.)

788t. I. B. Fleischer: *The entropy of an infinite probability space.*

Let $f(t)$ be a nonincreasing function on the unit interval. Define the entropy of a finite probability space as the expectation of $f(p)$ and that of an infinite space X as the supremum of entropies of the finite spaces obtained by restricting p to a finite Borel subfield of X. Theorem: let $\{A\}$ be the atoms of the measure algebra of X and N be their complement. Suppose $f(t)$ is $O(t^{-1})$ at zero. Then the entropy of X is $\sum p(A)f(p(A))+p(N)\lim_{t\to 0+}f(t)$. In particular, Shannon's logarithmic entropy can be finite only for a purely atomic space. This work was partly done while the author was employed at the Bell Telephone Laboratories. (Received July 10, 1957.)

789. H. M. Gurk: *Single-server, time-limited queues.*

Lindley (The Theory of Queues with a Single Server, Proc. Cambridge Phil. Soc. vol. 48 (1952) pp. 277–289) has solved the general case of a single server conservative queuing system with general arrival interval and service time distributions. His method provides the waiting time distribution of a customer (steady-state conditions assumed) by means of an integral equation. His results are extended here so as to cover the general case of time-limited queues, i.e., a customer will wait no more than a fixed time T_0. Assuming independent arrival intervals and service times, then if $A(t)$, $S(s)$, and $W(x)$ are respectively the arrival interval, service time, and waiting time distribution functions, then for $0\leqq x<T_0$, $W(x)=\int_0^\infty [S(x-t)+\int_{x-t}^\infty(1-A(s+t-x))dS(s)]dW(t)+\int_x^{T_0-}\int_0^\infty(1-A(s+t-x))dS(s)dW(t)+\int_0^{T_0-}\int_{T_0-t}^\infty I(s+t, T_0, x)dS(s)dW(t)$ where for $t>T_0 I(t, T_0, x)=\int_0^{t-T_0}[1-A(t-\xi-x)+I(t-\xi, T_0, x)]dA(\xi)$. If $S(s)$ is an exponential distribution, then the last double integral in the equation for $W(x)$ can be replaced by $+[1-W(T_0-)]\cdot\int_0^\infty(1-A(T_0+s-x))dS(s)$. $W(x)$ has been determined for several examples of $S(s)$ and $A(t)$. In particular the Poisson arrival, exponential service case is given. (Received July 8, 1957.)

790. J. G. Kemeny and J. L. Snell (p): *Semimartingales of Markov chains.*

An absorbing Markov chain is one that has a set of "boundary" states, where the process ends with probability 1. There is given a finite absorbing Markov chain, and non-negative values assigned to the elements of the boundary. An upper semimartingale is a function on the states of the chain, such that the expected value of

794. Edgar Reich: *Asymptotic properties of queueing times when the arrival epochs are arbitrary.*

Given the sequences $\{s_n\}$, $s_n \geq 0$ (the service periods), and $\{t_n\}$, $0 = t_1 \leq t_2 \leq \cdots$ (the arrival instants), one obtains the sequence of queueing times by the relations $q_0 = 0$, $q_{n+1} = \max\ (0,\ q_n + s_n + t_n - t_{n+1})$. The prime problem is to find asymptotic results for $\{q_n\}$ in terms of $\{t_n\}$ and $\{s_n\}$. Classical results of Lindley, and Kiefer-Wolfowitz deal with the case where $\{t_{n+1} - t_n\}$, $\{s_n\}$ are independent sequences of mutually independent equidistributed random variables. We study the case where the classical assumptions are relaxed, and $\{t_n\}$ and $\{s_n\}$ are arbitrary sequences of real numbers, for which only certain averages exist. The results yield additional information in the classical case, and may also be applied to situations where the classical assumptions are violated, e.g., when $\{t_n\}$ are the events of a Poisson process of variable density. Here one finds conditions under which (i) $Eq_n = O(1)$, and (ii) $q_n = O(n)$ w. p. 1. (Received June 14, 1957.)

795. J. M. Shapiro: *Sums of powers of independent random variables.*

For $n = 1, 2, \cdots$, let $S_n = x_{n1} + \cdots + x_{nk_n} - A_n$ be the sum of independent infinitesimal random variables minus suitably chosen constants A_n and let $F_n(x)$ be the distribution function of S_n. For $r > 1$ let $S_n^r = |x_{n1}|^r + \cdots + |x_{nk_n}|^r - B_n(r)$ where $B_n(r)$ are suitably chosen constants and let $F_n^r(x)$ be the distribution function of S_n^r. Let $F(x)$, $F^r(x)$ and $H(x)$ be distribution functions. Suppose (*) $\lim_{n\to\infty} F_n^r(x) = F^r(x)$ and $\lim_{r\to\infty} F^r(x) = H(x)$. Then it is shown that $H(x)$ is the distribution function of the sum of two independent random variables, one Gaussian and the other Poisson (including their degenerate cases). Other results are obtained. In particular let $F_n(x) \to F(x)$ and let $G(x)$ be the nondecreasing function determined by the Lévy-Khintchine formula for the characteristic function of (the infinitely divisible) distribution $F(x)$. Suppose $G(x)$ is nonincreasing for $x < -1$ and $x > 1$. Then if either (1) the random variables (x_{nk}) are symmetric, (2) (x_{nk}) are positive random variables and $A_n = 0$, or (3) for each n, $x_{n1}, \cdots, x_{nk_n}$ are identically distributed and $A_n = 0$, it follows that for suitably chosen constants $B_n(r)$, (*) holds where $H(x)$ is a Poisson distribution (or an improper distribution). (Received July 8, 1957.)

TOPOLOGY

796*t*. F. W. Anderson and R. L. Blair: *Characterizations of certain lattices of functions.*

Let K be a chain in its interval topology and let X be a compact Hausdorff space. A sublattice L of $C(X, K)$, the lattice of all K-valued continuous functions on X, is *characterizing* in case whenever $x \neq y$ in X and $f, g \in L$, there is an $h \in L$ such that $h(x) > f(x)$ and $h(y) < g(y)$. If K is conditionally complete and dense-in-itself, then a sublattice L of $C(X, K)$ is a *Q-sublattice* in case L contains a set Q of constant functions order-dense in K. Necessary and sufficient conditions are given in order that an abstract lattice be lattice-homomorphic to a characterizing Q-sublattice of $C(X, K)$ for some X and K. Also given are necessary and sufficient conditions that this homomorphism be an isomorphism. With certain further restrictions on X and K a characterization of $C(X, K)$ itself is obtained. In particular, a new solution of Birkhoff's Problem 81 [*Lattice theory*, rev. ed., 1948] is given. (Received July 11, 1956.)

the function after one step from any state is greater than or equal to the value of the function at the state. A martingale is a semimartingale for which equality holds. A representation theorem is developed for all non-negative semimartingales with the prescribed boundary values in terms of martingales of modified chains. A modified chain is one obtained by adding interior states to the boundary, and assigning zero to these. The representation is unique and leads to a simple geometric interpretation. With the exception of certain degenerate cases, the non-negative upper semimartingales form a convex cubic polyhedron. In degenerate cases the polyhedron reduces to smaller polyhedra of the same dimension, including a simplex in the most degenerate case. Certain extensions and applications are discussed. In particular, the representation theorem yields a representation of discrete subharmonic functions in terms of harmonic functions. (Received May 22, 1957.)

791*t*. J. G. Kemeny and J. L. Snell: *A fundamental matrix for regular Markov chains.*

Let P be transition matrix of a regular finite Markov chain. Let L be the limit of P^n. The matrix Z is defined as $(I-P+L)^{-1}$. Let a_j be the limiting probability of being in state j. Let s_{ij}^n be the mean number of times that the process is in state j, given that it starts in state i. It is shown that $s_{ij}^n-na_j$ approaches a limit, and this limit is the ij-entry of $Z-I$. It is also shown that other basic quantities for the Markov chain can be expressed simply in terms of Z. For example, if W is the matrix whose ij-entry is z_{jj}, and D the diagonal matrix with $1/a_j$ as jj-entry, then the mean number of steps to go from state i to state j is the ij-entry of $(I-Z+W)D$. A similar expression exists for the variance of this quantity. It is well known that the variance of the function giving the number of times that the process is in state j in the first n steps, normalized by $n^{1/2}$, approaches a limit independent of the starting state. It is shown that this limiting value is $a_j(2z_{jj}-a_j-1)$. (Received June 26, 1957.)

792*t*. D. J. Newman and M. S. Klamkin: *Expectations for sums of powers.*

Numbers are chosen from the interval (0, 1) in a random unbiased way, and the process is continued until the sum of their nth powers exceeds 1. The expected number of such choices is then determined. For $n=1$, the problem has been solved by W. Weissblum in which case the answer is e. In this note we solve the general problem in terms of definite integrals and find, thereby, the asymptotic behavior as $n\to\infty$. Heuristically, it might be reasoned, the "average" number is 1/2 and so the expected number should be around 2^n. This is not the case and in fact it turns out that the expected number E_n is asymptotic to ne^{γ}. (Received July 9, 1957.)

793*t*. Valdemars Punga: *Random sampling applied to the solutions of systems of linear equations.*

The paper discusses essentially the Monte Carlo type method devised by J. von Neumann and S. M. Ulam for the inversion of matrices, but slightly modifying and simplifying the method applies it directly to the solutions of systems of linear equations of the type: $x_i=a_{i1}x_1+a_{i2}x_2+\cdots, +a_{i,i-1}x_{i-1}+a_{i,i+1}x_{i+1}+\cdots, +a_{in}x_n+A_i$, $i=1, 2, \cdots, n$, where coefficients a_{ij} are positive and $a_{i1}+a_{i2}+\cdots, +a_{i,i-1}+a_{i,i+1}+\cdots, +a_{in}<1$. A_i can be any positive or negative numbers. (Received July 10, 1957.)

797. Steve Armentrout: *A study of certain plane-like spaces without the use of arcs.*

In this paper, a space T satisfying certain axioms, denoted by 0, I, II, III, IV, and V, is studied, and it is shown that certain propositions which hold in a space satisfying axioms 0 and 1–5 of R. L. Moore's *Foundations of point set theory* also hold in the space T. No argument in the development makes use of an arc. Axioms 0, I, and III are axioms 0, 1′(4) [Axiom 1′ with condition (5) omitted], and 3, respectively, of *Foundations*, and Axiom II is the following: If P is a point of a region R, there is a connected domain D, lying in R and containing P, such that if X is a point of D, there is a compact continuum lying in R and containing P and X. A closed chain of compact continua is a certain finite collection of compact continua; a J-set is a continuum which is the sum of the sets of such a collection. The J-sets are, in a certain sense, substitutes for simple closed curves. Axioms IV and V are statements about J-sets and their complementary domains, analogous to axioms 4 and 5 of *Foundations*. (Received July 5, 1957.)

798t. Joseph Auslander: *Mean-L-stable systems.*

Let X be a compact metric space with metric ρ and let T be a self-homeomorphism of X. The system (X, T) is called mean-L-stable provided that for every $\epsilon>0$ there exists $\delta>0$ such that $\rho(x, y)<\delta$ implies $\rho(xT^n, yT^n)<\epsilon$ for all n except a set of upper density less than ϵ. If δ can be chosen so that E is vacuous, (X, T) is called uniformly-L-stable. The points x and y are called proximal if $\inf_n \rho(xT^n, yT^n)=0$; otherwise they are called distal. It is shown that the set of distinct proximal pairs in a mean-L-stable system has invariant measure zero in $X \times X$. "Proximal" is an equivalence relation, and the quotient system is shown to be uniformly-L-stable. A minimal mean-L-stable system containing a point distal to all other points is locally almost periodic. The following decomposition theorem is obtained: $X=\bigcup_\alpha N_\alpha$, where each N_α is closed invariant and contains precisely one minimal set M_α. Every equivalence class of mutually proximal points is represented in just one M_α, and $\bigcup_\alpha M_\alpha$ has invariant measure one. (Received July 15, 1957.)

799. W. D. Barcus and J. P. Meyer (p): *On the suspension of the loop space.* Preliminary report.

Let X be a 1-connected topological space, Ω the space of loops on X, $S\Omega$ the suspension of Ω. Define a map p: $S\Omega \to X$ by $p(t, \omega)=\omega(t)$, $t \in I$, $\omega \in \Omega$; p is dual to the inclusion i: $X \subset \Omega SX$ used to study the homotopy suspension. Transform p into a fibre map by the standard method. Then it can be shown that the fibre has the (weak) homotopy type of the join $\Omega * \Omega$. The homology sequence of the fibering is closely related to G. W. Whitehead's sequence containing the homology suspension (Ann. of Math. vol. 62 (1955) p. 254), but has an extra term. One application is the following corollary, which extends Whitehead's result by one dimension: *A cohomology operation of type* $(n, q; \pi, G)$, $q \leqq 3n$, *is additive if and only if it is a suspension.* (Received July 12, 1957.)

800. R. H. Bing: *Conditions under which a surface in E^3 is tame.*

Suppose $X_0, X_1, X_2, \cdots$ is a sequence of homeomorphic sets in a metric space. Then $X_1, X_2, \cdots$ is said to converge homeomorphically to X_0 if there is a sequence of numbers $\epsilon_1, \epsilon_2, \cdots$ converging to 0 and a sequence of homeomorphisms $h_1, h_2, \cdots$ such that h_i takes X_i onto X_0 and moves no point by more than ϵ_i. Suppose S is a continuum (closed connected set) in E^3 which is a 2-manifold (locally topologically like E^2). Then E^3-S has two components D_1, D_2. It is shown that S is tame (there is a homeomorphism of E^3 onto itself taking S onto a polyhedron) if there is a sequence of 2-manifolds in D_1 converging homeomorphically to S and a sequence of 2-manifolds in D_2 converging homeomorphically to S. (Received July 9, 1957.)

801*t*. R. H. Bing: *E^3 does not contain uncountably many mutually exclusive wild surfaces.*

A surface S (closed set which is a 2-manifold) is wild in E^3 if there is no homeomorphism of E^3 onto itself that takes S onto a polyhedron (sum of a locally finite collection of simplexes). It is shown that if G is an uncountable collection of mutually exclusive surfaces in E^3, then not more than a countable number of elements of G are wild. This is proved as follows: If an element of G is wild, this element has a component that is wild because locally tame sets are tame. Hence if G contains uncountably many wild elements, there exists an uncountable collection G' of mutually exclusive wild connected surfaces. Some element of G' would contain sequences converging homeomorphically to it from both sides. But a surface is tame if there are sequences converging to it homeomorphically from both sides. (Received July 9, 1957.)

802*t*. William Browder: *Homology of loops of certain H-spaces.*

Let X be a simply connected H-space whose Pontrjagin ring $H_*(X)$ is an exterior algebra on odd dimensional generators $x_1, x_2, \cdots$, where $\dim x_i \leq \dim x_{i+1}$ and the number of x_i's such that $\dim x_i < n$ is finite for any n, (coefficients will always be assumed to be an integral domain or a field). Then in the spectral sequence of homology for the space of paths of X, E^n is an anti-commutative algebra (Hopf algebra for field coefficients) for $n \geq 2$, in particular $E^2 = H_*(\Omega) \otimes H_*(X)$ (see J. C. Moore, Boletin de la Sociedad Matematica Mexicana vol. 1 (1956) pp. 28–37). *Theorem*: $H_*(\Omega)$ is a polynomial ring generated by $y_1, y_2, \cdots$, with the property that $\sigma(y_i) = x_i$, where σ is the homology suspension associated with an acyclic fibre space. The theorem is first proved for fields, using the structure theorem of Borel for Hopf algebras over perfect fields (A. Borel, Ann. of Math. vol. 57 (1953) pp. 115–206). The result is then extended to integral domains using the homomorphism induced by the canonical homomorphism of the integral domain into its quotient field. (Received July 8, 1957.)

803*t*. Morton Brown: *Continuous collections of higher-dimensional hereditarily indecomposable continua.*

R. H. Bing has shown Trans. Amer. Math. Soc. vol. 71 (1951) pp. 267–273 that there exists in Euclidean N-space an $N-1$ dimensional hereditarily indecomposable continuum with exactly two complementary domains which separates Euclidean N-space irreducibly. The author shows that Euclidean N-space with the origin deleted is the sum of a continuous collection of such continua. (Received July 15, 1957.)

804. C. E. Burgess: *Chainable continua and indecomposability.*

Let M denote a compact connected metric space. If for each positive number ϵ there exists a finite sequence of open sets $D_1, D_2, \cdots, D_n$ covering M such that each D_i has a diameter less than ϵ and $D_i \cdot D_j \neq 0$ only if $|i-j| < 2$, then M is said to be *linearly chainable*. If this is modified so that $D_1 \cdot D_n \neq 0$, then M is said to be *circularly* *chainable*. In order that M should be either indecomposable or 2-indecomposable, it is sufficient that any one of the following four conditions be satisfied: (1) M is both circularly chainable and unicoherent; (2) M is circularly chainable and is irreducible between some two points; (3) M is circularly chainable and is not multicoherent; (4) every proper subcontinuum of M is circularly chainable. If M is linearly chainable, then in order that M should be circularly chainable, it is necessary and sufficient that M should be either indecomposable or 2-indecomposable. (Received July 10, 1957.)

805. Eugenio Calabi: *Fibre bundles techniques in the modification of complex analytic structure.* Preliminary report.

Let $Gr(p, q; C)$ denote the complex Grassmann manifold of p-dimensional linear subspaces V_p of a $(p+q)$-dimensional vector space V_{p+q} over the complex field C. The complement of a $Gr(n+1, n-1; C)$ in $Gr(n+1, n; C)$ $(n=1, 2, \cdots)$ is naturally a fibre bundle over $Gr(n, n; C)$ with a complex affine n-space A_n as fibre, and the group V_n of translations of A_n as structure group. The restriction B_n of this bundle to a suitable open, everywhere dense submanifold H_n of $Gr(n, n; C)$ is a trivial bundle in terms of its real analytic structure, but not in terms of its complex one. In fact, if Λ is a subgroup of V_n generated by $2n$ real-linearly independent translations, then Λ acts on B_n. Factoring Λ out, one obtains a $2n$-torus bundle over H_n; the restriction of the latter to a compact, complex submanifold K_n of H_n for $n \geqq 2$ yields a compact, complex manifold whose Chern classes are distinct from those of the product of K_n by a complex $2n$-toral group. The complex structure of other complex vector- and toral bundles can be modified by similar techniques. (Received July 10, 1957.)

806. M. L. Curtis: *A note on Kosiński's r-spaces.*

Following Kosiński (Fund. Math. vol. 43 (1955) 111–124) we call a point x in a space X an r-point if x has arbitrarily-small neighborhoods U such that for each $y \in U$ there is a deformation retraction of $\bar{U} - y$ onto $\bar{U} - U$. A space X is an r-space if it is finite dimensional, compact metric and each point is an r-point. Problem 7 in Kosiński's paper asks if $(a, b) \in A \times B$ being an r-point of $A \times B$ implies that a and b are r-points of A and B respectively. We answer this question in the negative by giving a 4-dimensional finite polyhedron P^4 which is not an r-space but is such that its cartesian product $P^4 \times S^1$ with a circle S^1 is an r-space. It is not known if $P^4 \times S^1$ is a topological manifold. (Received July 5, 1957.)

807. M. R. Demers and Herbert Federer (p): *On 2 dimensional Lebesgue area.* II.

Suppose f is any continuous map of a 2 dimensional finitely triangulable space into Euclidean n-space. It is proved that the Lebesgue area of f does not exceed the sum of the Lebesgue areas of the $C_{n,2}$ principal projections of f into 2-space. The argument uses a new result concerning the local structure of monotone images of finitely triangulable spaces, and a general theorem connecting the stable values of a map into k-space with the stable values of the sections of this map corresponding to a fibering of k-space into parallel $k-1$ dimensional planes. (Received May 21, 1957.)

808*t*. Edward Fadell: *On extending a theorem to Serre fiber spaces.*

The objective of this paper is to extend the following theorem (Hurewicz-Fadell, Proc. Nat. Acad. Sci. vol. 43 (1957) pp. 241–245) to fiber spaces in the sense of Serre. Let $\mathfrak{F} = \{X, B, p\}$ denote a fiber space with fiber F such that B is $(r-1)$-connected, $r>1$. Then in the associated spectral sequence $\{E_i, d_i\}$, $E_i = H(B, H(F))$ for

$2 \leq j \leq r$, $d_j = 0$ for $2 \leq j \leq r-1$, and the differential operator d_r is given by a cap product $d_r(h) = \gamma \cap h$, $h \in H(B, H(F))$, where γ is the characteristic cohomology class of B and $\pi_r(B)$ and $H(F)$ are suitably paired. In the above cited reference crucial use was made of continuous maps $\tilde{\lambda}: \tilde{B} \times F \rightarrow X$ ($\tilde{B}$ is the space of paths starting at the base point in B) and $\bar{\lambda}: X^I \rightarrow X^I$ induced by a lifting function λ for $\mathfrak{F}$. For Serre fiber spaces, lifting functions are no longer available and the extension of the above theorem requires the use of the Covering Homotopy Theorem to define, directly at the singular chain level, chain mappings analogous to those induced by the continuous maps $\tilde{\lambda}$ and $\bar{\lambda}$. (Received June 14, 1957.)

809*t*. R. H. Fox and J. W. Milnor: *Singularities of 2-spheres in 4-space and equivalence of knots.*

The boundary $\dot{\sigma}$ of a small 4-simplex σ around a point x of an oriented polyhedral surface F in oriented 4-space will intersect F in an oriented simple closed curve C. If C is knotted in the 3-sphere $\dot{\sigma}$ then x is a ***singular point*** and the knot type k of C is the ***singularity*** at x. Let k^{-1} denote the knot type obtained from k by reversing the orientation of C and taking its mirror image. Define k and l to be ***equivalent*** if there exists a polyhedral 2-sphere in 4-space having only two singular points, one of type k and one of type l^{-1}. Then the equivalence classes of knots form an abelian group G under the usual product operation. A collection $k_1, \cdots, k_n$ of knot types occurs as the collection of singularities of some 2-sphere iff the product $k_1 \cdots k_n$ is equivalent to the trivial knot. In order that k and l be equivalent it is necessary that the product of their Alexander polynomials have the form $a(t)a(1/t)$ for some integral polynomial $a(t)$. Consequently G is not finitely generated. G contains elements of order 2; it is conjectured that G also contains elements of order >2. (Received July 15, 1957.)

810. E. E. Grace: *The existence of cut sets in totally nonaposyndetic continua.*

Results of F. Burton Jones (Amer. J. Math. vol. 70 (1948) pp. 403–413) on the existence of weak cut points in certain nonaposyndetic compact metric continua are extended to a class of connected topological spaces which have a certain generalized completeness property. Stronger cutting properties are obtained and the use of an extension of the idea of nonaposyndesis yields weak cut sets in some cases where there are no cut points. (Received July 8, 1957.)

811*t*. Edward Halpern: *On the cohomology of certain loop spaces.*

Let J be a principal ideal domain. A ***monogenic twisted polynomial J-algebra*** of ***binomial type*** and ***height*** k, ($2 \leq k \leq \infty$), is free J-module generated by a sequence of elements $x_0, x_1, \cdots, x_{k-1}$ and multiplication defined by $x_m x_n = (m, n)x_{m+n}$ for $m+n<k$, ((m, n) is the binomial coefficient $(m+n)!/(m!n!)$), and $x_m x_n = 0$ for $m+n \geq k$; it is denoted by $J[x_1; k, (m, n)]$. Let X be a topological space with (singular) cohomology $H^*(X, J) = J[x]/(x^k)$, $2 \leq k \leq \infty$, with x of even degree d. Let Ω denote the space of loops at a base point of X. (a) If J has characteristic zero then $H^*(\Omega, J) \cong \Lambda_J(z_1) \otimes J[z_2; \infty, (m, n)]$, where $z_1, z_2 \in H^*(\Omega, J)$ have degrees $d-1$ and $kd-2$ respectively. (b) If J has characteristic $p \neq 0$ then $H^*(\Omega, J) \cong \Lambda_J(z_1) \otimes_{i \geq 0} J[z_{2p^i}; p, (m, n)]$, where $z_1, z_{2p^i} \in H^*(\Omega, J)$ have degrees $d-1$ and $p^i(kd-2)$ respectively, ($\otimes_{i \geq 0}$ denotes the "weak" tensor product). In particular, the theorem applies to complex and quaternionic projective n-spaces ($1 \leq n \leq \infty$) and the Cayley plane. (Received June 7, 1957.)

812. Melvin Henriksen (p) and J. R. Isbell: *Some properties of compactifications*. I.

A *compactification* of a space X is a compact space containing a dense subspace homeomorphic with X. (All spaces mentioned herein are assumed to be completely regular.) We are concerned with those properties $\mathcal{P}$ of spaces such that for all spaces X(*) *if the complement of X in one of its compactifications has $\mathcal{P}$, then the complement of X in any of its compactifications has $\mathcal{P}$.* These include compactness, local compactness, σ-compactness, countable compactness, the Lindelof property, paracompactness, and countable paracompactness. A continuous onto mapping $f\colon X \to Y$ that has a continuous extension over some compactification AX of X onto some compactification BY of Y that maps $AX - X$ homeomorphically onto $BY - Y$ is called a *meshing map*. A closed continuous onto mapping $f\colon X \to Y$ such that for each $y \in Y$, $f^{-1}(y)$ is compact is called a *fitting map*. A property $\mathcal{P}$ such that whenever $f\colon X \to Y$ is a meshing (resp. fitting) map, then X has $\mathcal{P}$ iff Y has $\mathcal{P}$ is called a *meshing* (resp. *fitting*), *property*. $\mathcal{P}$ satisfies (*) iff it is a meshing property. Every meshing map (resp. fitting property) is a fitting map (resp. meshing property), but not conversely. All of the properties listed above are actually fitting properties. (Received July 5, 1957.)

813*t*. Melvin Henriksen and J. R. Isbell: *Some properties of compactifications*. II.

(Terminology is as in the preceding abstract.) βX denotes the Stone-Čech compactification of the space X. A space X is said to have property $\mathcal{P}$ *at infinity* if $\beta X - X$ has property $\mathcal{P}$. If $\mathcal{P}$ is a meshing property, then so is $\mathcal{P}$ at infinity. $\mathcal{P}$ is a fitting property iff $\mathcal{P}$ at infinity is. Compactness at infinity is just local compactness. X is locally compact at infinity iff the set $R(X)$ of all points of X at which X fails to be locally compact is compact. In general, for any meshing property $\mathcal{P}$, X has property $\mathcal{P}$ at infinity iff $R(X)$ has property $\mathcal{P}$. X is σ-compact at infinity iff X is an absolute G_δ. X is Lindelöf at infinity iff every compact subset of X is contained in a compact set of countable character. Thus every metrizable space is Lindelöf at infinity. A countable product of spaces σ-compact (resp. Lindelöf) at infinity is σ-compact (resp. Lindelöf) at infinity. (Received July 5, 1957.)

814*t*. J. G. Horne, Jr.: *On O_w-ideals in $C(X)$*.

Let S be a set with relation R. An *R-ideal* is a subset $J \subset S$ satisfying (1) $f \in J$, $h \in S$ and hRf imply $h \in J$; (2) $f_1, f_2 \in J$ imply there exists $e \in J$ such that $f_i \, R \, e$ for $i = 1, 2$. The family of all proper R-ideals admits the dual Stone topology. Now suppose S is a commutative semigroup with identity. In previous notes (see Bull. Amer. Math. Soc. Abstract 63-1-140) we have considered O-ideals where $O = \{(f, e) : fe = f\}$. In this note we consider O_w-ideals where $O_w = \{(f, e) : f \in eS\}$. If S is the multiplicative semigroup of a ring with identity then the O_w-ideals are precisely the ring ideals which are O_w-directed. Let X be a topological space and let $C(X)$ denote the ring of real-valued continuous functions on X. THEOREM 1. *If $J \subset C(X)$ is an ideal such that $f \in J$ implies $|f|^{1/2} \in J$, then J is an O_w-ideal.* COR. 1. *The prime ideals and the m-closed ideals of $C(X)$ are O_w-ideals.* COR. 2. *The image under a multiplicative isomorphism of a prime or m-closed ideal is ditto.* THEOREM 2. *Every ideal of $C(X)$ is an O_w-ideal iff $x \in X$ and $f \in C(X)$ imply there is a neighborhood U of x such that either $f | U \geqq 0$ or $f | U \leqq 0$.* (See Bull. Amer. Math. Soc. Abstracts 60-4-462 and 61-2-183 of Gillman and Henriksen). (Received July 8, 1957.)

815t. J. G. Horne, Jr.: *On primary ideals in* $C(X)$.

We use the notation of the previous abstract. A primary ideal is an ideal which is contained in at most one maximal ideal. For $N \subset C(X)$ let $L(N) = \{f: fe = f \text{ for some } e \in N\}$. The collection of maximal ideals of $C(X)$ is denoted by $\mathfrak{N}$, the topological space consisting of $\mathfrak{N}$ and the dual Stone topology by $\mathfrak{N}_\Gamma$ (see above abstract). An ideal P is *prime-like* if the conditions $f \notin P$, $fe = f$ and $he \in P$ imply $h \in P$. If N is a prime-like ideal then $L(N)$ is a maximal O-ideal. Gillman and Henriksen proved that the maximal O-ideals and the prime ideals of $C(X)$ are primary (Trans. Amer. Math. Soc. vol. 77 (1956) p. 342). THEOREM 1. *For an ideal* $P \subset C(X)$ *these are equivalent*: (1) P *contains a maximal* O-*ideal*; (2) P *is a primary ideal*; (3) P *is a prime-like ideal*. An ideal $N \subset C(X)$ is *fixed* if there exists $x \in X$ such that $f(x) = 0$ for all $f \in N$. Suppose X is a completely regular G_δ-space. THEOREM 2. *The following conditions are equivalent for a maximal ideal* $N \subset C(X)$: (1) N *is fixed*; (2) N *is the union of its principal primary ideals*; (3) N *is an isolated point of* $\mathfrak{N}_\Gamma$; (4) $L(N)$ *is contained in a proper principal ideal*. N *is bounded* by f in the sense of Pursell (Pacific J. Math. vol. 5 (1955) p. 964) iff the principal ideal $[f]$ contains $L(N)$. (Received July 8, 1957.)

816. W. E. Malbon: *Invariants for quasi-compact mappings*.

A study is made of topological properties which are invariant under quasi-compact mappings where the image space is T_1 and has properties M and/or E. A T_1 space has property M provided every sequence converges to at most one point. A T_1 space has property E provided for any limit point x of a set K there exists a sequence S of infinitely many distinct points of K such that S converges to x. While Hausdorff spaces have property M and weakly separable spaces property E, bicompact T_1 spaces with properties M and E exist which are neither weakly separable nor Hausdorff. Examples of the results are: (1) for any compact metric space A each associated decomposition space A^* is compact and metric if and only if A^* has property M; (2) for any open (quasi-compact with compact, connected point inverses) mapping $f(A) = B$, the property of being locally compact, separable and metric is invariant if and only if B has property M; (3) for $f(A) = B$, A having properties M and E and B property E, f compact (in the limit point sense) is equivalent to f closed and $f^{-1}(b)$, b in B, compact (in the limit point sense). (Received July 3, 1957.)

817. Frank Raymond: *Duality theorem for homology manifolds*.

Let X be a compact, connected, finite dimensional, Hausdorff space. Let H_p denote the pth singular homology with coefficients in a group or field, G. If (1) for each x in X, $H_p(X, X-x) = 0$, $p \neq n$, and $= G$, for $p = n$, (2) for each x in X, $H_n(X) \to H_n(X, X-x) \to 0$ exact, (3) for each proper closed subset F of X, the injection of $H_n(F)$ into $H_n(X)$ is 0, then there exists an isomorphism of the Čech cohomology sequence, coefficients in G, A closed, $\to H^{n-p}(X, A) \to H^{n-p}(X) \to H^{n-p}(A) \to H^{n-p+1}(X, A) \to$ onto the singular homology sequence, coefficients in $G \to H_p(X-A) \to H_p(X) \to H^p(X, X-A) \to H_{p-1}(X-A) \to$ and the corresponding vertical diagram is commutative. This is essentially H. Cartan's duality theorem. It follows that: X is a Peano space, has the invariance of domain property, and interior and frontier points are characterized by dimension. X is a Cantor manifold, if n-dimensional. If X is n-dimensional, lc_1^{n-1}, field coefficients, then X is an orientable n-*gcm* in the sense of Wilder. If X is an LC_1^n (homotopy) orientable metric n-*gcm* (Wilder), then X is a homology manifold in the above sense. The nonorientable case can also be treated. A similar duality theorem can be obtained using Čech homology theory.

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818. J. H. Roberts: *Arcs of homeomorphisms.*

Suppose X is compact, metric, and of finite dimension $\leq n$. For each positive integer k let H_k be the space of all homeomorphisms of X into I_k, the k-dimensional interval. The topology for H_k is the metric topology given by the formula: $d(f, g) = \max d(f(x), g(x))$, xX. Theorem: *If $k \geq 2n+2$ then H_k is arc-wise connected, both locally and globally.* Simple examples show that the theorem is false for $k < 2n+2$. (Received July 10, 1957.)

819. R. H. Rosen: *On tree-like continua and irreducibility.*

The following theorems, which are generalizations of theorems proved by R. H. Bing for snake-like continua, are proved. I. Every monotone upper-semi-continuous decomposition of a compact metric tree-like continuum gives a compact metric tree-like continuum in the hyper-space. Definition: A tree-like continuum M is said to be k-branched if for each ϵ greater than 0, M can be covered by an ϵ-tree-chain with exactly k end links and k is the least such integer. II. Every compact metric tree-like continuum which is k-branched is irreducible about some set of k points. In order to prove II. the following result is used: III. If M is a compact metric nondegenerate continuum such that M contains a subcontinuum K such that M is not the sum of two proper subcontinua both of which intersect K, then there is a point q so that for any point p in K, M is irreducible from p to q. (Received May 22, 1957.)

820t. Walter Rudin: *Perfect subsets of compact spaces.*

If a compact Hausdorff space X contains a perfect subset P (i.e., P is compact, nonempty, and has no isolated point), then X contains a separable perfect set. (Equivalently, X contains a countable set which is dense in itself.) PROOF: There exists a continuous real function f on X which maps P onto the unit interval I, and there exists a compact subset E of P, minimal with respect to the property that $f(E) = I$. This E is perfect. For every rational $r \in I$, choose one $x_r \in E$ for which $f(x_r) = r$, and let S be the set of all x_r so chosen. Then S is countable, the minimality of E implies that E is the closure of S, so that E is separable. REMARK: Nevertheless, the set of all $x \in E$ such that $f(x) = t$ may have 2^c points, for every $t \in I$, where c denotes the power of the continuum. This happens, for instance, when X is the Čech compactification of a countable discrete space. (Received June 12, 1957.)

821. W. L. Strother (p) and L. E. Ward, Jr.: *Fixed points from neighborhood retracts.*

The space of subsets of an absolute retract is known to be an absolute retract and hence to have the fixed point property. This note establishes that the space of subsets of a connected neighborhood retract has the fixed point property. Whether this space of subsets is an absolute retract remains an open question. (Received July 3, 1957.)

822t. L. E. Ward, Jr.: *A fixed point theorem for multi-valued functions.*

A continuum ($=$compact connected Hausdorff space) is said to be *topologically chained* if each pair of distinct points is contained in a subcontinuum with exactly two non-cutpoints. The topologically chained and hereditarily unicoherent continua are characterized in terms of their inherent order properties. Such continua contain the generalized trees (and hence the dendrites) as a subclass. Using their order properties it is shown that each topologically chained, hereditarily unicoherent continuum has the fixed point property for continuous multi-valued functions. This is a type of Scherrer fixed point theorem which extends recent results of Wallace, Capel and Strother, Borsuk, Plunkett, and the author. (Received July 10, 1957.)

823. R. F. Williams: *Lebesgue area zero, dimension, zero, and fine-cyclic elements.*

If $f: X \to E_n$ is a map where X is compact, metric and 2-dimensional, then f can be uniformly approximated by "quasi-linear" maps $\phi = gh$, where h is a barycentric α-map of X onto a realized nerve $\alpha(X)$, and $g: \alpha(X) \to E_n$. Thus Lebesgue area, $L(f, X)$, can be defined. Theorem: $L(f, X) = 0$ if and only if, f can be uniformly approximated by maps f' such that either (1) $f'(X)$ is the sum of a finite number of line intervals, (2) f' can be factored through a space of dimension $\leqq 1$, or (3) f' is quasi-linear and the elementary area, $a(f, X)$, is zero. Theorem: if X is a continuous curve of finite degree of multi-coherence, then $L(f, X) = 0$ if and only if the middle space of the monotone, light factorization of f, is of dimension $\leqq 1$, or, equivantly, has no fine-cyclic element. This last generalizes a theorem of T. Rado, *Length and area*, p. 467. For definitions and properties of fine-cyclic elements, see L. Cesari [Bull. Amer. Math. Soc. Abstract 61-6-698] and C. Neugebauer [Bull. Amer. Math. Soc. Abstract 63-2-340]. (Received June 21, 1957.)

824. C. T. Yang: *On maps of spheres into euclidean spaces.*

In this note it is proved that given any map of an $(n+1)$-sphere into the euclidean n-space there exist three points of the sphere which are mapped into a single point and such that the spherical distance between any two of these points is equal to a preassigned number d, $0 < d \leqq 2\pi/3$. This result includes a theorem of Kakutani (Ann. of Math., 1942) as a special case. Moreover, this result can be generalized just as the well-known Borsuk-Ulam's theorem generalized by Hopf (Portugaliae Math., 1944). (Received July 10, 1957.)

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BOOK REVIEWS

Mechanik der Kontinua. By G. Hamel. Edited by I. Szabo. Stuttgart, Teubner, 1956. 210 pp. DM 29.70.

This is a time of great specialized compendia in the field of mechanics: we mention as examples the two-volume work of more than 1,000 pages on compressible fluid flow edited by Howarth, 1953, the 800 page work on the same subject edited by Sears, 1954, comprehensive monographs on partial topics of the theory of plasticity, and so on.

Here is a book written by a master in the field of mechanics which brings on 200 pages selected topics of the Mechanics of Continua. About half of the book, seven sections, deal with ideal fluid flow (compressible and incompressible), five sections with viscous flow, a last section is on deformable systems in general. The titles of the sections show the variety of subjects: 1. Fundamentals, 2. One-dimensional motion: sound and shock, 3. Potential flow without jets, 4. Plane potential flow with jets, 5. Rotational flow, 6. Two-dimensional steady compressible flow, 7. Potential flow. 8. The Navier Stokes equations, 9. Laminar motions, 10. Exact solutions, 11. Boundary layer theory, 12. Turbulence, 13. Elastic vibrations.

The editor tells us that Hamel's lectures on continuum mechanics were the ones most enjoyed by himself as well as by his students. We feel that the importance of G. Hamel as a great man in mechanics is not so much in this field as in that of the mechanics of a finite number of degrees of freedom, in stereomechanics, and in the fundamentals of mechanics. One can however understand the enthusiasm of students for a presentation which is interesting, original, concentrated without being too difficult, which on the firm basis of unfailing physical understanding handles with ease the necessary mathematics, —going again and again a few steps further than the usual presentations; we mention as one example the chapter on exact solutions of the Navier Stokes equations. On the other hand the author's derivation and discussion of the characteristics in compressible potential flow is almost confusing, and the presentation of plane deformation theory in plasticity seems complicated and not too clear.

These comments are however of minor importance. On the whole the book is stimulating to a high degree, rigourous without being pedantic and it has much to give to the student of mechanics on any level.

HILDA GEIRINGER

Fundamental concepts of algebra. By Claude Chevalley. New York, Academic Press, 1956. 8+241 pp. $6.80.

Chevalley has written a text-book, and his mathematical personality permeates every paragraph. Readers of his *Algebraic functions of one variable* who agreed at the time with André Weil's dictum "algebraic austerity can go no further" may decide that a counterexample has been produced. The book is tight, unified, direct, severe; relentlessly and uncompromisingly it pursues its ends: out of the simplest basic notions of algebra to build up with perfect precision the theory of the multilinear algebras of modules and to discuss those particular multilinear algebras which have found applications in topology and differential geometry. Group and ring theory are down to the irreducible minimum, field theory is completely absent; in their place, modules and their tensor products and the algebras one constructs from them: tensor, exterior, symmetric. The unity is monolithic. Gone is the discursive rambling of previous texts. This one marches unswerving and to its own music. It is presented by Chevalley as a serious effort to "adapt modern algebra teaching to present-day requirements"; since it represents thereby the first real departure in English from the van der Waerden tradition in first-year graduate algebra texts, it should be considered in some detail.

The general approach to the subject matter is that of Bourbaki's first three algebra chapters, but there are significant differences in content and treatment (Chevalley is often more general). As for the style, Bourbaki emerges from the comparison a warm, compassionate, and somewhat elderly gentleman.

The first chapter of the book is an Overture: in these first twenty pages are set forth the major themes of the entire book. They are devoted to the monoid (*née* semigroup with identity) and her cortege—submonoid, quotient monoid and homomorphism, product and free monoid. Two basic modern techniques appear and are emphasized from the very beginning here. One is the proof that A and B are isomorphic by constructing opposing maps f and g such that $f \circ g$ and $g \circ f$ are the identity. The other is the universal mapping property characterization, here used to define the free monoid and soon to be ubiquitous. This emphasis on mappings is one of the most characteristic features of the new algebra; the older "identifications" are now explicit natural mappings, and what earlier required brow-wrinkling now needs only diagram-chasing. So Chevalley appropriately puts in here two little diagrams and three little paragraphs explaining them. In view of the importance of diagrams to this sort of algebra, probably a few more should have been included later on as a suggestion to the novice how he might best follow some of the more complicated theorems.

Chapter two reviews the same concepts for a group, adding a discussion of homogeneous spaces and stability sets, a basic concept curiously ignored by algebra texts. The central chapter three defines a ring (with identity) and passes at once to the notion of a module M, an additive group with a ring R of operators. Insofar as possible M is general (not finitely generated or free) and R is not assumed commutative. Semi-simple modules are those in which every submodule is a direct summand; these are decomposed as usual. Hom (M, N) and $M \otimes N$ are introduced, then R is made commutative so multilinear maps can be discussed. The pages at this point start to look pretty gruesome, though the worst-looking lemmas turn out to be sheep in wolves' clothing—things like the general distributivity law for multilinear maps. The author's insistence on perfect precision of utterance make some rather intricate verbal pirouettes necessary to finally produce, say, the multinomial formula. Modules fly thick and fast, and one begins to understand what a poor Strasbourg goose whose liver is being fattened up for pâté must feel like. But after one more section of technique discussing the modules naturally derived from M by changing the ground ring R to a homomorphic image R', we can finally relax with vector spaces, matrices and their equivalence transformations, and linear maps. Here the insistence on unity both of methods and ideas means that certain classical topics—the basis theorem for a module when R is a principal ideal ring and the resulting canonical forms for a matrix, for example—cannot be included. The diagonalizable matrices are however accessible and are discussed; determinants, of course, must wait for the exterior algebra.

After a sidelong glance at the general (non-associative) algebra, we pass into the final and lengthiest section of the book: a treatment of four special associative algebras is given which provides a thorough working out of the basic techniques the author has been insisting on from the beginning. The pattern is somewhat the same for three of them, the tensor, exterior, and symmetric algebras: characterization by a universal mapping property, existence, and basic properties with respect to the module operations of chapter three. Everything of course is done as invariantly as possible, and extraordinary measures are occasionally adopted to avoid completely any ad hoc computation, no matter how simple.

The handling of the exterior and Grassmann algebras is interesting. Chevalley makes a distinction between them because the underlying module M is general. The exterior algebra $E(M)$ is the usual one universal with respect to maps ϕ such that $[\phi(m)]^2=0$, while the Grassmann algebra is defined to be the dual module $[E(M)]^*$, endowed with a multiplication defined by

$$\phi \wedge \psi = (\phi \otimes \psi) \circ U$$

where U is the map of $E \rightarrow E \otimes E$ induced by the diagonal map of $M \rightarrow M \times M$. There is a natural homomorphism of $E(M^*)$ into $[E(M)]^*$ which is an isomorphism if M is free with a finite basis so that the two algebras are isomorphic if M is in particular a finite dimensional vector space. This is the classical case, and Chevalley discusses it subsequently in some detail.

Another original feature of this last chapter is the treatment of the Pfaffian of a skew-symmetric bilinear form. Classically one takes a skew-symmetric matrix A and shows in a few lines by direct verification that its determinant is the square of another polynomial in the matrix entries, called the Pfaffian of A, $Pf(A)$. A longer and less explicit, but also less mysterious procedure is to use first an analogue of the Gram-Schmidt process to select a basis of the vector space (or free module), relative to which the matrix has a simple canonical form whose determinant is obviously a square. Then one takes the original skew-symmetric matrix to have independent transcendental entries, and the theorem follows easily [see, for example, Artin, *Geometric algebra*]. Chevalley proceeds as follows. The original skew-symmetric bilinear form γ on M gives as usual a map of $M \rightarrow M^*$; let $\gamma_x \in M^*$ be the image of $x \in M$. He shows that the linear form γ_x extends uniquely to a derivation d_x of degree -1 of the exterior algebra E over M. If we now let L_x denote left multiplication by x and define

$$\Omega(x_1 x_2 \cdots x_p) = (L_{x_1} + d_{x_1}) \circ \cdots \circ (L_{x_p} + d_{x_p}) \ [1],$$

then Ω extends linearly to be an automorphism of E. Now for $t \in E$, let $\epsilon(t)$ be the "constant" term of t, and let Γ be the natural extension of the bilinear form γ to $E \times E$. Then Chevalley proves that

$$\Gamma(t, t) = (\epsilon(\Omega(t)))^2,$$

so that if in particular M is a finite dimensional vector space, and t is the top-dimensional "volume element" of E, $e_1 e_2 \cdots e_n$, then the above relation is exactly

$$\det A = (Pf(\bar{A}))^2.$$

(The explicit polynomial expression for the Pfaffian is tucked away as a line in one of the exercises, and the above used as the definition.)

The chapter finishes with a discussion of the symmetric algebra on M. If in particular M is free, the symmetric algebra is just a polynomial ring, and this gives Chevalley another chance to use derivations; this time they lead somewhat less excitingly to Taylor's formula in n variables, with which the book concludes.

There are a number of exercises after each chapter—well over a hundred in all. Almost none of then are routine, nor are there many in the nature of specific examples; most of them are not easy. Many provide significant extensions and applications of the theory, and are the equivalent of many additional pages of text. Such, for instance, would be the exercises on the derived groups of a group, projective and injective modules, projective limits, quadratic forms and Clifford algebras, and representations of Lie algebras.

The importance of multilinear algebra as it appears here is hardly to be questioned, lying as it does at the bottom of the modern bundle and cohomological methods in algebra, topology, and differential geometry, and this book is the only English exposition I know of. Bourbaki is easier to read, but there is a forbidding amount of it, not all relevant. Chevalley is more compact and would be a fine book to precede say the Cartan-Eilenberg *Homological algebra* or many of the Cartan and Sophus Lie seminar notes from Paris. Except for some of the material in the last chapter, all of it is genuinely fundamental: the prospective reader can at least be assured that his sweat will not be wasted. This basic and essential *usefulness* of the book should be kept in mind as overshadowing any critical remarks made below.

In considering the treatment the book presents of its subject, one must recognize that it is extremely abstract, and the level of abstraction each man likes in his mathematics is as personal a taste as the amount of perfume he likes on his wife. When linear transformations made their comeback over matrices, it was easy to point to the shortened proofs and to the gain in clarity resulting from the triumph of abstraction over algorithm. An intense sans-culottism has since made the subject perhaps a bit top-heavy in concepts (after all, we still do have a multiplication table). Each reader will have to decide for instance whether Chevalley's seven page intrinsic treatment of the Pfaffian here is the height of beauty and elegance, or of absurdity, and if the former, whether the associated aroma is that of ripe bananas or of freshly-roasted coffee.

Considering the book from the standpoint of the student who wishes to learn multilinear algebra, an outstanding feature is the patient and insistent way in which the basic techniques are applied over and over again. One cannot emphasize too strongly the beautiful unity of the book: the pruning has been severe, perhaps, but at least one will not forget the essentials that have been left. On the other hand, the expository style in which it is written will certainly make the book abominably hard for a beginner, unreasonably hard, I should say. Not because it is unmotivated and dogmatic; as Chevalley points out, it could hardly be otherwise, and anyway, the average bright young graduate students seems to have the universal appetite of a goat. Rather it is simply because throughout the book the helping hand which could point out what is essential in a proof or definition is conspicuously and deliberately withheld. Not until page 223 does the author graciously unbend (perhaps as a reward to the reader for accompanying him so far) and confide that a complicated-looking theorem on the preceding page is only saying that a polynomial in n variables can be looked at as a polynomial in r variables whose coefficients are polynomials in the remaining $n-r$ variables. Some clue as to what is at the back of his mind may be gleaned from the following ringing utterance which ends the preface:

"This is an exercise in rectitude of thought, of which it would be futile to disguise the austerity."

The voice that we hear resounding is that of an Old Testament prophet, but the mental attitude is more like a tenth-grade Latin teacher's, reeking with the old theory of formal disciplines. Thinking rigorously demands first of all a firm grasp of the concepts, otherwise the sort of proof-following which passes for thinking is only a very sophisticated version of computation-checking. It is the difference between a rat running physically through a maze and a man running his pencil through one of the Sunday supplement mazes: one has the over-all understanding, the other does not. It is downright unfair for an older generation which learned these ideas in an intuitive fashion in which they were well-adapted for thought to foist off on a younger one, in the name of rigorous thinking and without any further explanation, such a construction as this one of a free abelian monoid $(\overline{F}, \overline{\Psi})$ on a set of generators S, which occurs at the tender page of 21:

Let N be the additive monoid of integers $\geqq 0$. For any $x \in S$, set $N_x = N$, and form the weak product $\prod^{*}_{x\in S} N_x$. This is a commutative monoid $\overline{F}$. Let ζ_x be the natural injection of N into $\overline{F}$ corresponding to the index x; if $x \in S$, set $\overline{\Psi}(x) = \zeta_x(1)$. Then $\overline{\Psi}$ is a mapping of S into $\overline{F}$. . . etc.

Granted that something like this must be said if one wants to be perfectly precise, nevertheless it obscures completely the *freeness* of the free abelian monoid. If everyone, Chevalley I dare say included, thinks of such a thing as the set of finite unordered formal products of elements of S, why can't this be said in another line or two?

We come finally to the question of the unorthodox subject matter of the book, considered as a first year graduate algebra text. Granting the importance of multilinear algebra, it only follows that it might reasonably be taught in the first year, and not that it necessarily should be. If it is, this means that such things as field theory and much of ring theory (factorizations, Noetherian rings) must go. It is a little hard to wave goodbye—algebra is thereby cut off from number theory and algebraic geometry, its two great classical estuaries. We get to be sure the cohomology theory, but no fields to apply it to.

However, as Chevalley remarks, one cannot teach everything in one year; let us consider multilinear algebra on its own merits. The following remarks are quite frankly subjective. Against this subject as an introduction to algebra, I would argue its dullness. It is a bit difficult to analyze precisely in what this dullness consists. Of course such a theorem as the general associativity law for tensor products is dull, but I mean more than that. Somehow, difficult as some of the material is, one never gets the feeling of advancing in depth; the difficulties lie in keeping track of the ever more complicated piling of module and map, and not as they do in, say, number theory in the inherent intricacy of the God-given structure. We have the feeling of not getting anywhere—we keep studying the same old things about ever more elaborate constructions. We keep on squaring pieces of wood in ever more curious sizes, but we never get to build the table. There is never any *einfall*. One never reads a proof and says "How clever!," one never sees a whole structure revealed by a theorem with a burst of insight—one just keeps plodding along for 200 pages and finally learns what a polynomial really is. Courses and books to accompany them must not do this, especially in early graduate work; they must reach climaxes. There is no Jordan-Hölder theorem, no Galois theorem, no basis theorem for abelian groups, no Wedderburn theorem to be found here—the subject just doesn't contain any. It's a tool subject, and tools should be kept in the closet, no matter how shiny they are.

Just for this reason, Chevalley's book should be welcomed. Now that it has appeared, multilinear algebra need not be taught in courses; the mature student of modern mathematics can learn the subject by himself. He has a fast, well-organized, meaty textbook, and he will be in good shape if he is able to find a friend who will explain now and then what is really going on inside it.

ARTHUR MATTUCK

BRIEF MENTION

Theory of Lie groups I. By C. Chevalley. Princeton University Press, 1946. 232 pp. $2.75 (paper).

Integration. By E. J. McShane. Princeton University Press, 1944. 402 pp. $2.95 (paper).

These are unchanged reissues, in paperback edition, of two out-of-print volumes in the Princeton Mathematical Series.

Quantum mechanics. By H. A. Kramers. New York, Interscience, 1957. 16+496 pp. $12.50.

A translation by D. ter Haar of monographs originally published in 1937 under the titles *Die Grundlagen der Quantentheorie* and *Quantentheorie des Elektrons und der Strahlung*.

Bibliography of Russian mathematics books. By G. E. Forsythe. New York, Chelsea, 1956. 106 pp. $3.95.

A list of over 600 titles of books in pure and applied mathematics published in Russian or Ukrainian since 1930.

Colloque sur la théorie des nombres, held at Brussels, December 1955. Paris, Masson, 1956. 208 pp. 2400 fr.

This includes papers by L. J. Mordell, Ch. Pisot, H. E. Richert, G. Ricci, J. Popken, A. Errera, K. F. Roth, P. Erdős, H. Davenport, H. Delange, J. G. van der Corput, and J. Teghem.

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, held at Berkeley, Calif., Dec. 1954 *and July–Aug.* 1955. Ed. by J. Neyman. Berkeley and Los Angeles, University of California Press, 1956. 5 vols.

Vol. 1: *Theory of statistics*. 10+208 pp. $6.00.
Vol. 2: *Probability theory*. 10+246 pp. $6.50.
Vol. 4: *Biology and problems of health*. 10+179 pp. $5.75.
Vol. 5: *Econometrics, industrial research, and psychometry*. 10+184 pp. $5.75.

Illinois Journal of Mathematics. Vol. 1, No. 1. March 1957.

A new quarterly journal of pure and applied mathematics, published by the University of Illinois, Urbana, Illinois, edited by R. Baer, J. L. Doob, A. H. Taub, G. W. Whitehead, and O. Zariski. The subscription price is $9.00 per volume ($5.00 for individual members of the American Mathematical Society).

RESEARCH PROBLEMS

5. V. L. Klee, Jr.: *Convex sets.*

In Euclidean n-space E, let C be convex, B the boundary of C, S the unit sphere $\{x: \|x\| = 1\}$, and D the set of all points of S which represent directions of line segments in B. (Specifically, $d \in D$ provided $\|d\| = 1$ and the segment $[-d, d]$ is parallel to some segment contained in B.) Then D is an F_σ set in S. Must D be of the first category in S? Must the $(n-1)$-measure of D be zero? (The answer is affirmative for $n = 2$, for then D is countable.) The same questions may be asked for more general classes of surfaces B. (Received January 18, 1957.)

6. V. L. Klee, Jr.: *Convex sets.*

In Euclidean n-space E, let K be a family of sets which includes all open convex sets, includes the union of each increasing sequence of its members, and includes the intersection of each decreasing sequence of its members. Must K include all convex Borel sets in E? (For $n = 2$, the answer is affirmative. See Duke Math. J. vol. 20 (1953) pp. 105–112.) (Received January 18, 1957.)

7. J. R. Isbell: *Commuting mappings of trees.*

Let T be a tree, i.e. a compact locally connected space in which every two points are joined by a unique arc. It can be seen that every commutative group Γ of homeomorphisms of T has a common fixed point. [By Zorn's lemma, it suffices to show that there is a proper subcontinuum which is mapped onto itself by every element of Γ. Observe that for f, g, in Γ, g maps the set S of fixed points of f into itself. So does g^{-1}; hence $g(S) = S$. Similarly g leaves invariant the least subcontinuum containing S. But this is all of T only if f leaves every end point of T fixed.] Is this true for commutative semigroups of continuous mappings? It is not known even for a semigroup generated by two mappings on an arc. (Received June 7, 1957.)

ERRATA, VOLUME 63

B. W. Brewer, *On certain character sums and related congruences.* Abstract 63-1-70.

Line 8 of the abstract: For "$\psi_4(Q)=3b(\alpha|p)$" read "$\psi_4(Q)=2b(\alpha|p)$."
Line 12 of the abstract: For "$Q\equiv\alpha^3\not\equiv\alpha^3$ mod p" read "$Q\equiv\alpha^3\not\equiv\alpha_1^3$ mod p."

David Zeitlin, *Plane parabolic functions.* Preliminary report. Abstract 63-2-271.

Line 1 of the abstract: "$\int_0{}^{x} ydx$" read "$\int_0^{x} ydx$."
Line 4 of the abstract: For "sinp$(-u)$" read ", sinp$(-u)$"
Line 6 of the abstract: Insert a comma at the end of the line.
Line 9 of the abstract: For "arccscp u" read ", arccscp u."
Line 12 of the abstract: For "odd n, $a_n+3^{-1}\sum_{m=0}^{n}\sum_{p=0}^{m} a_pa_{n-m}a_{m-p}=0$." read "odd $n>1$, $a_n+3^{-1}\cdot\sum_{m=0}^{n}\sum_{p=0}^{m} a_pa_{n-m}a_{m-p}=0$."

W. R. Abel and L. M. Blumenthal: *Metric arcs with a tangent property*. Abstract 63-4-534.

The "less than" sign that occurs in line 9 of the abstract should be replaced by a "greater than" sign.

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